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ON THE DERIVATIVES OF FUNCTIONS OF BOUNDED VARIATION

Abstract

Using a standard complete metric w on the set F of continuous functions of bounded variation on the interval $[0,1]$, we find that a typical function in F has an infinite derivative at continuum many points in every subinterval of $[0,1]$. Moreover, for a typical function in F, there are continuum many points in every subinterval of $[0,1]$ where it has no derivative, finite nor infinite. The restriction of the derivative of a typical function in F to the set of points of differentiability has infinite oscillation at each point of this set.

Let $C[0,1]$ denote the family of continuous real valued functions on the interval $[0,1]$ and let F denote the set of functions of bounded variation in $C[0,1]$.

It is known (see for example [B] or [C]) that with respect to the uniform metric on $C[0,1]$, a typical function in $C[0,1]$ has a unilateral infinite derivative at continuum many points in each subinterval of $[0,1]$, even though it has no finite unilateral derivative at any point. We wondered if some sort of analogue can be constructed for F. Problems of finding such an analogue are two-fold: the uniform metric is not complete on F, and functions in F are differentiable almost everywhere. So we define

$$w(f,g) = \big|f(0) - g(0)\big| + \text{total variation of } f - g \text{ on } [0,1]\,.$$

The proof that w is a complete metric on F is well-known (see [R, p. 147], for example).

With respect to the metric w, we will show that a typical function in F has infinite derivatives at continuum many points in each subinterval of $[0,1]$. For any residual set S, we will find that a typical $f \in F$ satisfies $f'(x) \in S$ almost

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everywhere. For any subset E of $(0,1)$, with exterior measure 1, we will show that the restriction to E of the Dini derivatives of any typical function in F are discontinuous at each point of E. All derivatives here are two-sided.

Theorem I. *For a typical $f \in F$, the set*

$$\{x \in I \ : \ |f'(x)| = \infty\}$$

has the power of the continuum for each subinterval I of $[0,1]$.

PROOF. Let $[c,d]$ be a subinterval of $[0,1]$. Let $k \in F$ and let ϵ be a positive number. Choose a subinterval $[a,b]$ of $[c,d]$ such that

$$V\big(k,[a,b]\big) < \frac{\epsilon}{8}\,.$$

(Here V denotes total variation.) Let f be a singular nondecreasing function in F, that vanishes on $[0,a]$, is constant on $[b,1]$ and such that

$$f(b) - f(a) = \frac{\epsilon}{2}\,.$$

(Lebesgue's singular function can be used to construct f; see [HS, (8.28)].) Then $w(k+f,k) = \epsilon/2$.

Now any function in the open ball with center $k+f$ and radius $\epsilon/8$ can be expressed $k+f+g$ where $g \in F$ and $w(g,0) < \epsilon/8$. Then

$$V\big(k+g,[a,b]\big) \le V\big(k,[a,b]\big) + V\big(g,[a,b]\big) < \frac{\epsilon}{8} + w(g,0) < \frac{\epsilon}{8} + \frac{\epsilon}{8} = \frac{\epsilon}{4}\,,$$

and

$$V\big(k+g,[a,b]\big) < \frac{\epsilon}{2} = f(b) - f(a)\,.$$

It follows from this and the fact that f is singular on $[a,b]$, that $k+f+g$ is not absolutely continuous on $[a,b]$ nor on $[c,d]$. Thus the set of functions in F that are not absolutely continuous on $[c,d]$ form a residual subset of F.

Finally, let $[c,d]$ run over all the subintervals of $[0,1]$ with rational endpoints and find that the set of functions in F that are absolutely continuous on no subinterval of $[0,1]$ form a residual subset of F. But such functions must have infinite derivatives at continuum many points in each subinterval of $[0,1]$. $\square$

Theorem II. *For every residual set of real numbers S, $f'(x) \in S$ almost everywhere for typical $f \in F$ (in particular, for such sets S of measure 0).*

Proof. Let p be a positive number and let X be a closed nowhere dense subset of $\mathbb{R}$. It suffices to prove that the set of all $g \in F$ for which

$$m\Big\{x \in (0,1) \ : \ g'(x) \in X\Big\} \geq p$$

is a nowhere dense subset of F.

