INROADS

Real Analysis Exchange
Vol. 36(1), 2010/2011, pp. 161–168

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THE DISTRIBUTION FUNCTION AND MEASURE PRESERVING MAPS

Abstract

Existence of measure preserving maps has been discussed in books on real analysis where the Axiom of Choice is instrumental. In this note we introduce a method to *construct* such maps. For our construction we use the distribution function and elementary differential equations.

1 Introduction.

The subject of measure preserving maps is addressed, for example, in [7], where existence of such maps between complete separable metric spaces is proved by applying the *Axiom of Choice*. So actual *formulas* of measure preserving maps are not given, hence this would not be very appealing to students. In this note, however, we introduce a method to *construct* infinitely many measure preserving maps from $\Omega \subseteq \mathbb{R}^n$ onto $[0, 1]$. Here Ω denotes a smooth and bounded domain satisfying $\mathcal{L}_n(\Omega) = 1$, where $\mathcal{L}_n$ denotes the n-dimensional Lebesgue measure in $\mathbb{R}^n$. The tools we use are the *distribution function* in conjunction with elementary differential equations.

The significance of measure preserving maps is partially due to their natural presence in real world phenomena. Here is an example from fluid mechanics. Let us consider an incompressible fluid occupying a bounded region D in $\mathbb{R}^2$. Denoting the velocity field of the flow by V, the motion of the flow is

Mathematical Reviews subject classification: Primary: 54C30, 34B05; Secondary: 35J25
Key words: measure preserving maps, distribution function, differential equations, Saint-Venant equation
Received by the editors June 10, 2009
Communicated by: Brian S. Thomson

mathematically formulated by the following initial value problem:

$$(IVP): \quad \begin{cases} \frac{d}{dt}\phi_t(x) = V(\phi_t(x)), & \text{in } D \\ \phi_0(x) = x \in D. \end{cases}$$

The solution of the (IVP), $\phi_t(x)$, represents the trajectory along which the fluid particle initially located at x travels. It is well known that $\phi_t : D \to D$ is a measure preserving map. In addition, ϕ_t is a diffeomorphism, if V is smooth. It is also known that the vorticity function associated to the flow, denoted $\omega_t(x)$, is constant along the trajectory $\phi_t(x)$. As a consequence, $\omega_t(x)$ and $\omega_0(x)$, the initial vorticity function, are equimeasurable in the sense that they verify:

$$\mathcal{L}_2(\omega_t^{-1}(U)) = \mathcal{L}_2(\omega_0^{-1}(U)),$$

for every Borel measurable set U in $\mathbb{R}$. For a thorough discussion on measure preserving maps in fluid mechanics the reader is referred to [1].

Another reason for the importance of measure preserving maps is that they appear in polar factorization of integrable vector valued functions, a concept that was introduced by Brenier, see [3] and [4], and may be interpreted as a generalization of an idea due to Ryff [8]. Here we briefly describe the polar factorization in a setting relevant to our purpose. Let $u \in L^1(X, \mu; \mathbb{R})$ be an integrable function defined on the measure space (X, μ), which is a measure-interval, see [2] for definition. Let $Y \subseteq \mathbb{R}$ be a Lebesgue measurable set satisfying $\mathcal{L}_1(Y) = \mu(X)$. The monotone rearrangement of u on Y, denoted $u^\sharp$, is the function $u^\sharp : Y \to \mathbb{R}$ that is a rearrangement of u, and satisfies $u^\sharp = \nabla\psi$, for some convex function $\psi : \mathbb{R} \to \mathbb{R}$. Let us note in passing that recently McCann [6] proved the uniqueness of the monotone rearrangement $u^\sharp$. We say u has a *polar factorization through* Y if there exists a measure preserving map $s : (X, \mu) \to (Y, \mathcal{L}_1)$ such that $u = u^\sharp \circ s$, almost everywhere in X. An interesting result of Burton and Douglas [2] states that if $u^\sharp$ is almost injective then u has a unique polar factorization.

2 Preliminaries.

As mentioned above Ω is assumed to be a fixed smooth and bounded domain in $\mathbb{R}^n$ with $\mathcal{L}_n(\Omega) = 1$. In case $n = 1$, we set $\Omega = [0, 1]$. We begin by recalling the following:

Definition. We say $\sigma : \overline{\Omega} \to [0, 1]$ is a measure preserving map if

$$\mathcal{L}_n(\sigma^{-1}(A)) = \mathcal{L}_1(A),$$

for every Borel set $A \subseteq [0,1]$. Here σ^{-1} denotes the pre-image.

