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Zbigniew Grande, Institute of Mathematics, Pedagogical University, ul. Chodkiewicza 30, 85-064 Bydgoszcz, Poland. e-mail: grande@wsp.bydgoszcz.pl

ON SOME SPECIAL NOTIONS OF APPROXIMATE QUASI-CONTINUITY

Abstract

Some special notions of approximate quasi-continuity and cliquishness are considered. Moreover, uniform, pointwise and transfinite convergence of sequences of such functions are investigated.

Let $\mathbb{R}$ be the set of all reals and let μ_e (μ) denote outer Lebesgue measure (Lebesgue measure) in $\mathbb{R}$. Let

$$d_u(A,x) = \limsup_{h\to 0^+} \mu_e(A\cap(x-h,x+h))/2h$$

$$(d_l(A,x) = \liminf_{h\to 0^+} \mu_e(A\cap(x-h,x+h))/2h)$$

denote the upper (lower) density of a set $A \subset \mathbb{R}$ at a point x. A point $x \in \mathbb{R}$ is called a density point of a set $A \subset \mathbb{R}$ if there exists a measurable (in the sense of Lebesgue) set $B \subset A$ such that $d_l(B,x) = 1$. The family

$$\mathcal{T}_d = \{A \subset \mathbb{R}; A \text{ is measurable and each } x \in A \text{ is a density point of } A\}$$

is a topology called the density topology [1].

Moreover, let $\mathcal{T}_e$ denote the Euclidean topology on $\mathbb{R}$. All considered functions will be real and defined on $\mathbb{R}$. A point x is a continuity point (an approximate continuity point) of a function f if it is a continuity point of f considered as an application from $(\mathbb{R},\mathcal{T}_e)$ $((\mathbb{R},\mathcal{T}_d))$ to $(\mathbb{R},\mathcal{T}_e)$. Denote by $C(f)$ (resp. $A(f)$) the set of all continuity (approximate continuity) points of f. Define the following families of functions:

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– $f \in \mathcal{A}_1$ ($f \in \mathcal{A}_2$) if for every point x, for every positive real η, and for every set $A \in \mathcal{T}_d$ containing x there is a point $u \in A \cap C(f)$ ($u \in A(f) \cap A$) such that $|f(u) - f(x)| < \eta$;

– $f \in \mathcal{A}_3$ ($f \in \mathcal{A}_4$) if for every point x, for every positive real η, and for every set $A \in \mathcal{T}_d$ containing x there is an open interval I such that $I \cap A \neq \emptyset$, $I \cap A \subset C(f)$ ($I \cap A \subset A(f)$) and $|f(u) - f(x)| < \eta$ for all points $u \in I \cap A$;

– $f \in \mathcal{A}_5$ ($f \in \mathcal{A}_6$) if for every nonempty set $A \in \mathcal{T}_d$ there is an open interval I such that $I \cap A \neq \emptyset$ and $I \cap A \subset C(f)$ ($I \cap A \subset A(f)$).

– $f \in \mathcal{A}_7$ if for each positive real η and for each nonempty set $A \in \mathcal{T}_d$ there is an open interval I such that $I \cap A \neq \emptyset$, $I \cap A \subset A(f)$ and $\operatorname{osc} f \leq \eta$ on $I \cap A$.

Moreover, a function f is said to be strongly quasi-continuous [strongly cliquish] (abbreviated s.q.c. [s.c]) at a point x if for every set $A \in \mathcal{T}_d$ containing x and for every positive real η there is an open interval I such that $I \cap A \neq \emptyset$ and $|f(t) - f(x)| < \eta$ for all $t \in A \cap I$ [$\operatorname{osc} f < \eta$ on $I \cap A$] ([4]).

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a function and let $x \in \mathbb{R}$ be a point. If there is an open set U such that $d_u(U, x) > 0$ and the restricted function $f|(U \cup \{x\})$ is continuous at x, then f is s.q.c. at x.

By elementary proofs, we obtain:

Remark 1. *If all functions f_n, $n = 1, 2, \ldots$, of some uniformly converging sequence $(f_n)_n$ are s.q.c. at a point x, then its limit f is also s.q.c. at x.*

Remark 2. *If $f \in \mathcal{A}_i$, $i = 1, 3, 4$, then f is s.q.c. at each point.*

Remark 3. *The inclusions*

$$\mathcal{A}_3 \subset \mathcal{A}_1 \subset \mathcal{A}_2; \;\; \mathcal{A}_3 \subset \mathcal{A}_4 \subset \mathcal{A}_2; \;\; \mathcal{A}_3 \subset \mathcal{A}_5 \cup \mathcal{A}_4 \subset \mathcal{A}_7 \subset \mathcal{A}_6$$

are true.

Since every function which is s.q.c. at each point is also almost everywhere continuous [4], we can observe that a function $f \in \mathcal{A}_1$ if and only if f is s.q.c. at each point x and that the families $\mathcal{A}_i$, $i = 1, 3, 4$, contain only almost everywhere continuous functions. It is obvious also that all functions belonging to the family $\mathcal{A}_5$ are almost everywhere continuous.

