Real Analysis Exchange
Vol. (), , pp. 205–222

D. J. Dewsnap, Southeastern Oklahoma State University, P.O. Box 4194, Durant, OK 74701-0609. e-mail: `ddewsnap@sosu.edu`

P. Fischer,*Department of Mathematics and Statistics, University of Guelph, Guelph, ON N1G 2W1, Canada. e-mail: `pfischer@uoguelph.ca`

INTERVAL MAPS AND KOENIGS' SEQUENCES

Abstract

Let f be an interval map in a neighborhood of the fixed point 0 with $-1 < \lambda = f'(0) < 0$. Continuity of f is not assumed at points other than the fixed point. It is shown that if either

$$f \circ f(x) \geq \lambda^2 x \text{ or } f \circ f(x) \leq \lambda^2 x$$

for each x in a neighborhood of 0, then the Koenigs' sequence $\{\phi_k\}$ defined by $\phi_k(x) = \dfrac{f^k(x)}{\lambda^k}$ converges uniformly to a limit ϕ in a neighborhood of 0 with $\phi(0) = 0$ and $\phi'(0) = 1$. Two examples are presented, the first of which is a C^1 map f with $f(0) = 0$ and $-1 < f'(0) < 0$ having a divergent Koenigs' sequence. The other example is a C^1 convex map g with $g(0) = 0$ and $0 < g'(0) < 1$ for which the associated Koenigs' sequence diverges and which has no orientation-reversing composition square root that is differentiable at 0.

1 Introduction

The Koenigs' sequence $\{\phi_k\}$ associated with an interval map $f : I \to I$ with fixed point 0 in I and $0 < |f'(0)| < 1$ is defined by $\phi_k(x) = \dfrac{f^k(x)}{(f'(0))^k}$ for each $x \in I$, where f^k denotes the k'th iterate of f. Our present study concentrates on an interval map $f : I \to I$ with fixed point 0 in I and $-1 < \lambda = f'(0) < 0$.

It is well-known that Koenigs' sequence converges on a local neighborhood of 0 if $f \in C^{1+\epsilon}$ for some $\epsilon > 0$. It is also well-known that $f \in C^1$ is insufficient

Key Words: Koenigs' sequences, iterative square roots
Mathematical Reviews subject classification: 39B12, 58F03
Received by the editors August 22, 1997
*Work supported in part by the NSERC of Canada under grant A-8421.

to guarantee convergence of Koenigs' sequence. We are interested in finding general conditions on f that ensure convergence of Koenigs' sequence on a neighborhood of 0. The limiting behavior of Koenigs' sequence is dependent on an interplay of the orbits of f and f^2. We show that when either

$$f \circ f(x) \geq \lambda^2 x \text{ or } f \circ f(x) \leq \lambda^2 x$$

on a neighborhood of 0, then Koenigs' sequence converges uniformly on a neighborhood of 0 to a limit ϕ with $\phi(0) = 0$ and $\phi'(0) = 1$. Furthermore, it is shown that ϕ is invertible on each orbit of f in a neighborhood of 0. Consequently, since ϕ satisfies the Schröder equation

$$\phi \circ f(x) = \lambda \phi(x)\,, \tag{1}$$

it conjugates f orbitwise to its linearization λx that is, $\phi \circ f \circ \phi^{-1}(x) = \lambda x$.

Another interest in studying convergence properties of Koenigs' sequences is the problem of existence of iterative roots. Consider a map g having fixed point 0 with $0 < \lambda^2 = g'(0) < 1$ and $-1 < \lambda < 0$. Suppose g satisfies either

$$g(x) \geq \lambda^2 x \text{ or } g(x) \leq \lambda^2 x$$

on a neighborhood of 0, and let f be an orientation-reversing iterative square root of g. A necessary condition for such an f to be differentiable at the fixed point 0, with $\lambda = f'(0)$, is that the limit ϕ of the Koenigs' sequence for g satisfy the Schröder equation as it appears in (1) on a neighborhood of 0.

1.1 Koenigs' Sequences

Suppose $f : X \to \boldsymbol{C}$ is an analytic function where X is a neighborhood of the origin in the complex plane, $f(0) = 0$, and $0 < |f'(0)| < 1$. G. Koenigs [3] showed that the Schröder equation [4], $\phi \circ f(z) = \lambda\, \phi(z)$ where λ is a scalar, has a unique local analytic solution ϕ given by

$$\phi(z) = \lim_{k\to\infty} \phi_k(z) = \lim_{k\to\infty} \frac{f^k(z)}{\lambda^k}\,, \tag{2}$$

where $\lambda = f'(0)$, $\phi(0) = 0$, and $\phi'(0) = 1$. The sequence $\{\phi_k\}$ is the Koenigs' sequence for f. The Schröder equation was introduced in a more general form by E. Schröder in 1871 [6] and has since been studied extensively; traditionally because of its connection with the problem of continuous iteration (see [9], p. 31). The Schröder equation is an eigenvalue equation of a composition operator. If ϕ is an invertible solution of the Schröder equation, then ϕ conjugates f to its linearization λz that is, $\phi \circ f \circ \phi^{-1}(z) = \lambda z$. There are numerous results concerning solutions of the Schröder equation and associated convergence of Koenigs' sequences. A number of authors [8], [4], [7], have considered the case when f is an interval map. These results pertain to maps that are continuous and strictly increasing on a neighborhood of the fixed point.

There is an independent motivation for investigating the limiting behavior of Koenigs' sequences [2]. The problem of obtaining smooth solutions of the Feigenbaum-Cvitanović functional equation is related to the problem of determining sufficient conditions on an interval self-map f to ensure convergence of Koenigs' sequence on a neighborhood of a stable fixed point 0 when $-1 < f'(0) < 0$. For a good introduction to this equation see [5].

2 Stable Fixed Points

Throughout this paper, I shall denote an interval of the reals of finite positive length and will be regarded as the underlying topological space. If $0 \in I$, then a neighborhood $\eta(0)$ shall refer to a subinterval of I having 0 as an interior point, with corresponding left-neighborhood defined by $\eta^-(0) = \eta(0) \cap \{x \in I | x < 0\}$ and right-neighborhood $\eta^+(0)$ similarly defined. We begin with a result that identifies an important class of functions.