So let T denote the set of all $g \in F$ for which $m\Big\{x \in (0,1) \ : \ g'(x) \in X\Big\} \geq p$. Let $k \in F \setminus T$. Then

$$m\Big\{x \in (0,1) \ : \ k'(x) \in X\Big\} < p\,.$$

There are positive numbers r and q such that

$$m\Big\{x \in (0,1) \ : \ \text{the distance from } k'(x) \text{ to } X \text{ is less than } q\Big\} = r < p\,.$$

Choose any $h \in T$. Then

$$m\Big\{x \in (0,1) \ : \ \big|k'(x) - h'(x)\big| \geq q\Big\} \geq p - r\,.$$

We apply the Vitali Covering Theorem to this set to find mutually disjoint intervals $[x_i, u_i]$ such that

$$\sum_i (u_i - x_i) \geq p - r$$

and for each index i,

$$\big|(k-h)(u_i) - (k-h)(x_i)\big| \geq \frac{q(u_i - x_i)}{2}\,.$$

Consequently,

$$\sum_i \big|(k-h)(u_i) - (k-h)(x_i)\big| \geq \frac{q(p-r)}{2}\,.$$

It follows that

$$w(h,k) \geq \frac{q(p-r)}{2}\,,$$

and T is a closed subset of F. It remains to prove that $F \setminus T$ is dense in F.

Let ϵ be a positive number. Let $(y_j)_{j=-\infty}^{\infty} \subset \mathbb{R} \setminus X$ be a sequence such that

$$\lim_{j \to -\infty} y_j = -\infty\,, \quad \lim_{j \to \infty} y_j = \infty \quad \text{and} \quad 0 \leq y_j - y_{j-1} < \epsilon \ \text{ for each } j\,.$$

Let $h_0 \in F$. For $x \in [0,1]$, define

$$f_1(x) = r_j - h_0'(x) \text{ where } j \text{ is such that } r_j > h_0'(x) \geq r_{j-1}\,.$$

Then $0 \leq f_1(x) < \epsilon$. Let f_2 be the indefinite integral of f_1:

$$f_2(x) = \int_0^x f_1(t)\,dt\,.$$

Then $0 \leq f_2'(x) \leq \epsilon$ almost everywhere and

$$w(f_2, 0) = V\big(f_2, [0,1]\big) = \int_0^1 f_2'(t)\,dt \leq \epsilon\,.$$

Also $f_2'(x)+h_0'(x)$ is in the set $\{r_j\} \subset \mathbb{R}\setminus X$ almost everywhere, so $f_2+h_0 \notin T$. Finally

$$w(f_2+h_0, h_0) = w(f_2, 0) \leq \epsilon\,.$$

Thus $F \setminus T$ is a dense open subset of F. □

Theorem III. *Let E be any subset of $[0,1]$ with exterior measure 1. Then the restriction to E of the Dini derivates of a typical function in F are discontinuous on E. Moreover, their oscillations at each point of E are infinite.*

Proof. Let I be an open subinterval of $[0,1]$ and J be an open subinterval of $\mathbb{R}$. Then it suffices to prove that the set of functions $g \in F$ for which

$$m\big\{x \in I \ :\ g'(x) \in J\big\} > 0$$

is an open dense subset of F. Let T denote the set of all $g \in F$ for which $m\{x \in I \ :\ g'(x) \in J\} = 0$. Take $h \notin T$. Then $m\{x \in I \ :\ h'(x) \in J\} > 0$. Let s and r be positive numbers such that

$$m\Big\{x \in I \ :\ \text{the distance from } h'(x) \text{ to } \mathbb{R}\setminus J \text{ is at least } s\Big\} = r > 0\,.$$