Approximately continuous functions are in Baire class 1, so they belong to $\mathcal{A}_7 \cap \mathcal{A}_2$. Since there are approximately continuous functions which are not almost everywhere continuous ([1]), in the families $\mathcal{A}_7$ and $\mathcal{A}_2$ there are functions which are not almost everywhere continuous. However, if $f \in \mathcal{A}_2 \cup \mathcal{A}_6$, then for every positive real η and for every measurable set A with $\mu(A) > 0$ there is a measurable subset $B \subset A$ such that $\mu(B) > 0$ and $\operatorname{osc} f \leq \eta$ on B. So, by Davies theorem ([2, 3]), every function $f \in \mathcal{A}_2 \cup \mathcal{A}_6$ is measurable (in the sense of Lebesgue). If $f \in \mathcal{A}_6$, then for every open interval I and for every positive real η there is an open interval $J \subset I$ such that $\operatorname{osc} f \leq \eta$ on J. Thus, the set $C(f)$ of arbitrary function $f \in \mathcal{A}_6$ is dense, and consequently it is a residual G_δ set. So, every function $f \in \mathcal{A}_6$ has the Baire property. However, there are functions $f \in \mathcal{A}_2$ which do not have the Baire property. Therefore, we adopt the following definition ([6]).

A function f is called approximately quasi-continuous at a point x if for every positive real η and for each set $A \in \mathcal{T}_d$ containing x there is a nonempty set $B \subset A$ belonging to $\mathcal{T}_d$ such that $|f(t) - f(x)| < \eta$ for all $t \in B$.

Observe that a function f is approximately quasi-continuous at every point if and only if $f \in \mathcal{A}_2$. In [6] it is proved that for every measurable function f there is a sequence of approximately quasi-continuous functions f_n, $n = 1, 2, \ldots$ such that $f = \lim_{n\to\infty} f_n$. So, if f is measurable without the Baire property, then there is an index k such that the function f_k does not have the Baire property. Such function f_k is in the family $\mathcal{A}_2$ as an approximately quasi-continuous function, but it doesn't have the Baire property. From the above it follows that

$$\mathcal{A}_2 \setminus \mathcal{A}_i \neq \emptyset, \quad i = 1, 3, 4, 5, 6, 7.$$

Since there is an approximately continuous function f with $\operatorname{int}(C(f)) = \emptyset$ ([1]) we obtain $\mathcal{A}_7 \setminus \mathcal{A}_5 \neq \emptyset$.

The function $f(x) = 0$ for $x \neq 0$ and $f(0) = 1$ belongs to $\mathcal{A}_5 \subset \mathcal{A}_7$, but it is not in $\mathcal{A}_2 \supset \mathcal{A}_1 \supset \mathcal{A}_3$. Moreover, it is not in $\mathcal{A}_4$. So,

$$\mathcal{A}_5 \setminus \mathcal{A}_i \neq \emptyset, \quad i = 1, 2, 3, 4.$$

If g is an approximately continuous function such that $g(\mathbb{R}) = [0, 1]$ and the set $C(f) = g^{-1}(0)$ is dense and its interior is empty ([1]), then the function $f(x) = g(x)$ for $x \neq 0$ and $f(0) = 2$ belongs to $\mathcal{A}_7 \setminus \mathcal{A}_i$ for $i = 1, 2, 3, 4, 5$.

As an example of function $f \in \mathcal{A}_1 \setminus \mathcal{A}_6$ we can take any strictly monotone function which is continuous from the right at every point and which is such that the set $\mathbb{R} \setminus C(f)$ is dense.

So, we obtain the following assertion.

Theorem 1. *The inclusions*

– $\mathcal{A}_3 \subset \mathcal{A}_1 \subset \mathcal{A}_2$; $\quad \mathcal{A}_3 \subset \mathcal{A}_4 \subset \mathcal{A}_2$*;*

– $\mathcal{A}_3 \subset \mathcal{A}_5 \subset \mathcal{A}_7$; $\mathcal{A}_3 \subset \mathcal{A}_4 \subset \mathcal{A}_7$;

are true. Moreover, each of the above inclusions is strict and

$$\mathcal{A}_1 \setminus \mathcal{A}_6 \neq \emptyset.$$

The inclusion $\mathcal{A}_7 \subset \mathcal{A}_6$ is evident.

Theorem 2. *The relation $\mathcal{A}_6 \setminus \mathcal{A}_7 \neq \emptyset$ is true.*

PROOF. Let $A_1 \subset I_1 = (0,1)$ be a Cantor set of positive measure. In the second step in every component $I_{1,n}$, $n = 1,2,\ldots$, of the set $I_1 \setminus A_1$ we find an open interval $J_{1,n}$ having the same centers as $I_{1,n}$ and such that $|J_{1,n}| < 4^{-2}|I_{1,n}|$, where the symbol $|J_{1,n}|$ denotes the length of the interval $J_{1,n}$. Next, in each open interval $J_{1,n}$ we find a Cantor set $A_{1,n} \subset J_{1,n}$ of positive measure.