Lemma 2.1. *Let $f : I \to I$ with $0 \in I$ and $f(0) = 0$. Let λ and $\varepsilon > 0$ satisfy $-1 < \lambda \pm \varepsilon < 0$. Then*

$$\lambda - \varepsilon < \frac{f(x)}{x} < \lambda + \varepsilon \tag{3}$$

for each $x \neq 0$ if and only if the orbits of f have the following properties:

1. *If $x < 0$, then $x(\lambda+\varepsilon)^{2k+1} < f^{2k+1}(x) < x(\lambda-\varepsilon)^{2k+1}$ for each $k \geq 0$, and $x(\lambda-\varepsilon)^{2k} < f^{2k}(x) < x(\lambda+\varepsilon)^{2k}$ for each $k > 0$.*
2. *If $x > 0$, then $x(\lambda-\varepsilon)^{2k+1} < f^{2k+1}(x) < x(\lambda+\varepsilon)^{2k+1}$ for each $k \geq 0$, and $x(\lambda+\varepsilon)^{2k} < f^{2k}(x) < x(\lambda-\varepsilon)^{2k}$ for each $k > 0$.*

PROOF. Assume that (3) holds for each $x \in I \backslash \{0\}$. It follows from (3), the f-invariance of I, and the choice of ε, that

$$0 < x(\lambda+\varepsilon) < f(x) < x(\lambda-\varepsilon) \leq |x| \tag{4}$$

for each $x \in I^-(0)$, and

$$-|x| \leq x(\lambda-\varepsilon) < f(x) < x(\lambda+\varepsilon) < 0 \tag{5}$$

for each $x \in I^+(0)$. Successively applying (4) and (5) yields 1.) and 2.). The converse follows from 1.) and 2.) with $k = 0$. □

If $f : I \to I$ has a fixed point $0 \in I$ with $-1 < f'(0) < 0$, then there is an f-invariant neighborhood $\eta(0)$ for which f satisfies the conclusions of Lemma 2.1 with $\lambda = f'(0)$. We now present a corresponding lemma for interval self-maps where the trajectories converge to the stable fixed point in a monotonic manner. In the following lemma, if $I^-(0)$ or $I^+(0)$ is empty, then it is understood that the result is only valid for the one-sided neighborhood of 0.

Lemma 2.2. *Let $f : I \to I$ with $0 \in I$ and $f(0) = 0$. Let λ and $\varepsilon > 0$ satisfy $0 < \lambda \pm \varepsilon < 1$. Then $\lambda - \varepsilon < \frac{f(x)}{x} < \lambda + \varepsilon$ for each $x \neq 0$ if and only if the orbits of f have the following properties:*

1. *If $x < 0$, then $x(\lambda + \varepsilon)^k < f^k(x) < x(\lambda - \varepsilon)^k$ for each $k > 0$.*
2. *If $x > 0$, then $x(\lambda - \varepsilon)^k < f^k(x) < x(\lambda + \varepsilon)^k$ for each $k > 0$.*

PROOF. For the given $0 < \lambda < 1$ and $\varepsilon > 0$ assume that (3) holds for each $x \in I\backslash\{0\}$. It follows from (3) and the choice of ε that

$$x < x(\lambda + \varepsilon) < f(x) < x(\lambda - \varepsilon) < 0 \tag{6}$$

for each $x \in I^-(0)$, and

$$0 < x(\lambda - \varepsilon) < f(x) < x(\lambda + \varepsilon) < x \tag{7}$$

for each $x \in I^+(0)$. Successively applying (6) and (7) yields 1.) and 2.). The converse follows from 1.) and 2.) with $k = 1$. □

If $f : I \to I$ has a fixed point $0 \in I$ with $0 < f'(0) < 1$, then there is an f-invariant neighborhood $\eta(0)$ wherein f satisfies the conclusions of Lemma 2.2 with $\lambda = f'(0)$. Functions satisfying either Lemma 2.1 or Lemma 2.2 will be referred to as being 0-local.

Definition 2.3. *A function f is* 0-local *on I if $f : I \to I$, $0 \in I$, $f(0) = 0$, and there is a λ and $\varepsilon > 0$ satisfying $0 < |\lambda| \pm \varepsilon < 1$ such that*

$$\lambda - \varepsilon < \frac{f(x)}{x} < \lambda + \varepsilon$$

for each $x \neq 0$.

The notion of c-locality can be defined in a similar manner for any real c. At times, it will be necessary to specify that f is 0-local on I with λ and the given ε. The next result, which follows from the definition, presents a useful property of 0-local functions.

Proposition 2.4. *If f is 0-local on I with λ, then f^n is 0-local on I with λ^n for each $n > 0$.*

A function satisfying the conditions of Lemma 2.1 is a strictly decreasing function of its orbits, and a function satisfying the conditions of Lemma 2.2 is a strictly increasing function of its orbits. The following lemma is an improvement of Lemma 2.1 and Lemma 2.2 for interval maps that are differentiable at the fixed point.

Lemma 2.5. *Let f be 0-local on I with $\lambda = f'(0)$ and ε. Then there is a sequence of functions $\{\tau_k\}_{k=1}^{\infty}$ such that $f^k(x) = x\prod_{\nu=1}^{k}(\lambda + \tau_\nu(x))$ for each $x \in I$ and $k > 0$, where $\tau_k : I \to (-\varepsilon, \varepsilon)$, $\tau_k(0) = 0$, τ_k is continuous at 0, and $\tau_k \xrightarrow{k} 0$ uniformly on I.*

PROOF. Define

$$\tau_k(x) = \begin{cases} \dfrac{f(f^{k-1}(x))}{f^{k-1}(x)} - \lambda\,, & \text{if } x \in I\backslash\{0\}\,; \\ 0\,, & \text{if } x = 0\,, \end{cases}$$

and note that $\tau_{k+1} = \tau_k \circ f$ for each $k > 0$. The formula for $f^k(x)$ is verified by substitution of the given formula for τ_k. We conclude from Lemma 2.1 and Lemma 2.2 that $\tau_k \xrightarrow{k} 0$ uniformly on I. The 0-locality of f and the differentiability of f at 0 imply that τ_k is continuous at 0. □

The final lemma of this section provides a means of constructing and representing a function in the form stated in the preceding lemma.