The following result is well known.

Proposition 1. *Every measure preserving map $\sigma : \overline{\Omega} \to [0,1]$ is surjective, modulo sets of measure zero.*

Proof. Observe that $\sigma^{-1}([0,1] \setminus \sigma(\overline{\Omega})) = \emptyset$, hence $\mathcal{L}_n(\sigma^{-1}([0,1] \setminus \sigma(\overline{\Omega})) = 0$. On the other hand, since σ is measure preserving, we have $\mathcal{L}_n(\sigma^{-1}([0,1] \setminus \sigma(\overline{\Omega})) = \mathcal{L}_1([0,1] \setminus \sigma(\overline{\Omega}))$. Thus we obtain $\mathcal{L}_1([0,1] \setminus \sigma(\overline{\Omega})) = 0$. Whence $[0,1] = \sigma(\overline{\Omega})$, modulo a set of $\mathcal{L}_1$-measure zero. □

For a (Lebesgue) measurable function $u : \overline{\Omega} \to \mathbb{R}^+ \equiv [0, \infty)$, the function $\lambda_u(t)$ defined by

$$\lambda_u(t) = \mathcal{L}_n(\{x \in \overline{\Omega} : \ u(x) \geq t\}), \quad \forall t \geq 0,$$

is called the distribution function of u. It is clear that $\lambda_u(t)$ is non-increasing and left continuous. Moreover, if u has no flat sections then $\lambda_u(t)$ is continuous.

Definition. We say $u : \overline{\Omega} \to \mathbb{R}^+$ has no flat sections if the graph of u has (Lebesgue) negligible flat sections; that is,

$$\mathcal{L}_n(\{x \in \overline{\Omega} : \ u(x) = \alpha\}) = 0, \ \ \forall \alpha \geq 0.$$

Lemma 1. *Suppose $u : \overline{\Omega} \to \mathbb{R}^+$ is continuous and has no flat sections. Then $\lambda_u : [u_m, u_M] \to [0,1]$ is a continuous bijection. Moreover, we have the following*

$$\int_0^1 \chi_E \circ \lambda_u^{-1}(t) \, dt = \int_\Omega \chi_E \circ u(x) \, dx, \tag{1}$$

for every $E \subseteq [0,1]$. Here, u_m and u_M denote the minimum and maximum values of u over $\overline{\Omega}$, respectively. Also, χ_E indicates the characteristic function of E; that is,

$$\chi_E(t) = \begin{cases} 1 & t \in E \\ 0 & t \notin E. \end{cases}$$

Proof. Since u has no flat sections it follows that $\lambda_u(t)$ is continuous. On the other hand, since u is continuous, we infer that $\lambda_u(t)$ is decreasing. Finally, observe that $\lambda_u(u_m) = 1$ and $\lambda_u(u_M) = 0$, hence $\lambda_u(t)$ is a continuous bijection from $[u_n, u_M]$ onto $[0,1]$.

Now we prove (1). Consider $E \subseteq [0,1]$ and observe that $\chi_{_E} \circ \lambda_u^{-1}(t) = \chi_{_{\lambda_u(E)}}(t)$. Thus,

$$\int_0^1 \chi_{_E} \circ \lambda_u^{-1}(t) \, dt = \mathcal{L}_1(\lambda_u(E)). \tag{2}$$

Case 1. $E = (a, b)$.

In this case $\mathcal{L}_1(\lambda_u(E)) = \mathcal{L}_1([\lambda_u(b), \lambda_u(a)])$, since $\lambda_u(t)$ is continuous and decreasing. Thus,

$$\begin{aligned}\mathcal{L}_1(\lambda_u(E)) = \lambda_u(a) - \lambda_u(b) &= \mathcal{L}_n(\{u \geq a\}) - \mathcal{L}_n(\{u \geq b\}) \\ &= \mathcal{L}_n(\{a \leq u < b\}) = \mathcal{L}_n(\{a < u < b\}),\end{aligned}$$

where in the last equality we have used the fact that u has no flat sections. We now obtain

$$\begin{aligned}\mathcal{L}_1(\lambda_u(E)) = \mathcal{L}_n(u^{-1}((a, b))) &= \int_\Omega \chi_{u^{-1}((a,b))}(x)\,dx \\ &= \int_\Omega \chi_{(a,b)} \circ u(x)\,dx = \int_\Omega \chi_E \circ u(x)\,dx.\end{aligned}$$

Therefore, from (2), we deduce (1).

