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AN OSCILLATION FUNCTION ON THE REAL LINE

Abstract

By means of a certain well known family $\mathcal{B}$ of subsets of $\mathbb{R}$ fulfilling two conditions we introduce some topologies on $\mathbb{R}$ (in Section 2 we consider the density topology). We observe that the family of the sets $\Omega_f(y) := \{x \in \mathbb{R};\ \omega_f(x) \geq y\}$ for an arbitrary bounded function $f : \mathbb{R} \to \mathbb{R}$ (where $\omega_f(x)$ is a kind of $\mathcal{B}$-oscillation of f) has three properties. Then we show that for each family $\{\Omega(y)\}_{y\in[0,1]} \subset 2^{\mathbb{R}}$ having similar properties and in addition fulfilling conditions M_1 and $\mathcal{U}'$ (known from the literature) there is a function $f : \mathbb{R} \to [0,1]$ such that $\Omega_f(y) = \Omega(y)$ for each $y \in [0,1]$. In Section 2 we prove some analogous result for the density topology. [1]

Let $\mathbb{R}$ denote the set of all real numbers.

1 $\mathcal{B}$-Oscillation

Definition 1. Let $\mathcal{B}_0^+ \subset 2^{\mathbb{R}}$ be a nonempty family of sets fulfilling the following conditions:

(1) if $B \in \mathcal{B}_0^+$, then for every $t > 0$, $B \cap (0,t) \in \mathcal{B}_0^+$,

(2) $B_1 \cup B_2 \in \mathcal{B}_0^+$ if and only if $B_1 \in \mathcal{B}_0^+$ or $B_2 \in \mathcal{B}_0^+$.

For every set $A \subset \mathbb{R}$ and $x \in \mathbb{R}$ put $A + x = \{y \in \mathbb{R}; \exists_{a\in A}\ (y = a + x)\}$ and $-A := \{y \in \mathbb{R} : -y \in A\}$.

Now we can define the family $\mathcal{B}_0^-$ as

$$\mathcal{B}_0^- := \left\{B \subset \mathbb{R} : -B \in \mathcal{B}_0^+\right\}$$

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and for each $x \in \mathbb{R}$

$$\mathcal{B}_x^+ := \left\{ B \subset \mathbb{R} : (B - x) \in \mathcal{B}_0^+ \right\},$$
$$\mathcal{B}_x^- := \left\{ B \subset \mathbb{R} : (-B - x) \in \mathcal{B}_0^+ \right\}.$$

For $x \in \mathbb{R}$, put $\mathcal{B}_x := \mathcal{B}_x^+ \cup \mathcal{B}_x^-$ and by $\mathcal{B}$ denote the family of all subsets B of $\mathbb{R}$ such that there exists $x_B \in \mathbb{R}$ and $B \in \mathcal{B}_{x_B}$.

Definition 2. (see [2]) We say that a family $\mathcal{B}$ fulfills condition M_1, if for every $x_0 \in \mathbb{R}$ and a set $B \in \mathcal{B}_{x_0}^+$ and for every family of sets $\{B_x\}_{x \in B}$ such that $B_x \in \mathcal{B}_x$ $(x \in B)$, the set $\bigcup_{x \in B} B_x$ belongs to the family $\mathcal{B}_{x_0}^+$.

Assume that the family $\mathcal{B}$ fulfills M_1. Let us define the operation "$\cdot$" in the following way: $\dot{A} := \{x \in \mathbb{R} : A \in \mathcal{B}_x\}$ for arbitrary $A \subset \mathbb{R}$. It is now possible to consider the closure operation "$^-$" for each subset $A \subset \mathbb{R}$: $\overline{A} := A \cup \dot{A}$. In fact, it is not difficult to check that the operation "$^-$ " satisfies the Kuratowski's axioms. Let τ denotes the topology on $\mathbb{R}$ generated by the operation "$^-$".