Lemma 2.6. *Let $\varepsilon > 0$, $0 < |\lambda| \pm \varepsilon < 1$, and let $\tau : I \to (-\varepsilon, \varepsilon)$ with $0 \in I$, $\tau(0) = 0$, and τ continuous at 0. If*

$$f(x) = x(\lambda + \tau(x)) \tag{8}$$

for each $x \in I$, then f is 0-local on I with $\lambda = f'(0)$ and ε. Let $\tau_1 = \tau$ and $\tau_{k+1} = \tau_k \circ f$ for each $k > 0$. Then $f^k(x) = x\prod_{\nu=1}^{k}(\lambda + \tau_\nu(x))$ for each $x \in I$ and $k > 0$, where $\tau_k : I \to (-\varepsilon, \varepsilon)$, $\tau_k(0)=0$, τ_k is continuous at 0, and $\tau_k \xrightarrow{k} 0$ uniformly on I.

PROOF. Since (8) holds for each $x \in I$,

$$\tau(x) = \begin{cases} \dfrac{f(x)}{x} - \lambda\,, & \text{if } x \in I\backslash\{0\}\,; \\ 0\,, & \text{if } x = 0\,. \end{cases}$$

By assumption $\tau : I \to (-\varepsilon, \varepsilon\ )$ and τ is continuous at 0. Consequently, (3) holds for each $x \in I\backslash\{0\}$ with $\lambda = f'(0)$. It follows that f is 0-local on I with $\lambda = f'(0)$ and ε. Since $\{\tau_k\}_{k=1}^{\infty}$ satisfies the recursion formula $\tau_{k+1} = \tau_k \circ f$, the proof is complete. □

Lemma 2.6 will be used to construct an interval map for which the associated Koenigs' sequence diverges everywhere except at the fixed point.

3 Convex Functions and Stable Fixed Points

A study of the convexity relationship between an interval self-map and its second iterate in a neighborhood of a stable fixed point will be useful for establishing sufficient conditions for convergence of Koenigs' sequences. We begin with a result that reveals a convexity relationship between a twice differentiable function f and its second iterate f^2 in a neighborhood of a stable fixed point 0 when $-1 < f'(0) < 0$.

Proposition 3.1. *Let $f : I \to I$ be twice differentiable with $0 \in I$, $f(0) = 0$, $-1 < f'(0) < 0$, and $f''(0) > 0$. If f'' is continuous at 0, then there is a neighborhood of 0 wherein f is strictly convex and f^2 is strictly concave.*

Proof. If $g = f^2$, then $g''(0) = f'(0)f''(0)(1 + f'(0))$ and therefore $f''(0)$ and $g''(0)$ have opposite signs. Since $f''(0) > 0$ and f'' is continuous at 0, there is a neighborhood of 0 wherein $f''(x) > 0$ and $g''(x) < 0$. □

The proof of Proposition 3.1 yields the following related result.

Proposition 3.2. *Let $f : I \to I$ be twice differentiable with $0 \in I$, $f(0) = 0$, either $f'(0) < -1$ or $f'(0) > 0$, and $f''(0) > 0$. If f'' is continuous at 0, then there is a neighborhood of 0 wherein f and f^2 are both strictly convex.*

If $f''(x) > 0$ for each x in a neighborhood of 0 but f'' is discontinuous at 0, then the conclusion of Proposition 3.1 is no longer ensured. The following example illustrates this fact.

Example 3.3. *Let*

$$f(x) = \begin{cases} x(x/2 + \mu), & \text{if } x \le 0\,; \\ x(x/2 + \mu) + \rho \int_0^x t^2 \sin(1/t)dt, & \text{if } x > 0\,, \end{cases}$$

where $-1 < \mu < 0$ and $1 + \mu < \rho < 1$. Therefore,

$$f'(x) = \begin{cases} \mu + x, & \text{if } x \le 0; \\ \mu + x + \rho x^2 \sin(1/x), & \text{if } x > 0\,, \end{cases}$$

and

$$f''(x) = \begin{cases} 1, & \text{if } x \le 0; \\ 1 + 2\rho x \sin(1/x) - \rho\cos(1/x), & \text{if } x > 0\,. \end{cases}$$

Clearly f is strictly convex for $x \le 0$, and $f''(x) > 1 - \rho - 2\rho x$ for each $x > 0$. Since $\rho < 1$, there is a $\delta > 0$ such that $f''(x) > 0$ for each $x \in [-\delta, \delta]$. Let f be 0-local on $\eta(0) \subseteq [-\delta, \delta]$. Then $\mu = f'(0)$, f'' is discontinuous at 0, and f is strictly convex on $\eta(0)$. Let $g = f^2$ and let $x_k = 1/2k\pi$ for $k > 0$. Then

$$\begin{aligned} g''(x_k) =& f''(f(x_k))(f'(x_k))^2 + f''(x_k)f'(f(x_k)) \\ =& (\mu + x_k)^2 + (1-\rho)(\mu + f(x_k)) \\ \xrightarrow{k}& \mu(1 + \mu - \rho) > 0\,, \end{aligned}$$

and therefore there is a $K > 0$ such that $g''(x_k) > 0$ for each $k \ge K$. Thus f^2 is not concave on any neighborhood of the fixed point 0.

The following proposition is closely associated with convergence of both $\{\phi_k\}$ and $\{\phi'_k\}$ on a neighborhood of the fixed point 0.

Proposition 3.4. *Let $f : I \to I$ with $0 \in I$, $f(0) = 0$, $-1 < \lambda = f'(0) < 0$, and $f''(0) > 0$. Then there is a neighborhood $\eta(0)$ wherein*

$$\lambda x < f(x) \text{ and } f \circ f(x) < \lambda^2 x$$

for each $x \ne 0$.

PROOF. Since $f''(0) > 0$, there is a neighborhood $\zeta(0)$ wherein $f'(x) < \lambda$ for each $x \in \zeta^-(0)$ and $\lambda < f'(x)$ for each $x \in \zeta^+(0)$. It follows from the Mean Value Theorem that $\lambda x < f(x)$ for each $x \in \zeta(0)\backslash\{0\}$. If $g = f^2$, then $g : I \to I$ with $g(0) = 0$, $0 < \lambda^2 = g'(0) < 1$, and $g''(0) = f'(0)f''(0)(1 + f'(0)) < 0$. Proceeding in a manner similar to the above shows that there is a neighborhood $\eta(0) \subseteq \zeta(0)$ with the required properties. □

The next proposition presents conditions under which concavity of f^2 implies convexity of f.