In general, in the k^{th} step ($k > 2$) we consider all components $I_{1,n_2,\ldots,n_k}$ of the set $J_{1,n_2,\ldots,n_{k-1}} \setminus A_{1,n_2,\ldots,n_{k-1}}$, $n_i \geq 1$ for $1 < i \leq k$, and we find open intervals $J_{1,n_2,\ldots,n_k} \subset I_{1,n_2,\ldots,n_k}$ having the same centers as $I_{1,n_2,\ldots,n_k}$ and such that

$$|J_{1,n_2,\ldots,n_k}| < 4^{-k}|I_{1,n_2,\ldots,n_k}|. \tag{1}$$

For $1 < i \leq k$ and $n_i = 1,2,\ldots$, let $A_{1,n_2,\ldots,n_k} \subset J_{1,n_2,\ldots,n_k}$ be a Cantor set of positive measure and let

$$B_{1,n_2,\ldots,n_k} = \{x \in A_{1,n_2,\ldots,n_k}; d_l(A_{1,n_2,\ldots,n_k}, x) = 1\}.$$

Let $f(x) = 1$ for $x \in B_{1,n_2,\ldots,n_k}$, whenever k is even and $n_i \geq 1$ for $1 < i \leq k$ and let $f(x) = 0$ otherwise on $\mathbb{R}$. We shall prove that $f \in \mathcal{A}_6$. Let $A \in \mathcal{T}_d$ be a nonempty set. Denote by B the union of all sets $B_{1,n_2,\ldots,n_k}$, where $k = 2,3,\ldots$ and $n_i = 1,2,\ldots$ for $1 < i \leq k$. Observe that $B \subset A(f)$. So, if $A \subset B$, then for every open interval I such that $I \cap A \neq \emptyset$ we have $I \cap A \subset A(f)$. If A is not a subset B, then there is a point $x \in A \setminus B$. If x is not in $\mathrm{cl}(B)$, where $\mathrm{cl}(B)$ denotes the closure of the set B, then f is continuous at x and for every interval $I \subset \mathbb{R} \setminus \mathrm{cl}(B)$ containing x we obtain that $I \cap A \neq \emptyset$ and $I \cap A \subset C(f) \subset A(f)$. So, we suppose that $x \in cl(B)$. If $x = 0$ or $x = 1$, then f is unilaterally continuous at x and there is an open interval $I \subset \mathcal{R} \setminus [0,1]$ with $I \cap A \neq \emptyset$. Consequently, in this case we have $f(t) = f(x) = 0$ for all $t \in I \cap A$. If there are indexes $n_2,\ldots,n_k$ such that $x \in A_{1,n_2,\ldots,n_k} \setminus B_{1,n_2,\ldots,n_k}$, then, by (1), there is an open interval

$$I \subset J_{1,n_2,\ldots,n_k} \setminus A_{1,n_2,\ldots,n_k} \setminus \bigcup_{n_{k+1}=1}^{\infty} J_{1,n_2,\ldots,n_{k+1}}$$

such that $I \cap A \neq \emptyset$ and $I \cap A \subset C(f) \subset A(f)$. If not, there is a sequence of indexes $n_2, \ldots, n_k, \ldots$, such that

$$x \in \bigcap_{k=1}^{\infty} I_{1,n_2,\ldots,n_k} = \bigcap_{k=1}^{\infty} J_{1,n_2,\ldots,n_k}.$$

Since $d_l(A,x) = 1$, there is an index m such that for each $k \geq m$ we have

$$\mu(A \cap I_{1,n_2,\ldots,n_k}) > |I_{1,n_2,\ldots,n_k}|/2. \tag{2}$$

By (1), there is an index $j > m$ such that for each $k \geq j$ we obtain

$$4|J_{1,n_2,\ldots,n_k}| < |I_{1,n_2,\ldots,n_k}|. \tag{3}$$

Fix $k > j$ and observe that from (2) and (3) it follows that

$$A \cap (I_{1,n_2,\ldots,n_k} \setminus cl(J_{1,n_2,\ldots,n_k})) \neq \emptyset.$$

Consequently, there is an open interval I such that $A \cap I \neq \emptyset$ and $I \cap A \subset C(f) \subset A(f)$. So, $f \in \mathcal{A}_6$.

For the proof that f is not in $\mathcal{A}_7$ it suffices to observe that $B \in \mathcal{T}_d$ and for every open interval I with $I \cap B \neq \emptyset$ the oscillation $\operatorname{osc} f$ on $I \cap B$ is greater than a for every positive real $a < 1$. □

Remark 4. *Observe that if $f \in \mathcal{A}_6$ is of the first Baire class, then $f \in \mathcal{A}_7$. Consequently, every approximately continuous function belongs to $\mathcal{A}_7$.*

For a family Φ let $\mathcal{B}(\Phi)$ ($\mathcal{B}_u(\Phi)$) denote the family of all limits of converging (of uniformly converging) sequences of functions from Φ. Moreover, let Q_s denote the family of all functions which are s.q.c. at every point. By Remark 1 the family $Q_s = \mathcal{A}_1$ is uniformly closed.