Definition 3. A number g is called a $\mathcal{B}$-limit number of a function $f : \mathbb{R} \to \mathbb{R}$ at a point x_0 if for every positive number ϵ

$$\{x \in \mathbb{R}; |f(x) - g| < \epsilon\} \in \mathcal{B}_{x_0}.$$

By $L(f, x)$ we denote the set of all $\mathcal{B}$-limit numbers of the function f at x.

It is known that for each bounded or locally bounded function f and every point $x \in \mathbb{R}$ there exists at least one $\mathcal{B}$-limit number of f at x but for every f and $x \in \mathbb{R}$ the set $L(f, x)$ is closed in the usual Euclidean topology on $\mathbb{R}$.

For a bounded function $f : \mathbb{R} \to \mathbb{R}$ let

$$m(f, x) = \min \{L(f, x) \cup \{f(x)\}],$$
$$M(f, x) = \max \{L(f, x) \cup \{f(x)\}].$$

We say that a function f is upper $\mathcal{B}$-semicontinuous (lower $\mathcal{B}$-semicontinuous) at a point x_0 if

$$M(f, x_0) \le f(x_0), \quad (m(f, x_0) \ge f(x_0)).$$

From theorem 14 in [2] we infer the following characterization. For an arbitrary bounded function f, the function $M(f, x)$ $(x \in \mathbb{R})$ is upper $\mathcal{B}$-semicontinuous if and only if the family $\mathcal{B}$ fulfills condition M_1 and similarly: the function $m(f, x)$ is lower $\mathcal{B}$-semicontinuous if and only if the family $\mathcal{B}$ fulfills condition M_1.

By the symbol $\omega_f(x)$ we denote the $\mathcal{B}$-oscillation of a bounded real function f at x defined as follows:

$$\omega_f(x) := M(f,x) - m(f,x).$$

Let us observe that for an arbitrary bounded function $f : \mathbb{R} \to \mathbb{R}$, which is upper $\mathcal{B}$-semicontinuous at each point $x \in \mathbb{R}$ and for each $a \in \mathbb{R}$ and $x_0 \in E_a := \{x; f(x) < a\}$ there exists such a set $V_{x_0} \in \tau$ (with $x_0 \in V_{x_0}$) such that for any $x \in V_{x_0}$, $f(x) < a$. Hence the set E_a is τ-open. And conversely, if the set E_a is τ-open for each $a \in \mathbb{R}$, then f is upper $\mathcal{B}$-semicontinuous in each point $x \in \mathbb{R}$. Therefore for a bounded function $f : \mathbb{R} \to \mathbb{R}$ the following properties are true:.

(1) The set $\Omega_f(y) := \{x : \omega_f(x) \geq y\}$ is τ-closed for each $y \in \mathbb{R}$.

(2) If $y_1 < y_2$, then $\Omega_f(y_2) \subset \Omega_f(y_1)$.

(3) The set $\bigcup_{y\in\mathbb{R}} [\Omega_f(y) \times \{y\}]$ is $\tau \times \tau_e$-closed, where τ_e denotes the Euclidean topology on $\mathbb{R}$.

Now let $\{\Omega(y)\}_{0\leq y\leq 1}$ be a nonempty family of nonempty subsets of $\mathbb{R}$ such that:

(α_1) the set $\Omega(y)$ is τ-closed for each $y \in [0,1]$,

(α_2) if $y_1 < y_2$, then $\Omega(y_2) \subset \Omega(y_1)$,

(α_3) the set $\bigcup_{y\in\mathbb{R}} [\Omega(y) \times \{y\}]$ is $\tau \times \tau_e$-closed,

(α_4) $\Omega(0) = \mathbb{R}$.

For each $y \in [0,1]$ put $\Omega(y) = A(y) \cup B(y)$, where $A(y) := \dot{\Omega(y)}$ and $B(y) := \Omega(y) \setminus A(y)$. Assume that the family $\mathcal{B}$ fulfills the following condition $\mathcal{U}'$ which is a particular case of the condition $\mathcal{U}$ from [4].