Remark. It is clear from the above argument that the assertion in the lemma also holds when E is any interval of the form $[a, b)$, $(a, b]$ or $[a, b]$.

Case 2. E is an open subset of $[0, 1]$.

Since any open subset of $[0, 1]$ can be written as a disjoint union of intervals of the form (a, b), $[a, b)$, (a, b) and $[a, b]$, the assertion follows immediately from Case 1, and the proceeding remark.

Case 3. E is a Borel subset of $[0, 1]$.

In this case we consider $S = \{F \subseteq [0, 1] : \text{ (1)is valid}\}$. It is easy to see that S is a σ-algebra. Hence, since S contains open sets, it must also contain the Borel sets as well. This completes the proof of the lemma. □

3 Main results.

Our main result is the following:

Theorem 1. *Suppose $u \in C(\overline{\Omega})$ is a positive function that has no flat sections. Then $\sigma := \lambda_u \circ u$ is a measure preserving map from Ω onto $[0, 1]$.*

PROOF. Let us fix $A \subseteq [0, 1]$. Then

$$\mathcal{L}_n(\sigma^{-1}(A)) = \mathcal{L}_n(u^{-1}(\lambda_u^{-1}(A))) = \int_\Omega \chi_{u^{-1}(\lambda_u^{-1}(A))}(x)\,dx,$$

where u^{-1} denotes the pre-image. Thus,

$$\mathcal{L}_n(\sigma^{-1}(A)) = \int_\Omega \chi_{\lambda_u^{-1}(A)} \circ u(x)\, dx = \int_\Omega \chi_{\lambda_u^{-1}(A)} \circ \lambda_u^{-1}(x)\, dx, \tag{3}$$

where in the last equality we have used Lemma 2, which is applicable since $\lambda_u^{-1}(A)$ is a Borel set, thanks to the continuity of λ_u. Note that $\chi_{\lambda_u^{-1}(A)} \circ \lambda_u^{-1}(x) = \chi_A(x)$, hence

$$\int_\Omega \chi_{\lambda_u^{-1}(A)} \circ \lambda_u^{-1}(x)\, dx = \mathcal{L}_1(A). \tag{4}$$

from (3) and (4) we infer $\mathcal{L}_n(\sigma^{-1}(A)) = \mathcal{L}_1(A)$, hence σ is measure preserving. □

Now using Theorem 1 we construct measure preserving maps. Let us first consider the case $n = 1$, so $\Omega = [0,1]$. The following simple boundary value problem helps us to construct infinitely many measure preserving maps from $[0,1]$ onto $[0,1]$:

$$(P_{\alpha,\beta}) \qquad \begin{cases} -u'' = 1 & \text{in } (0,1) \\ u(0) = \alpha, & u(1) = \beta, \end{cases}$$

where α, β are non-negative constants. Note that $(P_{\alpha,\beta})$ has a unique solution $u_{\alpha,\beta} \in C^2([0,1])$, which is non-negative and has no flat sections.

Example 1. Let $\alpha = 1$, $\beta = 0$. In this case $u(x) := u_{1,0}(x) = -x^2/2 - x/2 + 1$, and $\lambda_u(t) = (-1 + \sqrt{9-8t})/2$, for $t \in [0,1]$. Hence, $\sigma(x) = x$.

Example 2. Let $\alpha = 0$, $\beta = 1$. In this case $u(x) := u_{0,1}(x) = -x^2/2 + 3x/2$, and $\lambda_u(t) = (-1 + \sqrt{9-8t})/2$. Hence, $\sigma(x) = 1 - x$.

Example 3. Let $\alpha = \beta = 0$. In this case $u(x) := u_{0,0}(x) = -x^2/2 + x^2/2$, and $\lambda_u(t) = \sqrt{1-8t}$, for $0 \le t \le 1/8$. Hence, $\sigma(x) = |1 - 2x|$. Indeed, for every Borel set $A \subseteq [0,1]$, we have

$$\mathcal{L}_1(\sigma^{-1}(A)) = \int_{\sigma^{-1}(A)} dx = 2 \int_{\sigma_1^{-1}(A)} dx,$$

where $\sigma_1(x) = 1 - 2x$. On the other hand, using the change of variable formula, we obtain $\int_{\sigma_1^{-1}(A)} dx = \int_A \frac{1}{2}\, dx$. Thus, $\mathcal{L}_1(\sigma^{-1}(A)) = \int_A dx = \mathcal{L}_1(A)$.

Clearly every pair (α, β) induces a measure preserving map $\sigma_{\alpha,\beta}$. Our next result shows that there are only two monotone measure preserving maps from $[0,1]$ onto $[0,1]$; namely, $\sigma^1(x) = x$ and $\sigma^2(x) = 1 - x$.