Proposition 3.5. *Let $f : I \to I$, $0 \in I$, $f(0) = 0$, $-1 < f'(0) < 0$, and let f be differentiable with f' continuous at 0. If f^2 is concave on a neighborhood of 0, then f is convex on a neighborhood of 0.*

PROOF. Let f be 0-local on $\eta(0)$ with $f'(x) < 0$ for each $x \in \eta(0)$ and f^2 concave on $\eta(0)$. Let $x, y \in \eta^-(0)$ and $x < y$. We will show that $f'(x) \le f'(y)$. Since $(f^2)'(y) \le (f^2)'(x)$, we have

$$f'(y)f'(f(y)) \le f'(x)f'(f(x)), \text{ and therefore } \frac{f'(y)}{f'(x)} \le \frac{f'(f(x))}{f'(f(y))}\,.$$

Furthermore since $(f^2)'(f(x)) \le (f^2)'(f(y))$, it follows that

$$f'(f(x))f'(f^2(x)) \le f'(f(y))f'(f^2(y)) \quad \text{and} \quad \frac{f'(y)}{f'(x)} \le \frac{f'(f(x))}{f'(f(y))} \le \frac{f'(f^2(y))}{f'(f^2(x))}\,.$$

Continuing in this manner we obtain

$$\frac{f'(y)}{f'(x)} \le \frac{f'(f^{2k-1}(x))}{f'(f^{2k-1}(y))} \le \frac{f'(f^{2k}(y))}{f'(f^{2k}(x))} \le \lim_{k\to\infty} \frac{f'(f^{2k}(y))}{f'(f^{2k}(x))} = 1\,;$$

where in taking the limit we used the fact that f' is continuous at 0. We conclude that $f'(x) \le f'(y)$. Similarly, it can be shown that if $x, y \in \eta^+(0)$ and $x < y$, then $f'(x) \le f'(y)$. The continuity of f' at 0 implies that f' is increasing on $\eta(0)$. □

The proof of Proposition 3.5 shows that if f^2 is strictly concave on a neighborhood of 0, then f is strictly convex on a neighborhood of 0. The proof also indicates that with the additional assumptions that f is 0-local on I and $f'(x) < 0$ for each $x \in I$, then f^2 concave on I implies that f is convex on I. Some interesting aspects of Proposition 3.5 are highlighted in the next example.

Example 3.6. *Let $-1 < \mu < 0$, let α, β be positive real numbers, and let*

$$f(x) = \begin{cases} x(\alpha x + \mu), & \text{if } x \le 0; \\ x(\beta x + \mu), & \text{if } x > 0. \end{cases}$$

Then $f(0) = 0$, $f'(0) = \mu$, and f' is continuous. We are led to the following conclusions:

1. *If $\alpha/\beta \in (0, |\mu|) \bigcup (1/|\mu|, \infty)$, then there is an f-invariant neighborhood $\eta(0)$ for which f is strictly convex and f'' exists and is bounded on $\eta(0) \setminus \{0\}$; however, f^2 is not concave on any neighborhood of 0. Note that $f''(0)$ does not exist.*

2. *If $\alpha/\beta \in (|\mu|, 1/|\mu|)$, then there is an f-invariant neighborhood $\eta(0)$ for which f^2 is strictly concave, f is strictly convex, and f'' exists and is bounded on $\eta(0) \setminus \{0\}$. Note that $f''(0)$ exists if and only if $\alpha = \beta$.*

The proof of Proposition 3.5 yields the following result.

Proposition 3.7. *Let $f : I \to I$, $0 \in I$, $f(0) = 0$, $0 < f'(0) < 1$, and let f be differentiable with f' continuous at 0. If f^2 is concave on a neighborhood of 0, then f is concave on a neighborhood of 0.*

Note that under the assumptions of Proposition 3.7 if f^2 is strictly concave on a neighborhood of 0, then f is strictly concave on a neighborhood of 0.

4 A Preliminary Result

Consider a 0-local map g on I with $0 \in I^\circ$ and $0 < g'(0) < 1$. If $f \circ f = g$, then f is referred to as an iterative square root of g. An iterative square root f is orientation-reversing if $f(I^-(0)) \subseteq I^+(0)$ and $f(I^+(0)) \subseteq I^-(0)$. The existence of an orientation-reversing iterative square root of g is closely associated with convergence of Koenigs' sequences. We will now present a result that will lead to a description of sufficient conditions for uniform convergence of Koenigs' sequences. Note that $D^- f$ and $D_+ f$ refer to the derivates of f defined by

$$D^- f(x) = \limsup_{y \to x^-} \frac{f(y) - f(x)}{y - x} \quad \textit{and} \quad D_+ f(x) = \liminf_{y \to x^+} \frac{f(y) - f(x)}{y - x} \cdot$$

Lemma 4.1. *Let g be 0-local on I with $0 \in I^\circ$, $0 < \lambda^2 = g'(0) < 1$, and $g(x) \leq \lambda^2 x$ for each $x \in I$. If f is an orientation-reversing iterative square root of g, then $f(x)/x$ is increasing on each of the orbits of g. If $D^- f(0) \leq D_+ f(0)$, then $f(x)/x$ is increasing on each of the orbits of f.*

Proof. We first prove that $f(x)/x$ is increasing on the orbits of g on $I^-(0)$. If $x \in I^-(0)$, then

$$\frac{g(x)}{x} \geq \lambda^2 \geq \frac{g(f(x))}{f(x)} , \tag{9}$$

and multiplying (9) by $f(x)/g(x)$ gives

$$\frac{f(x)}{x} \leq \frac{g(f(x))}{g(x)} \text{ and equivalently } \frac{f(x)}{x} \leq \frac{f(f^2(x))}{f^2(x)} = \frac{f(g(x))}{g(x)} \cdot \tag{10}$$

The result follows from the latter inequality in (10) since $x < f^2(x) = g(x)$. The proof that $f(x)/x$ is increasing on each of the orbits of g on $I^+(0)$ is similar, except that we consider an $x \in I^+(0)$ and each of the above inequalities is reversed.

With the additional assumption that $D^- f(0) \leq D_+ f(0)$, we will now show that $f(x)/x$ is increasing on each of the orbits of f on I. It is sufficient to consider the orbit of an arbitrary $x \in I^-(0)$:

$$\frac{f(f^{2k}(x))}{f^{2k}(x)} \overset{k}{\nearrow} \mu \leq D^- f(0) \leq D_+ f(0) \leq \rho \overset{k}{\swarrow} \frac{f(f^{2k+1}(x))}{f^{2k+1}(x)}$$

for some real numbers μ and ρ. The conclusion follows. $\square$

Under the assumptions of Lemma 4.1 if f is differentiable at the fixed point, then $f(x)/x$ is increasing on each of the orbits of f. Note however that we can't conclude that f is 0-local on the entire interval I. If in the statement of Lemma 4.1 one assumes that $g(x) \geq \lambda^2 x$ for each $x \in I$, then $f(x)/x$ is a decreasing function of the orbits of g.