Each subset $A \subset \mathbb{R}$ can be represented as a sum of such A_1 and A_2 that:

(a) $A_1 \subset A$, $A_2 \subset A$, $A_1 \cap A_2 = \emptyset$,

(b) $\dot{A_1} = \dot{A}$, $\dot{A_2} = \dot{A}$.

We prove the following theorem.

Theorem 1. *Let $\mathcal{B}$ be an arbitrary family fulfilling conditions M_1 and $\mathcal{U}'$. Then for each family of subsets $\{\Omega(y)\}_{0\leq y\leq 1}$ of reals fulfilling conditions (α_1) - (α_4) there exists a function $f : \mathbb{R} \to [0,1]$ such that for any $0 \leq y \leq 1$ we have $\Omega(y) = \Omega_f(y)$.*

Let B_a stand for the set $\{y \in [0,1] : a \in B(y)\}$ and let F be the set of all $a \in \mathbb{R}$ for which $B_a \neq \emptyset$. Let $x_0 \in \mathbb{R}$ be an arbitrarily chosen point and

$$y_0 := \max\left\{p_Y\left[\mathcal{P}(x_0) \cap \Big(\bigcup_{0\le y\le 1} (A(y)\times\{y\})\Big)\right]\right\},$$

where $\mathcal{P}(x_0) := \{p \in \mathbb{R}\times\mathbb{R} : p = (x_0, y)\}$ and p_Y is the projection to Y axis. The point $x_0 \in A(y_0)$ because the set $\bigcup_{0\le y\le 1}(A(y)\times\{y\})$ is $\tau\times\tau_e$-closed. Conditions (α_1) - (α_4) and definition of y_0 easily imply that there exists such h_0 that

$$(x_0 - h_0, x_0 + h_0) \cap p_X\left[\Big(\bigcup_{y\in[0,1]} (B(y)\times\{y\})\Big) \cap \Big(\bigcup_{y_0'\le y} (\mathbb{R}\times\{y\})\Big)\right] \notin \mathcal{B},$$

where y_0' is an arbitrarily chosen number from $(y_0, 1]$ and p_X is projection to the X axis.

To prove our theorem it is sufficient to define the function f by

$$f(x) := \begin{cases} \sup\{y \in [0,1] :\ x \in \Omega(y)\} & \text{for } x \in A_1 \cup F, \\ 0 & \text{for } x \in \mathbb{R}\setminus(A_1\cup F), \end{cases}$$

where the condition $\mathcal{U}'$ was applied for $A = \{x : \sup\{y :\ x \in A(y)\} > 0\}$.

Prove the inclusion $\Omega(y) \subset \Omega_f(y)$. Let y_0 be some number from $(0,1]$ and $x \in \Omega(y_0)$.

(I_1) Suppose $x \notin F$ and $y_0' := \sup\{y \in [0,1] : x \in A(y)\}$. From (α_3) and definition of f we obtain that $x \in \Omega(y_0')$ and $y_0' = \max[L(f,x) \cup \{f(x)\}] = f(x)$. Since $\omega_f(x) = y_0'$, we infer that $x \in \Omega_f(y_0')$ and because $y_0' \ge y_0$, then $x \in \Omega_f(y_0)$.

(I_2) If $x \in F$, then we have inequality $y_0' < f(x)$ and again from (α_3) and definition of f it follows that $x \in \Omega(f(x))$. Since $\omega_f(x) = f(x) > y_0$, hence $x \in \Omega_f(y_0)$.

The proof of the converse inclusion is obvious.

2 Approximate Oscillation

In this section we find a necessary and sufficient condition for a family of sets to be the family of associated sets of approximate oscillation.