Theorem 2. *Suppose σ is a measure preserving map from $[0,1]$ onto $[0,1]$. In addition, suppose σ is continuous and monotone, then either $\sigma = x$ or $\sigma = 1 - x$.*

PROOF. We first assume σ is increasing. Since σ is surjective it follows that $\sigma(0) = 0$ and $\sigma(1) = 1$. Consider $x \in (0,1)$, and let $0 < t$ be small enough to ensure $x + t \in [0,1]$. Then clearly we have

$$\frac{\sigma(x+t) - \sigma(x)}{t} = \frac{1}{t}\mathcal{L}_1(\sigma([x, x+t])) = \frac{1}{t}\mathcal{L}_1([x, x+t]) = 1,$$

where in the second equality we have used the fact that since σ is bijective, σ^{-1} is measure preserving from $[0,1]$ onto $[0,1]$ as well. Thus,

$$\lim_{t \to 0^+} \frac{\sigma(x+t) - \sigma(x)}{t} = 1.$$

Similarly one can show

$$\lim_{t \to 0^-} \frac{\sigma(x+t) - \sigma(x)}{t} = 1.$$

Hence $\sigma'(x) = 1$ in $(0,1)$. So, $\sigma(x) = x + C$. Recalling $\sigma(0) = 0$, we infer $C = 0$, hence $\sigma(x) = x$, as desired. For the case when σ is decreasing, one uses similar arguments as above and shows that $\sigma'(x) = -1$, so using the fact that this time $\sigma(0) = 1$, we infer $\sigma(x) = 1 - x$. $\square$

Example 4. Let $B(0,R)$ denote the n-dimensional ball, $n \geq 2$, centered at the origin with radius R. We construct a measure preserving map from $B(0,R)$ onto $[0,1]$. Let u be the unique solution of the following Saint-Venant boundary value problem:

$$\begin{cases} -\Delta u = 1, & \text{in } B(0,R) \\ u = 0, & \text{on } \partial B(0,R), \end{cases}$$

where Δ denotes the standard Laplacian operator; that is, $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$. It is easy to see that $u(x) = (R^2 - \|x\|_2)/(2n)$, where $\|\cdot\|_2$ stands for the usual n-dimensional Euclidean distance. Note that $u \geq t$ is the ball centered at the origin with radius $\sqrt{R^2 - 2nt}$. Whence, $\lambda_u(t) = \omega_n(R^2 - 2nt)^n/2$, where ω_n denotes the volume of the unit n-ball. Thus $\lambda_u \circ u(x) = \omega_n \|x\|_2$. To check that σ indeed is a measure preserving map, it suffices to show that $\mathcal{L}_n(\sigma^{-1}((a,b))) = b - a$. But this follows from

$$\mathcal{L}_n(\sigma^{-1}((a,b))) = \mathcal{L}_n\left(B(0, (\frac{b}{\omega_n})^{1/n}) \setminus B(0, (\frac{a}{\omega_n})^{1/n})\right) = b - a,$$

as desired.

The above example can be generalized as follows.

Example 5. Let $f(x)$ be a radial function on $B(0,R)$; that is, $f(x) = \phi(r)$, where $r = \|x\|_2$. We also assume $f(x)$ is positive and smooth. Consider the following boundary value problem:

$$\begin{cases} -\Delta u = f(x), & \text{in } B(0,R) \\ u = 0, & \text{on } \partial B(0,R). \end{cases}$$

It is well known, see for example [5], that

$$u(x) = \frac{1}{(n\omega_n^{1/n})^2} \int_{\omega_n r^n}^{\omega_n R^n} \xi^{\frac{2}{n}-2} \left(\int_0^{\xi} f^{\Delta}(s)\, ds \right) d\xi,$$

where $f^{\Delta}(s)$ is defined by

$$f^{\Delta}(s) = \inf\{\alpha \geq 0 : \ \lambda_f(\alpha) \leq s\}.$$

Therefore $\sigma(x) = \lambda_u \circ u(x)$ defines a measure preserving map from $B(0,R)$ onto $[0,1]$.

Remark. Note that the positiveness of $f(x)$, in Example 5, guarantees that $u(x)$ has no flat sections.

Acknowledgment. The author would like to thank Professor Jiří Neustupa from the Czech Academy of Sciences, Prague, for his comments on the original version of the paper. He also likes to thank the anonymous referees for their comments which helped to improve the presentation.

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