5 Convergence of Koenigs' Sequences

If f is 0-local on I with $-1 < f'(0) < 0$, then $\phi_k(x) < 0$ for each $x < 0$, $\phi_k(0) = 0$, $\phi_k(x) > 0$ for each $x > 0$, and ϕ_k is strictly increasing on each of the orbits of f on I. Thus, if $\{\phi_k\}$ converges to a limit ϕ, then $\phi(0) = 0$ and, as we shall see, ϕ is strictly increasing on each of the orbits of f on I. We now present the principal result.

Theorem 5.1. *Let f be defined on a neighborhood of 0 with $f(0) = 0$ and $-1 < \lambda = f'(0) < 0$. If either $f \circ f(x) \leq \lambda^2 x$ or $f \circ f(x) \geq \lambda^2 x$ on a neighborhood of 0, then Koenigs' sequence converges uniformly to a limit ϕ on a neighborhood of 0 with $\phi(0) = 0$ and $\phi'(0) = 1$. The limit ϕ is invertible on each orbit of f in a neighborhood of 0 and conjugates f orbitwise to its linearization λx, that is $\phi \circ f \circ \phi^{-1}(x) = \lambda x$.*

Proof. We assume without loss of generality that f is 0-local on I with ε. We present the proof only for the case when $f \circ f(x) \leq \lambda^2 x$ for each $x \in I$. The proof for the other case is similar. Let $x \in I^-(0)$. Since f satisfies the conditions of Lemma 4.1 and f is differentiable at 0, it follows that

$$\frac{f(f^{2k}(x))}{f^{2k}(x)} \overset{k}{\nearrow} \lambda \overset{k}{\swarrow} \frac{f(f^{2k+1}(x))}{f^{2k+1}(x)} \tag{11}$$

where the convergence is uniform on $I^-(0)$. Consequently, for each $k \geq 0$ we have

$$\frac{f^2(f^{2k+1}(x))}{f^{2k+1}(x)} \leq \lambda^2\,,\ \lambda^2 \leq \frac{f^2(f^{2k}(x))}{f^{2k}(x)}\, and \frac{f(f^{2k}(x))}{f^{2k}(x)} \leq \lambda\,. \tag{12}$$

Multiplication of the first inequality in (12) by $f^{2k+1}(x)/\lambda^{2k+3}$ gives

$$\phi_{2k+1}(x) \leq \phi_{2k+3}(x) \tag{13}$$

for each $k \geq 0$. Similar manipulation of the latter two inequalities in (12) shows that for each $k \geq 0$,

$$\phi_{2k+2}(x) \leq \phi_{2k}(x) \text{ and } \phi_{2k+1}(x) \leq \phi_{2k}(x)\,. \tag{14}$$

Dividing (11) by λ yields

$$\frac{f(f^{2k+1}(x))/\lambda}{f^{2k+1}(x)} \overset{k}{\nearrow} 1 \overset{k}{\swarrow} \frac{f(f^{2k}(x))/\lambda}{f^{2k}(x)}$$

and equivalently

$$\frac{\phi_{2k+2}(x)}{\phi_{2k+1}(x)} \overset{k}{\nearrow} 1 \overset{k}{\swarrow} \frac{\phi_{2k+1}(x)}{\phi_{2k}(x)}\,, \tag{15}$$

where the convergence is uniform on $I^{-}(0)$. We conclude from (13), (14), and (15), that $\lim_{k\to\infty}\phi_k(x)=\phi(x)$ exists and

$$\phi_{2k+1}(x) \overset{k}{\nearrow} \phi(x) \overset{k}{\swarrow} \phi_{2k}(x) \tag{16}$$

for each $x\in I^{-}(0)$. To prove that the convergence in (16) is uniform on $I^{-}(0)$ we consider

$$\phi_{2k}(x)-\phi_{2k+1}(x)=|\phi_{2k}(x)|\left|1-\frac{\phi_{2k+1}(x)}{\phi_{2k}(x)}\right|\cdot \tag{17}$$

Since f is 0-local on I with ε, then

$$|\phi_{2k}(x)|\le|\phi_1(x)|=\left|\frac{f(x)}{\lambda}\right|<\frac{|x||\varepsilon-\lambda|}{|\lambda|}\le\frac{|I^{-}(0)||\varepsilon-\lambda|}{|\lambda|}\cdot \tag{18}$$

Let $\delta>0$ be given. It follows from the uniform convergence in (15) that there is a $K(\delta)\ge 0$ such that

$$\left|1-\frac{\phi_{2k+1}(x)}{\phi_{2k}(x)}\right|<\delta\left(\frac{|\lambda|}{|I^{-}(0)||\varepsilon-\lambda|}\right) \tag{19}$$

for every $x\in I^{-}(0)$ and $k>K$. We conclude from (16), (17), (18), and (19), that $\phi_{2k}(x)-\phi_{2k+1}(x)<\delta$ for every $x\in I^{-}(0)$ and $k>K$. Since $\phi_k(0)=0$ for each $k\ge 0$, $\phi(0)=0$. The proof that $\{\phi_k\}$ converges uniformly on $I^{-}(0)\cup\{0\}$ is complete. In an entirely similar manner it can be shown that $\{\phi_k\}$ converges uniformly on $I^{+}(0)\cup\{0\}$. The fact that $\phi_1(x)\le\phi(x)\le\phi_0(x)$ for each $x\in I$ and $\phi_1'(0)=1=\phi_0'(0)$ shows that $\phi'(0)=1$.

To prove that ϕ is strictly increasing, and therefore invertible, on each orbit of f in $I^{+}(0)$, it is sufficient to prove that $\phi(f^2(x))<\phi(x)$ for an arbitrary $x\in I^{+}(0)$. Let $x\in I^{+}(0)$. Since f is 0-local on I with ε, we have

$$0<f^{2k}(x)(\lambda+\varepsilon)^2<f^2(f^{2k}(x))<f^{2k}(x)(\lambda-\varepsilon)^2$$

for each $k\ge 0$. It follows that $\phi_{2k}(f^2(x))<\phi_{2k}(x)(\lambda-\varepsilon)^2$ for each $k\ge 0$, and therefore $\phi(f^2(x))\le\phi(x)(\lambda-\varepsilon)^2<\phi(x)$, which concludes the proof for $x\in I^{+}(0)$. The proof for $x\in I^{-}(0)$ is similar. An alternate proof uses the fact that ϕ satisfies the Schröder equation. $\square$

The proof of Theorem 5.1 yields the following corollary.