Let $E \subset \mathbb{R}$ and $x_0 \in \mathbb{R}$. The upper outer density of E at the point x_0 is the number

$$\overline{D_{x_0}(E)} = \limsup_{h \to 0+} \frac{|E \cap (x_0 - h, x_0 + h)|}{2h},$$

where $|\cdot|$ denotes outer measure.

Let $\mathcal{B}_x = \mathcal{U}_x$ be the family of all sets for which x *is not* a dispersion point (i. e. $\overline{D_x}(E) > 0$). For $\mathcal{B} = \mathcal{U}$ the number g is called an approximate limit number ($\mathcal{U}$-limit number). Let $L_{\mathcal{U}}(f,x)$ denote the set of all $\mathcal{U}$-limit numbers of a function f at a point x.

For a bounded function $f : \mathbb{R} \to \mathbb{R}$ we write

$$m_{\mathcal{U}}(f,x) = \min\{L_{\mathcal{U}}(f,x) \cup \{f(x)\}\}, \quad M_{\mathcal{U}}(f,x) = \max\{L_{\mathcal{U}}(f,x) \cup \{f(x)\}\}.$$

We say that the function f is upper $\mathcal{U}$-semicontinuous (lower $\mathcal{U}$-semicontinuous) at a point x_0 if $M_{\mathcal{U}}(f,x_0) \leq f(x_0)$, $(m_{\mathcal{U}}(f,x_0) \geq f(x_0))$.

The $\mathcal{U}$-oscillation of a bounded function $f : \mathbb{R} \to \mathbb{R}$ at a point $x \in \mathbb{R}$ is a function $\omega_f(x) = M_{\mathcal{U}}(f,x) - m_{\mathcal{U}}(f,x)$.

Let τ_s denote the density topology on $\mathbb{R}$ ([1], [5]) and τ_e the natural topology. It is easy to see that for an arbitrary bounded function $f : \mathbb{R} \to \mathbb{R}$, which is upper $\mathcal{U}$-semicontinuous at each point of $\mathbb{R}$ and for each $a \in \mathbb{R}$ and $x_0 \in E_a = \{x : \ f(x) < a\}$ there exists a set $V_0 \in \tau_s$ $(x_0 \in V_0)$ such that $f(x) < a$ for every $x \in V_0$. (It follows from the Lebesgue Density Theorem.) Hence the set E_a is τ_s-open.

Conversely, if the set E_a is τ_s-open for each $a \in \mathbb{R}$, then f is upper $\mathcal{U}$-semicontinuous at each point $x \in \mathbb{R}$. It is easy to see ([2], [3]), that the following facts hold for each bounded function $f : \mathbb{R} \to \mathbb{R}$:

(1) The set $\Omega_f(y) = \{x : \ \omega_f(x) \geq y\}$ is τ_s-closed for each $y \in \mathbb{R}$.

(2) If $y_1 < y_2$, then $\Omega_f(y_2) \subset \Omega_f(y_1)$.

(3) The set $\bigcup_{y \in \mathbb{R}} (\Omega_f(y) \times \{y\})$ is $\tau_s \times \tau_e$-closed on the plane $\mathbb{R} \times \mathbb{R}$.

It follows from the Lebesgue Density Theorem that every nonempty τ_s-closed set D can be represented as a sum of two disjoint subsets D_1, D_2, the first one consisting of all points of density of D and the second one satisfying $|D_2| = 0$.

Let $\{\Omega(y)\}_{0 \leq y \leq 1}$ be a nonempty family of subsets of $\mathbb{R}$ such that:

(α_1) The set $\Omega(y)$ is τ_s-closed for each $y \in [0,1]$,

(α_2) If $y_1 < y_2$, then $\Omega(y_2) \subset \Omega(y_1)$,

(α_3) The set $\bigcup_{y \in [0,1]} (\Omega(y) \times \{y\})$ is $\tau_s \times \tau_e$-closed.

(α_4) $\Omega(0) = \mathbb{R}$.