Theorem 3. *The equality $\mathcal{B}_u(\mathcal{A}_3) = Q_s$ is true.*

PROOF. Let $f \in Q_s$ be a function. It suffices to prove that for every positive real η there is a function $g \in \mathcal{A}_3$ such that $|f - g| \leq \eta$. Fix a real $\eta > 0$. Let

$$E = \{y; \mu(cl(f^{-1}(y))) > 0\}.$$

Since f is almost everywhere continuous, the set E is countable. There is a sequence $(c_k)_{k=-\infty}^{\infty}$ of reals $c_k \in \mathbb{R} \setminus E$ such that $0 < c_{k+1} - c_k < \eta/2$ for all integers k and

$$\mathbb{R} = \bigcup_{k=-\infty}^{\infty} [c_k, c_{k+1}).$$

If x is such that $c_k \le f(x) < c_{k+1}$, then let $h(x) = c_k$. If h is not s.q.c. at x and $h(x) = c_k$, then we put $g(x) = c_{k-1}$. Otherwise let $g(x) = h(x)$. Since f is s.q.c. at each point $x \in \mathbb{R}$, the function g is the same on $\mathbb{R}$. But the image $g(\mathbb{R})$ is a discrete set, so we obtain $g \in \mathcal{A}_3$. Evidently, $|f - g| \le \eta$. □

Since $\mathcal{A}_3 \subset \mathcal{A}_4 \subset Q_s$, we obtain the following.

Corollary 1. *The equality $\mathcal{B}_u(\mathcal{A}_4) = Q_s$ is true.*

Remark 5. *The equality $\mathcal{B}_u(\mathcal{A}_2) = \mathcal{A}_2$ is true.*

Proof. Fix $f \in \mathcal{B}_u(\mathcal{A}_2)$. There is a sequence of functions $f_n \in \mathcal{A}_2$, $n = 1, 2, \ldots$, which converges uniformly to f. Fix a point x, a positive real η and a set $A \in \mathcal{T}_d$ containing x. Let k be an index for which $|f_k - f| < \eta/4$. Since $f_k \in \mathcal{A}_2$, there is a point $u \in A \cap A(f_k)$ such that $|f_k(u) - f_k(x)| < \eta/4$. But $u \in A(f_k)$, so there is a measurable subset $B \subset A$ such that $\mu(B) > 0$ and $|f_k(t) - f_k(u)| < \eta/4$ for each point $t \in B$. The function f is measurable as the limit of the sequence of measurable functions f_n. Consequently, there is a point $w \in B \cap A(f) \subset A \cap A(f)$. Moreover, we have

$$\begin{aligned} |f(w) - f(x)| \le & |f(w) - f_k(w)| + |f_k(w) - f_k(u)| + |f_k(u) - f_k(x)| \\ & + |f_k(x) - f(x)| < \eta/4 + \eta/4 + \eta/4 + \eta/4 = \eta. \end{aligned}$$ □

Let $\mathcal{C}_{ae}$ denote the family of all functions f for which $\mu(\mathbb{R} \setminus C(f)) = 0$. We have the following.

Remark 6. *The equality $\mathcal{B}_u(\mathcal{A}_5) = \mathcal{C}_{ae}$ is true.*

Proof. It suffices to prove that for every $f \in \mathcal{C}_{ae}$ and for every positive real η there is a function $h \in \mathcal{A}_5$ with $|f - h| \le \eta$. Fix $f \in \mathcal{C}_{ae}$ and a positive real η. Define the function h the same as that in the proof of Theorem 4 and observe that $h \in \mathcal{A}_5$, because it is almost everywhere continuous and its image $h(\mathbb{R})$ is a discrete set. □

Theorem 4. *A function $f \in \mathcal{B}_u(\mathcal{A}_7)$ if and only if it is strongly cliquish at each point $x \in \mathbb{R}$.*

Proof. Assume, to the contrary, that there is a function $f \in \mathcal{B}_u(\mathcal{A}_7)$ such that there are a nonempty set $A \in \mathcal{T}_d$ and a positive real η with $\operatorname{osc} f > \eta$ on $A \cap I$ for every open interval I such that $I \cap A \neq \emptyset$. Since $f \in \mathcal{B}_u(\mathcal{A}_7)$, there is a function $g \in \mathcal{A}_7$ such that $|f - g| < \eta/4$. For every open interval I such that $I \cap A \neq \emptyset$ we obtain that $\operatorname{osc} g > \eta/2$ on the set $I \cap A$. But $g \in \mathcal{A}_7$, so there is an open interval I such that $I \cap A \neq \emptyset$ and $I \cap A \subset A(g)$ and $\operatorname{osc} g < \eta/2$

on $I \cap A$. This contradiction shows that if $f \in \mathcal{B}_u(\mathcal{A}_7)$, then f is s.c. at each point.