Corollary 5.2. *Let f be 0-local on I with $-1<\lambda=f'(0)<0$. If $f\circ f(x)\le\lambda^2x$ for each $x\in I$, then $\phi_{2k+1}(x)\overset{k}{\nearrow}\phi(x)\overset{k}{\swarrow}\phi_{2k}(x)$, and if $f\circ f(x)\ge\lambda^2x$ for each $x\in I$, then $\phi_{2k}(x)\overset{k}{\nearrow}\phi(x)\overset{k}{\swarrow}\phi_{2k+1}(x)$. The convergence is uniform in both cases with $\phi(0)=0$ and $\phi'(0)=1$.*

Functions satisfying the conditions of Theorem 5.1 are presented in the following proposition.

Proposition 5.3. *Let $p(x) = x(\alpha x + \lambda)$ and $q(x) = x(\beta x + \lambda)$ for each $x \in \mathbb{R}$ with $-1 < \lambda < 0$ and $0 < \alpha < \beta < -\alpha/\lambda$. If f is a function satisfying $p(x) \le f(x) \le q(x)$ on a neighborhood of 0, then $f'(0) = \lambda$ and $f \circ f(x) \le \lambda^2 x$ on a neighborhood of 0.*

The authors thank the referee for the following simplified proof of Proposition 5.3.

PROOF. It is clear that $f'(0) = \lambda$. Using the fact that q is decreasing on a neighborhood of 0, we get

$$f(f(x)) \le q(f(x)) \le q(p(x)) = \lambda^2 x + (\lambda\alpha + \lambda^2\beta)x^2 + \mathcal{O}(x^3)\,,$$

and it follows immediately from the conditions on α and β that $\lambda\alpha + \lambda^2\beta < 0$. □

Functions satisfying the conditions of Proposition 3.4 also satisfy the conditions of Theorem 5.1. Other examples include those described in Proposition 3.1 and Proposition 3.5. We have the following additional corollary of Theorem 5.1.

Corollary 5.4. *Let g be 0-local on I with $\lambda^2 = g'(0)$ and $-1 < \lambda < 0$. Let f be an orientation-reversing iterative square root of g with $\lambda = f'(0)$. If $g(x) \le \lambda^2 x$ for each $x \in I$, then the Koenigs' sequence for g satisfies*

$$\lambda\phi_k(x) \stackrel{k}{\nearrow} \lambda\phi(x) = \phi(f(x)) \stackrel{k}{\swarrow} \phi_k(f(x))\,;$$

and if $g(x) \ge \lambda^2 x$ for each $x \in I$, then

$$\phi_k(f(x)) \stackrel{k}{\nearrow} \phi(f(x)) = \lambda\phi(x) \stackrel{k}{\swarrow} \lambda\phi_k(x)\,.$$

The convergence is uniform on I in both cases, with $\phi(0) = 0$ and $\phi'(0) = 1$.

Without prior knowledge of the existence of an orientation-reversing iterative square root of g, the following result can be obtained.

Proposition 5.5. *Let g be defined on a neighborhood of 0 with $g(0) = 0$ and $0 < \lambda^2 = g'(0) < 1$. If $g(x) \le \lambda^2 x$ on a neighborhood of 0, then for each $x < 0$ and $y > 0$ on a neighborhood of 0 the Koenigs' sequence for g satisfies*

$$-\infty \le \phi(x) \stackrel{k}{\swarrow} \phi_k(x) < 0 = \phi(0) \le \phi(y) \stackrel{k}{\swarrow} \phi_k(y)\,.$$

If $g(x) \ge \lambda^2 x$ on a neighborhood of 0, then for each $x < 0$ and $y > 0$ on a neighborhood of 0 the Koenigs' sequence for g satisfies

$$\phi_k(x) \stackrel{k}{\nearrow} \phi(x) \le \phi(0) = 0 < \phi_k(y) \stackrel{k}{\nearrow} \phi(y) \le \infty\,.$$

Proof. We assume without loss of generality that g is 0-local on I. If $g(x) \le \lambda^2 x$ for each $x \in I$, then for each $x \in I$ and $k \ge 0$ we have $g(g^k(x)) \le \lambda^2 g^k(x)$ and therefore

$$\phi_{k+1}(x) = \frac{g(g^k(x))}{\lambda^{2(k+1)}} \le \frac{\lambda^2 g^k(x)}{\lambda^{2(k+1)}} = \phi_k(x) \; .$$

The conclusion follows from the above inequality along with the fact that $\phi_k(x) < 0$ for each $x \in I^-(0)$, $\phi_k(0) = 0$, and $\phi_k(x) > 0$ for each $x \in I^+(0)$. The proof for the case where $g(x) \ge \lambda^2 x$ is similar. □

Let g be convex and 0-local on a closed interval I with $0 \in I^\circ$ and $0 < \lambda^2 = g'(0) < 1$. It is well-known that there exists an orientation-reversing iterative square root of g on I. As well, g satisfies the conditions of Proposition 5.5. It follows from Theorem 5.1 that a necessary condition for the existence of an orientation-reversing iterative square root f that is differentiable at 0 is that the limit ϕ of the Koenigs' sequence for g satisfy the Schröder equation (1) for each $x \in I$.

6 Divergence of Koenigs' Sequences

We will now construct a 0-local map f with $-1 < \lambda = f'(0) < 0$ for which the associated Koenigs' sequence diverges everywhere except at the fixed point. Once this has been accomplished, we will then see that f can easily be redefined to be a C^1 map with divergent Koenigs' sequence.