For each $y \in [0,1]$ let $\Omega(y) = A(y) \cup B(y)$, where $A(y)$ is the set of all density points of $\Omega(y)$ and $B(y) = \Omega(y) \setminus A(y)$.

The main result of this section is the following.

Theorem 2. *For every family $\{\Omega(y)\}_{0 \le y \le 1}$ fulfilling conditions (α_1) – (α_4) there exists a function $f : \mathbb{R} \to [0,1]$ such that for each $0 \le y \le 1$ we have $\Omega(y) = \Omega_f(y)$.*

Notice, that if for some $y' \in (0,1]$, $x \in A(y')$, then $x \in A(y)$ for every $0 \le y < y'$. Similarly, if $x \in B(y'')$, for some $y'' < y'$, then $x \in B(y)$ for each $y'' < y < y'$. For each $a \in \mathbb{R}$ define the set B_a by

$$B_a = \{y \in [0,1] : \ a \in B(y)\} .$$

Let F be the set of all $a \in \mathbb{R}$, for which B_a is nondegenerate interval.

Lemma 1.

$$|F| = 0.$$

Proof of lemma. Take an arbitrary point $x_0 \in F$ and $h > 0$. Put

$$y_0 = \inf\Big\{y : \ ([(x_0 - h, x_0 + h) \setminus A(y)] \times \{y\}) \cap \bigcup_{a \in F} (\{a\} \times B_a) \neq \emptyset\Big\}$$

and take the following sequence of sets.

$$\begin{aligned} W_1 =& p_X\Big\{\Big[\big((x_0 - h, x_0 + h) \setminus A(y_0 + \frac{1-y_0}{2})\big) \\ & \times \Big\{y_0 + \frac{1-y_0}{2}\Big\}\Big] \cap \bigcup_{a\in F} (\{a\} \times B_a)\Big\} \\ W_2 =& p_X\Big\{\bigcup_{k=1}^{2^2}\Big[\big((x_0 - h, x_0 + h) \setminus A(y_0 + \frac{k(1-y_0)}{2^2})\big) \\ & \times \Big\{y_0 + \frac{k(1-y_0)}{2^2}\Big\}\Big] \cap \bigcup_{a\in F} (\{a\} \times B_a)\Big\} \end{aligned}$$

. . . .

$$\begin{aligned} W_{2n} =& p_X\Big\{\bigcup_{k=1}^{2^{2n}}\Big[\big((x_0 - h, x_0 + h) \setminus A(y_0 + \frac{k(1-y_0)}{2^{2n}})\big) \\ & \times \Big\{y_0 + \frac{k(1-y_0)}{2^{2n}}\Big\}\Big] \cap \bigcup_{a\in F} (\{a\} \times B_a)\Big\} \end{aligned}$$

. . . .
It is easy to verify that

$$0 \leq |(x_0 - h, x_0 + h) \cap F| \leq |W_1| + \sum_{n=1}^{\infty} |W_{2n}| \,.$$

From the Lebesgue theorem it is clear that

$$|(x_0 - h, x_0 + h) \cap F| = 0.$$

Since the number $h > 0$ was chosen arbitrarily, $\overline{D_{x_0}(F)} = 0$. So F consists of upper outer dispersion points of F. Once again using the Lebesgue theorem we obtain that $|F| = 0$. □

To prove Theorem 2 let $A = \{x : \sup\{y\colon x \in A(y)\} > 0\}$. It is known ([6]) that A can be represented as a sum of two subsets A_1 and A_2 such that

(a) $A_1 \cap A_2 = \emptyset$,

(b) $|A_1| = |A|\,, \quad |A_2| = |A|$.

Our function f can be now defined as follows:

$$f(x) = \begin{cases} \sup\{y \in [0,1] : \; x \in \Omega(y)\} & \text{for } x \in A_1 \cup F, \\ 0 & \text{for } x \in \mathbb{R} \setminus (A_1 \cup F)\,. \end{cases}$$

The rest of the proof follows the lines of the proof of Theorem 1.

□

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