Now, suppose the function f is s.c. at each point $x \in \mathbb{R}$. Fix a positive real η. We shall prove that there is a function $g \in \mathcal{A}_7$ such that $|f - g| < \eta$. Let I_1 be an open interval with rational endpoints such that $\operatorname{osc} f < \eta$ on I_1. Fix an ordinal number $\alpha > 1$ and suppose that for every ordinal number $\beta < \alpha$ there is an open interval I_β with rational endpoints such that $\mu(I_\beta \setminus G_\beta) > 0$, where $G_\beta = \cup_{\gamma<\beta} I_\gamma$, and $\operatorname{osc} f < \eta$ on the set

$$H_\beta = \{x \in I_\beta \setminus G_\beta; d_l(I_\beta \setminus G_\beta, x) = 1\}.$$

Since the function f is s.c. at every point, there is an open interval I_α with rational endpoints such that $\mu(I_\alpha \setminus G_\alpha) > 0$ and $\operatorname{osc} f < \eta$ on the set H_α. By transfinite induction we find a transfinite sequence of such open intervals $(I_\alpha)_{\alpha<\alpha_0}$ with rational endpoints, where α_0 is the first ordinal number for which $\mu(\mathbb{R} \setminus G_{\alpha_0}) = 0$. Since the family of all open intervals with rational endpoints is countable, the ordinal number α_0 is also countable. For every $\alpha < \alpha_0$ we find a point $x_\alpha \in H_\alpha$ and let $g(x) = f(x_\alpha)$ for $x \in H_\alpha$ and $g(x) = f(x)$ otherwise on $\mathbb{R}$. If $x \in H_\alpha$, then

$$|f(x) - g(x)| = |f(x) - f(x_\alpha)| < \eta.$$

In the remaining case $f(x) = g(x)$, so $|f - g| < \eta$. For the completeness of the proof it suffices to show that the function $g \in \mathcal{A}_7$. For this, fix a nonempty set $A \in \mathcal{T}_d$ and a positive real ε. There is an ordinal number $\beta < \alpha_0$ such that $I_\beta \cap A \neq \emptyset$ and $I_\alpha \cap A = \emptyset$ for $\alpha < \beta$. Then $\emptyset \neq A \cap I_\beta \subset H_\beta$, since otherwise we have $G_\beta \cap A \neq \emptyset$, a contradiction. Consequently, $g(x) = f(x_\beta)$ for each point $x \in I_\beta \cap A$. So, $\operatorname{osc} f = 0 < \varepsilon$ on $I_\beta \cap A$. $\square$

Remark 7. *The function f from the proof of Theorem 2 is such that $f \in \mathcal{A}_6 \setminus \mathcal{B}_u(\mathcal{A}_7)$, since* $\operatorname{osc} f = 1$ *on the sets $I \cap B$ for every open interval I with $I \cap B \neq \emptyset$.*

Observe that if $f \in \mathcal{B}_u(\mathcal{A}_6)$, then f is measurable and the set $\mathbb{R} \setminus C(f)$ is of the first category. Moreover, if

$$\text{ap-osc}\, f(x) = \inf\{\operatorname{osc}_A f; \emptyset \neq A \in \mathcal{T}_d,\ \ x \in A\},$$

then we have the following.

Theorem 5. *If a function f belongs to $\mathcal{B}_u(\mathcal{A}_6)$, then for every positive real η and for every nonempty set $A \in \mathcal{T}_d$ the set $\{x \in A; \text{ap-osc}\, f(x) \geq \eta\}$ is nowhere dense in A.*

Proof. Assume, to the contrary, that there are a function $f \in \mathcal{B}_u(\mathcal{A}_6)$, a nonempty set $A \in \mathcal{T}_d$, and a positive real η such that for every open interval J with $J \cap A \neq \emptyset$ there is a point $x \in J \cap A$ at which ap-osc $f(x) \geq \eta$. Since $f \in B_u(\mathcal{A}_6)$, there is a function $g \in \mathcal{A}_6$ such that $|f - g| < \eta/3$. Observe that if ap-osc $f(x) \geq \eta$, then ap-osc $g(x) \geq \eta/4$. So, for every open interval J with $J \cap A \neq \emptyset$ there is a point $x \in J \cap A$ at which ap-osc $g(x) \geq \eta/4$. But $g \in \mathcal{A}_6$, so there is an open interval J with $J \cap A \neq \emptyset$ and $J \cap A \subset A(f)$. Consequently, for each point $x \in J \cap A$ we obtain ap-osc $g(x) = 0 < \eta/4$, a contradiction. □

Problem. Characterize the class $\mathcal{B}_u(\mathcal{A}_6)$.

Remark 8. *Since every function $f \in \mathcal{A}_2$ is measurable and every measurable function is the limit of a sequence of approximately quasi-continuous functions (which belong to $\mathcal{A}_2$) ([6]), we obtain that $\mathcal{B}(\mathcal{A}_2)$ is the family of all measurable functions.*

In [8] Mauldin shows that $f \in \mathcal{B}(\mathcal{C}_{ae})$ if and only if there are a function g of Baire class 1 and an F_σ-set A of measure zero such that $\{x; f(x) \neq g(x)\} \subset A$.