Example 6.1. *Let $-1/e < \lambda < 0$, $0 < \varepsilon < -\lambda$, $0 < \sigma < \log(1 - \varepsilon/\lambda)$, and let*

$$f(x) = x\lambda\, e^{\sigma P(x)} \text{ for each } x \in I \, ,$$

where the interval I and the function P are defined as follows: Let $a_0 < 0$ and $a_k = a_0\, \lambda^k \prod_{\nu=1}^k e^{\sigma/\nu}$ for each $k > 0$. Let $I = [a_0, a_1]$, and define

$$P(x) = \begin{cases} \dfrac{1}{2k+1}\,, & \text{if } x \in [a_{2k}, a_{2k+2})\,,\ k \ge 0\,; \\ 0\,, & \text{if } x = 0\,; \\ \dfrac{1}{2k+2}\,, & \text{if } x \in (a_{2k+3}, a_{2k+1}]\,,\ k \ge 0\,. \end{cases}$$

Then f is 0-local on I with $\lambda = f'(0)$, and the Koenigs' sequence for f satisfies

$$\lim_{k\to\infty} \phi_k(x) = \begin{cases} -\infty\,, & \text{if } x \in [a_0, 0)\,; \\ 0\,, & \text{if } x = 0\,; \\ +\infty\,, & \text{if } x \in (0, a_1]\,. \end{cases}$$

PROOF. It follows that $|a_{k+1}| < |a_k|$ for each $k \ge 0$ and $a_{2k} \overset{k}{\nearrow} 0 \overset{k}{\swarrow} a_{2k+1}$. Thus $P : I \to [0,1]$, $P(0) = 0$, and P is continuous at 0. Now define the function $\tau(x) = \lambda\left(e^{\sigma P(x)} - 1\right)$ for each $x \in I$. Then $\tau : I \to (-\varepsilon,\, 0\,]$, $\tau(0) = 0$, and τ is continuous at 0. Let

$$f(x) = x(\lambda + \tau(x)) = x\lambda\, e^{\sigma P(x)}$$

for each $x \in I$. By Lemma 2.6 we conclude that f is 0-local on I with $\lambda = f'(0)$. For each $k \ge 0$,

$$f(a_k) = a_k \lambda\, e^{\sigma P(a_k)} = a_0\, \lambda^{k+1}\, e^{\sigma \sum_{\nu=1}^{k+1} 1/\nu} = a_{k+1}$$

and therefore $f^k(a_0) = a_k$. Since f is strictly decreasing on I, it follows that for each $k \ge 0$,

$$f([a_{2k}, a_{2k+2})) \subset (a_{2k+3}, a_{2k+1}] \text{ and } f((a_{2k+3}, a_{2k+1}]) \subset [a_{2k+2}, a_{2k+4})\,.$$

Let $P_1 = P$ and $\tau_1 = \tau$. Applying Lemma 2.6, we define the sequence $\{\tau_k\}_{k=1}^{\infty}$ by setting $P_{k+1} = P_k \circ f$ and $\tau_{k+1} = \tau_k \circ f$ for each $k > 0$. Thus,

$$P_k(x) = \begin{cases} \dfrac{1}{2n+k}\,, & \text{if } x \in [a_{2n}, a_{2n+2})\,,\ n \ge 0\,; \\ 0\,, & \text{if } x = 0\,; \\ \dfrac{1}{2n+1+k}\,, & \text{if } x \in (a_{2n+3}, a_{2n+1}]\,,\ n \ge 0\,, \end{cases}$$

for each $k > 0$. Therefore $P_k : I \to [0, 1/k]$, $P_k(0) = 0$, P_k is continuous at 0, and $P_k \overset{k}{\to} 0$ uniformly on I. For each $k > 0$,

$$\tau_k(x) = \begin{cases} \lambda\big(e^{\frac{\sigma}{2n+k}} - 1\big)\,, & \text{if } x \in [a_{2n}, a_{2n+2})\,,\ n \ge 0\,; \\ 0\,, & \text{if } x = 0\,; \\ \lambda\big(e^{\frac{\sigma}{2n+1+k}} - 1\big)\,, & \text{if } x \in (a_{2n+3}, a_{2n+1}]\,,\ n \ge 0\,, \end{cases}$$

and therefore $\tau_k : I \to [\lambda(e^{\sigma/k} - 1), 0\,] \subset (-\varepsilon, 0\,]$, $\tau_k(0) = 0$, τ_k is continuous at 0, and $\tau_k \overset{k}{\to} 0$ uniformly on I. In accordance with Lemma 2.6, for each $x \in I$ and $k > 0$ we have

$$f^k(x) = x\prod_{\nu=1}^{k}(\lambda + \tau_\nu(x)) = x\lambda^k \prod_{\nu=1}^{k} e^{\sigma P_\nu(x)} = x\lambda^k e^{\sigma \sum_{\nu=1}^{k} P_\nu(x)},$$

and

$$\phi_k(x) = \frac{f^k(x)}{\lambda^k} = x\, e^{\sigma \sum_{\nu=1}^{k} P_\nu(x)}\,. \tag{20}$$

If $z \in I^-(0)$, then $z \in [a_{2n}, a_{2n+2})$ for some $n \geq 0$. Using (20) gives

$$\phi_k(z) = z\, e^{\sigma \sum_{\nu=1}^{k} 1/(2n+\nu)} \xrightarrow{k} -\infty.$$

Similarly, if $z \in I^+(0)$, then $z \in (a_{2n+3}, a_{2n+1}]$ for some $n \geq 0$ and therefore

$$\phi_k(z) = z\, e^{\sigma \sum_{\nu=1}^{k} 1/(2n+1+\nu)} \xrightarrow{k} +\infty. \qquad \square$$

For an interval map f with $f(0) = 0$ and $-1 < f'(0) < 0$ it is well-known that Koenigs' sequence converges if $f \in C^{1+\epsilon}$ for some $\epsilon > 0$. We will now illustrate by means of an example that C^1 is insufficient to ensure convergence of Koenigs' sequence. The map defined in Example 6.1 can easily be redefined to be a C^1 function with divergent Koenigs' sequence. This is achieved in the following manner.

Example 6.2. *Let $-1/e < \lambda < 0$, $0 < \varepsilon < -\lambda$, $0 < \sigma < \log(1 - \varepsilon/\lambda)$, and let*

$$f(x) = x\lambda\, e^{\sigma P(x)} \text{ for each } x \in I\,,$$

where the interval I and the function P are defined as follows: Let $a_0 < 0$ and $a_k = a_0\, \lambda^k \prod_{\nu=1}^{k} e^{\sigma/\nu}$ for each $k > 0$. Let $I = [a_0, a_1]$, and define $P(0) = 0$ and $P(a_k) = 1/(k+1)$ for each $k \geq 0$. Let P be C^1 on $I \setminus \{0\}$ with P' strictly decreasing on both $I^-(0)$ and $I^+(0)$. Then f is C^1 and 0-local on I with $\lambda = f'(0)$ and f has a divergent Koenigs' sequence.