Theorem 6. *The equality $\mathcal{B}(\mathcal{A}_5) = \mathcal{B}(\mathcal{C}_{ae})$ is true.*

Proof. Since every $f \in \mathcal{A}_5$ belongs to $\mathcal{C}_{ae}$, by Mauldin's theorem we obtain the inclusion $\mathcal{B}(\mathcal{A}_5) \subset \mathcal{B}(\mathcal{C}_{ae})$. Let $f \in \mathcal{B}(\mathcal{C}_{ae})$ be a function. By Mauldin's theorem there are a function g of the first class of Baire and an F_σ-set A of measure zero such that $\{x; f(x) \neq g(x)\} \subset A$. Let $h = f - g$. Then $h(x) = 0$ for each point x which is not in A. There are closed sets A_n, $n = 1, 2, \ldots$, such that $A_1 \subset \ldots \subset A_n \subset \ldots$ and $A = \bigcup_n A_n$. For $n = 1, 2, \ldots$ let $h_n(x) = h(x)$ for $x \in A_n$ and let $h(x) = 0$ otherwise on $\mathbb{R}$. Since every set A_n, $n = 1, 2, \ldots$, is closed and of measure zero, it is nowhere dense and consequently every function $h_n \in \mathcal{A}_5$, $n = 1, 2, \ldots$. For the function g there is a sequence of continuous functions g_n, $n = 1, 2, \ldots$, such that $g = \lim_{n\to\infty} g_n$. Observe that every function $f_n = g_n + h_n$, $n = 1, 2, \ldots$, belongs to $\mathcal{A}_5$ as the sum of the continuous function g_n and the function h_n belonging to $\mathcal{A}_5$. Since

$$f = g + h = \lim_n g_n + \lim_n h_n = \lim_n f_n,$$

the proof is complete. □

In [5] the following theorem is proved.

Theorem 7. *Let f be a function such that there is a Baire 1 function g such that for every positive real η and for each point x such that $|f(x) - g(x)| \geq \eta$ there is a closed interval $I(x)$ containing x and such that $\mu(I(x) \setminus cl(\{t; |f(t) - g(t)| \geq \eta\})) = 0$. Then there is a sequence of functions $f_n \in Q_s$ such that $\lim_n f_n = f$.*

Since every function g from Q_s is the limit of a sequence of functions from $\mathcal{A}_3$ which uniformly converges to g, every function satisfying the hypothesis of the above theorem belongs to $\mathcal{B}(\mathcal{A}_3)$.

We can prove the following.

Theorem 8. *Let f be a function such that there are a Baire 1 function g and an F_σ set B of measure zero such that $\{t; f(t) \neq g(t)\} \subset B$ and for every positive real η and for each point x such that $|fx) - g(x)| \geq \eta$ the upper density $d_u(cl(\{t; |f(t) - g(t)| \geq \eta\}), x) = 0$. Then there is a sequence of functions $f_n \in \mathcal{A}_3$, $n = 1, 2, \ldots$, which converges to f.*

Proof. Let $h = f - g$ and let $B = \cup_n B_n$, where every set B_i is closed and $B_i \subset B_{i+1}$ for $i = 1, 2, \ldots$. For $n = 1, 2, \ldots$ let $A_n = \{x; |h(x)| \geq 1/n\}$. Fix a positive integer n. By our hypothesis there is a family of disjoint closed intervals $I_{k,l,i}$, $k \leq n$, $l, i = 1, 2, \ldots$ such that

- $I_{1,l,i} \subset \mathbb{R} \setminus (\mathrm{cl}(A_n) \cap B)$ for $l, i \geq 1$;
- $I_{k,l,i} \subset \mathbb{R} \setminus \mathrm{cl}(A_n)$ for $1 < k \leq n$, $l, i = 1, 2, \ldots$;
- for each $k \leq n$ the inclusion
$$I_{k,l,i} \subset A(\mathrm{cl}(A_k) \cap B_k, 1/k)(= \{t; \inf\{|u - t|; u \in \mathrm{cl}(A_k) \cap B_k\} \leq 1/n\}$$
is true for $l, i \geq 1$;
- if $k \leq n$, $l \geq 1$ and $x \in \mathrm{cl}(A_k) \cap B_k$, then $d_u\big(\bigcup_{i=1}^\infty I_{k,l,i}, x\big) > 0$;
- for each $k \leq n$ and for each $x \in \mathbb{R} \setminus (A_k \cap B_k)$ there is an open set U containing x such that the set $\{(k, l, i); U \cap I_{k,l,i} \neq \emptyset\}$ is finite.

Next, in every interval $\mathrm{int}(I_{k,l,i})$, $k \leq n$, $l, i = 1, 2, \ldots$, we find a closed interval $J_{k,l,i}$ such that if $k \leq n$, $l \geq 1$ and $x \in \mathrm{cl}(A_k) \cap B_k$, then $d_u\big(\bigcup_{i=1}^\infty J_{k,l,i}, x)\big) > 0$. Let $(w_{1,l})_{l=1}^\infty$ be a sequence of all rationals with $w_{1,1} = 0$ and for $k > 1$ let $(w_{k,l})_{l=1}^\infty$ be a sequence of all rationals belonging to the interval $[-1/(k-1), 1/(k-1)]$ with $w_{k,1} = 0$. Put

$$h_n(x) = \begin{cases} w_{k,l} & x \in J_{k,l,i}, k \leq n, l, i \geq 1 \\ h(x) & x \in A_n \cap B_n \\ \text{linear on the components} & \\ \text{of the sets } I_{k,l,i} \setminus J_{k,l,i}, & l, i \geq 1,\ k \leq n \\ 0 & \text{otherwise on } \mathbb{R}. \end{cases}$$

Evidently, the function h_n is continuous at each point x which is not in $\mathrm{cl}(A_n)\cap B_n$. Fix a positive real η, a point $x\in \mathrm{cl}(A_n)\cap B_n$ and a set $A\in \mathcal{T}_d$ containing x. If there is an integer $k\le n$ such that $x\in A_k\cap B_k$, then there is a rational $w_{k,l}$ such that $|h(x)-w_{k,l}|<\eta$. Since $d_u\big(\bigcup_i J_{k,l,i},x\big)>0$, there is an interval $J_{k,l,i}$ such that $J_{k,l,i}\cap A\ne\emptyset$. Every point $t\in A\cap J_{k,l,i}$ is a continuity point of h_n and

$$|h_n(t)-h_n(x)|=|w_{k,l}-h(x)|<\eta.$$

In the remaining case we obtain that $h_n(x)=0$ and $x\in \mathrm{cl}(A_k)\cap B_k$ for some positive integer k. Since $w_{k,1}=0$ and since $d_u\big(\bigcup_i J_{k,1,i},x\big)>0$, there is an interval $J_{k,1,i}$ such that $J_{k,1,i}\cap A\ne\emptyset$. For each point $t\in J_{k,1,i}\cap A$ the function h_n is continuous at t and $|h_n(t)-h_n(x)|=0<\eta$. So, $h_n\in\mathcal{A}_3$.

Now we will prove that $\lim_{n\to\infty}h_n=h$. If there is a positive integer n with $x\in A_n\cap B_n$, then $h_k(x)=h(x)$ for $k\ge n$. If not, we have $h(x)=0$. Fix a positive real η and a positive integer n with $1/n<\eta$. If $x\in \mathrm{cl}(A_m)\cap B_m$ for some positive integer m, then $h_k(x)=0$ for all $k\ge m$. So, we suppose that x is not in $\mathrm{cl}(A_m)\cap B_m$ for $m\ge 1$. Since x is not in the set $cl(A_n)\cap B_n$, there is a positive integer $m>n$ such that $|x-y|>1/m$ for every point $y\in \mathrm{cl}(A_n)$. Consequently,

$$h_k(x)\le 1/(m-1)\le 1/n<\eta$$

for every $k\ge m$. This completes the proof that $h=\lim_{n\to\infty}h_n$.

Since g is a Baire 1 function, there is a sequence of continuous functions g_n such that $g=\lim_{n\to\infty}g_n$. Evidently, the functions $f_n=h_n+g_n$, $n=1,2,\ldots$, belong to the family $\mathcal{A}_3$ and

$$\lim_{n\to\infty}f_n=\lim_{n\to\infty}h_n+\lim_{n\to\infty}g_n=h+g=f. \qquad\square$$

Corollary 2. *If the function $f:\mathbb{R}\longrightarrow\mathbb{R}$ is almost everywhere continuous, then $f\in\mathcal{B}(\mathcal{A}_3)$.*

Denote by $\mathcal{P}_s$ the family of all functions which are s.c. at each point.

Problem. Is it true that $\mathcal{P}_s\cap\mathcal{B}(\mathcal{C}_{ae})=\mathcal{B}(\mathcal{A}_3)$?

Now, we will investigate the transfinite convergence of sequences. Let ω_1 denote the first uncountable ordinal number. A transfinite sequence of functions f_α, $\alpha<\omega_1$, converges to a function f ($\lim_\alpha f_\alpha=f$) if for each point x there is an ordinal number $\beta<\omega_1$ such that $f_\alpha(x)=f(x)$ for each countable ordinal $\alpha>\beta$.

Theorem 9. *Let $\mathcal{K}$ be a family of functions such that if a function f is not in $\mathcal{K}$, then there is a countable set A such that for every function $g\in\mathcal{K}$ there is a point $x\in A$ with $g(x)\ne f(x)$. Then the limits of all converging transfinite sequences of functions from the family $\mathcal{K}$ belong to $\mathcal{K}$.*

PROOF. Let $(f_\alpha) \in \mathcal{K}$, $\alpha < \omega_1$, and let $\lim_\alpha f_\alpha = f$. Suppose, to the contrary, that f is not in $\mathcal{K}$. Then there is a countable set $A = \{x_1, x_2, \ldots\}$ such that for each function $g \in \mathcal{K}$ there is a point $x \in A$ with $g(x) \neq f(x)$. For each positive integer n there is a countable ordinal number β_n such that $f_\alpha(x_n) = f(x_n)$ for $\beta_n < \alpha < \omega_1$. There is a countable ordinal number β such that $\beta_n < \beta$ for all positive integers n. So, $f_\beta(x_n) = f(x_n)$ for $n = 1, 2, \ldots$. Since $f_\beta \in \mathcal{K}$, we obtain a contradiction. □