Proof. That such a P exists follows from the special nature of the sequence $\{a_k\}$. Note that P' is necessarily negative on $I^-(0)$ with $\lim_{x\to 0^-} P'(x) = -\infty$ and P' is positive on $I^+(0)$ with $\lim_{x\to 0^+} P'(x) = \infty$. We have

$$f'(x) = \begin{cases} \lambda, & \text{if } x = 0\,; \\ \lambda e^{\sigma P(x)}\left(1 + \sigma x P'(x)\right), & \text{otherwise}\,. \end{cases}$$

It is evident from the formula for f' and the fact that P is continuous at 0 with $P(0) = 0$, that to prove the continuity of f' from the right at 0 it suffices to show that $\lim_{x\to 0^+} xP'(x) = 0$. If $x \in I^+(0)$, then $x \in (a_{2k+3}, a_{2k+1}]$ for some $k \geq 0$. Since P' is positive and strictly decreasing on $I^+(0)$,

$$\begin{aligned} xP'(x) \;&\leq\; a_{2k+1} P'(a_{2k+3}) < a_{2k+1} \frac{P(a_{2k+3})}{a_{2k+3}} = \frac{a_0 \lambda^{2k+1} \prod_{\nu=1}^{2k+1} e^{\sigma/\nu}}{a_0 \lambda^{2k+3}(2k+4) \prod_{\nu=1}^{2k+3} e^{\sigma/\nu}} \\ &= \frac{e^{-\sigma\left(\frac{1}{2k+2} + \frac{1}{2k+3}\right)}}{\lambda^2(2k+4)} \xrightarrow{k} 0\,. \end{aligned}$$

The proof that f' is continuous from the left at 0 is similar. It then follows that $f \in C^1$ on I. The 0-locality of f on I with $\lambda = f'(0)$ and divergence of the associated Koenigs' sequence is proven in a similar manner as for the previous example. In particular, note that $f(a_k) = a_{k+1}$ for each $k \geq 0$ as in the previous example. $\square$

Let g be a C^1 convex 0-local map on a closed interval I with $0 \in I^\circ$ and $0 < \lambda = g'(0) < 1$. There exists an orientation-reversing iterative square root of g on I. As well, g satisfies the conditions of Proposition 5.5. It follows from Theorem 5.1 that an orientation-reversing iterative square root of g is non-differentiable at the fixed point 0 if the Koenigs' sequence for g diverges on $I^+(0)$. We conclude with an example of such a map g. Note that it is sufficient to define g on $I^+(0) \cup \{0\}$ with the resulting Koenigs' sequence diverging on $I^+(0)$.

Example 6.3. *Let*

$$g(x) = \begin{cases} x\left(\lambda - \dfrac{1}{\log(x)}\right), & \text{if } x \in (0, e^{-1/\varepsilon}]\,; \\ 0\,, & \text{if } x = 0\,, \end{cases}$$

where $0 < \lambda < 1$ *and* $0 < \varepsilon < 1 - \lambda$. *Then* g *is* 0*-local on* $[0, e^{-1/\varepsilon}]$ *with* $\lambda = g'_+(0)$ *and* g' *is strictly increasing on* $[0, e^{-1/\varepsilon}]$. *The Koenigs' sequence for* g *diverges for each* $x \in (0, e^{-1/\varepsilon}]$.

PROOF. The 0-locality of g follows from the choice of ε. Clearly g' is strictly increasing on $[0, e^{-1/\varepsilon}]$ and $\lambda = g'_+(0)$. Using the standard reorganization, we see that the Koenigs' sequence satisfies

$$\phi_k(x) = \frac{g^k(x)}{\lambda^k} = \frac{g(x)}{\lambda} \cdot \frac{g(g(x))}{\lambda g(x)} \cdots \frac{g(g^{k-2}(x))}{\lambda g^{k-2}(x)} \cdot \frac{g(g^{k-1}(x))}{\lambda g^{k-1}(x)}\,,$$

and for our specific g,

$$\phi_k(x) = x(1 - \frac{1}{\lambda \log(x)})(1 - \frac{1}{\lambda \log(g(x))}) \cdots (1 - \frac{1}{\lambda \log(g^{k-1}(x))})\,.$$

To show that this sequence diverges it is enough to prove that the series

$$\sum_{k=0}^{\infty} \frac{1}{|\log(g^k(x))|} \tag{21}$$

diverges. It follows from the definition of g that $g^k(x) > x\lambda^k$ for each $x \in (0, e^{-1/\varepsilon}]$, which yields

$$\log(g^k(x)) > k\log(\lambda) + \log(x) \text{ and hence } k\log(\tfrac{1}{\lambda}) - \log(x) > |\log(g^k(x))|\,.$$

Thus,

$$\frac{1}{k\log\left(\frac{1}{\lambda}\right)-\log(x)}<\frac{1}{|\log(g^k(x))|}.$$

In view of the above inequality, we conclude that the series in (21) diverges. □

7 Acknowledgments

The authors wish to thank the referee, whose suggestions have considerably improved and simplified the paper. In particular, generalizations of Theorem 5.1 and Proposition 5.3 were pointed out, as well as the existence of a continuously differentiable function which has no orientation-reversing iterative square root that is differentiable at the fixed point.

References

[1] D. J. Dewsnap, *Orbitally convex functions*, Ph.D. Thesis, University of Guelph, Guelph, Canada, 1996.

[2] P. Fischer, *On the Feigenbaum functional equation,* (to be submitted).

[3] G. Koenigs, Recherches sur les intégrales de certaines équations fonctionnelles, *Ann. Sci. Ec. Norm. Sup. (3)*, **1** (1884), Supplement, 3–41.

[4] M. Kuczma,*Note on Schröder's functional equation*, J. Austral. Math. Soc., **4** (1964), 149–51.

[5] O. E. Lanford, *Smooth transformations of intervals*, Séminaire Bourbaki (1980-1981), Lecture Notes in Mathematics, 901, Springer-Verlag, Berlin and New York, (1981), 36–54.

[6] E. Schröder, *Über iterierte Funtionen,* Math. Ann., **3** (1871), 296–322.

[7] E. Seneta, *On Koenigs' ratios for iterates of real functions*, J. Austral. Math. Soc., **10** (1969), 207–13.

[8] G. Szekeres, *Regular iteration of real and complex functions*, Acta. Math., **100** (1958), 203–58.

[9] G. Targonski, *Seminar on functional operators and equations*, Lecture Notes in Math., 33, Springer-Verlag, Berlin-Heidelberg-New York, 1967.