Remark 9. *Observe that the families P_s and $\mathcal{A}_i$, $i = 1, 3, 5$, satisfy the hypothesis of the above Theorem 9.*

PROOF. If f is not in $\mathcal{A}_i$, $i = 1$ or 3 or 5, then every countable set A such that the set $\{(x, f(x)); x \in A\}$ is dense in the graph of the function f satisfies all requirements. □

Theorem 10. *Assume the Continuum Hypothesis HC. For every function f there is a transfinite sequence of functions $f_\alpha \in \mathcal{A}_2$, $\alpha < \omega_1$, such that $f = \lim_\alpha f_\alpha$.*

PROOF. Let $(x_\alpha)_{\alpha<\omega_1}$ be a transfinite sequence of all reals numbers. Fix an ordinal number $\alpha < \omega_1$ and let $(t_n)_n$ be a sequence of all numbers x_β with $\beta \leq \alpha$ such that $t_i \neq t_j$ for $i \neq j$, $i, j = 1, 2, \ldots$. For every positive integer n there are closed intervals I_n, J_n and a closed set A_n such that

- t_n is an endpoint of I_n and J_n;
- $J_n \subset I_n$ and $|J_n| = |I_n|/2$;
- $A_n \subset J_n \setminus \{t_k; k \neq n \text{ and } k = 1, 2, \ldots\}$;
- $I_n \cap A_k = \emptyset$ for $k < n$;
- $d_u(A_n, t_n) > 0$;
- $\mu(A_n) < |J_n|/4^n$.

For a construction of such I_n, J_n and A_n it suffices to find a nowhere dense closed set

$$A_n \subset \mathbb{R} \setminus \bigcup_{k \neq n} \{t_k\} \setminus \bigcup_{k<n} A_k$$

such that $d_u(A_n, t_n) > 0$ and next fix some closed intervals I_n and J_n satisfying all requirements.

Denote by B_n the set of all points $t \in A_n$ at which $d_u(A_n, t) > 0$. Let

$$f_\alpha(x) = \begin{cases} f(t_n) & \text{if } x \in B_n, n = 1, 2, \dots \\ 0 & \text{otherwise on } \mathbb{R} \end{cases}$$

Then $f_\alpha \in \mathcal{A}_2$ and $f = \lim_\alpha f_\alpha$.

Theorem 11. *If functions $f_\alpha \in \mathcal{A}_4$ for $\alpha < \omega_1$ and $f = \lim_\alpha f_\alpha$, then $f \in \mathcal{A}_4$.*

PROOF. Assume, by a contrary, that f is not in $\mathcal{A}_4$. Since by Theorems 1, 9 and Remark 9 the function $f \in \mathcal{A}_1$, there are a positive real η, a point x and a set $A \in \mathcal{T}_d$ such that $x \in A$ and for every open interval I with $I \cap A \neq \emptyset$ and $f(I \cap A) \subset (f(x) - \eta, f(x) + \eta)$ there is a point $t \in I \cap A$ at which the function f is not approximately continuous. Let $(x_n)_n$ be a sequence of points such that the set $\{(x_n, f(x_n)); n = 1, 2, \dots\}$ is dense in the graph of the function f and for each open interval I with $I \cap A \neq \emptyset$ and $f(I \cap A) \subset (f(x) - \eta, f(x) + \eta)$ there is a point $x_{n(I)} \in I \cap A$ at which f is not approximately continuous. There is a countable ordinal number β such that $f_\alpha(x_n) = f(x_n)$ for all countable ordinal numbers $\alpha \geq \beta$ and $n = 1, 2, \dots$. Consequently, the functions f and f_β are almost everywhere equal. Since $f \in \mathcal{A}_1$, there is an open interval I such that $I \cap A \neq \emptyset$ and $f(A \cap I) \subset (f(x) - \eta, f(x) + \eta)$. From the relation $f_\beta \in \mathcal{A}_4$ we obtain that there is an open interval $J \subset I$ such that $J \cap A \neq \emptyset$, $f_\beta(J \cap A) \subset (f(x) - \eta, f(x) + \eta)$ and the function f_β is approximately continuous at every point $t \in J \cap A$. Let a point $u = x_k \in A \cap J$ be such that the function f is not approximately continuous at u. Since $f_\beta(u) = f(u)$ and the functions f and f_β are equal at almost all points, the function f_β must be approximately continuous at u, a contradiction. □

Observe that there are Baire 1 functions which are not in $\mathcal{A}_6$. Since every Baire 1 function is the limit of a transfinite sequence of approximately continuous functions (see [7]), then the classes $\mathcal{A}_i$, $i = 6, 7$, are not closed under the transfinite convergence. But if a function f is the limit of a transfinite sequence of functions $f_\alpha \in \mathcal{A}_6$ ($f_\alpha \in \mathcal{A}_7$), then f is pointwise discontinuous (pointwise discontinuous on each nonempty set belonging to $\mathcal{T}_d$).

Problem. Characterize the functions being the limits of transfinite sequences of functions belonging to $\mathcal{A}_i$, $i = 6, 7$.

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