Let $g \in T$. So

$$m\Big\{x \in I \ :\ \big|h'(x) - g'(x)\big| \geq s\Big\} \geq r\,.$$

We use the Vitali Covering Theorem on this set to find pairwise disjoint intervals $[a_i, b_i]$ such that

$$\big|(h-g)(b_i) - (h-g)(a_i)\big| \geq \frac{s(b_i - a_i)}{2} \text{ for each } i$$

and $\sum_i (b_i - a_i) \geq r$. Hence

$$w(h,g) \geq V\big(h-g,[0,1]\big) \geq s \sum_i \frac{b_i - a_i}{2} \geq \frac{rs}{2},$$

and T is a closed subset of F.

Now let $g_0 \in T$, and let p be a positive number. Let $[a,b]$ be a subinterval of I for which $V(g_0,[a,b]) < p/4$. It is easy to construct a function $g_1 \in F$ that coincides wit g_0 on $[0,a]$ and on $[b,1]$, for which

$$V\big(g_1,[a,b]\big) < \frac{p}{2} \quad \text{and} \quad m\big\{x \in [a,b] \;:\; g_1'(x) \in J\big\} > 0.$$

Hence $g_1 \notin T$ and

$$w(g_0,g_1) = V\big(g_1 - g_0,[a,b]\big) \leq V\big(g_1,[a,b]\big) + V\big(g_0,[a,b]\big) < \frac{p}{2} + \frac{p}{4} < p.$$

So T is a nowhere dense closed set. □

Note that the set F_1 of nondecreasing functions in F is a closed subset of F. So F_1 is a complete metric space under w in its own right. The Theorems I, II and III are also true with F_1 in place of F by essentially the same arguments.

Let $g \in F_1$ and assume that $D^+g < \infty$ on a second category subset of $[0,1]$. It follows that there is a second category set E such that the set

$$\left\{\frac{g(x+p)-g(x)}{p} \;:\; p>0,\; x \in E\right\}$$

is bounded. Let I be a subinterval of $[0,1]$ in which E is dense. By continuity, the difference quotient of g is bounded on I. On the other hand, it is easy to prove that the set of all functions in F_1 with bounded difference quotient on I is a first category subset of F_1. It follows that the set of all $g \in F_1$ such that $D^+g(x) = \infty$ on a residual subset of $[0,1]$ is a residual subset of F_1. Likewise it is easy to prove that the set of all functions in F_1 with difference quotient bounded away from 0 on I is a first category subset of F_1. By an analogous argument it follows that the set of all $g \in F_1$ such that $D_+g(x) = 0$ on a residual subset of $[0,1]$ is a residual subset of F_1. The corresponding statements can be proved for D^-g and D_-g. We conclude with:

Proposition 1. *For a typical $f \in F_1$, the set*

$$\Big\{x \in (0,1) \;:\; D^+f(x) = D^-f(x) = \infty \;\; \textit{and} \;\; D_+f(x) = D_-f(x) = 0\Big\}$$

is a residual subset of $[0,1]$. Thus typical $f \in F_1$ have unilateral derivatives, finite or infinite, on at most a first category subset of $[0,1]$.

For $f \in F_1$ we define the four sets:

- $A_f = \{x \in (0,1) \ : \ D_-f(x) = D_+f(x) = 0\}$,
- $B_f = \{x \in (0,1) \ : \ D^-f(x) = D^+f(x) = \infty\}$,
- $C_f = \{x \in (0,1) \ : \ D_-f(x) = 0 \ \text{ and } \ D^+f(x) = \infty\}$,
- $D_f = \{x \in (0,1) \ : \ D^-f(x) = \infty \ \text{ and } \ D_+f(x) = 0\}$.

(The idea is that in each set there is one restriction on the left and one on the right.) For typical $f \in F_1$, we know that $A_f \cup B_f \cup C_f \cup D_f$ is a residual subset of $[0,1]$.

Is there a strictly increasing singular function f for which $A_f \cup B_f \cup C_f \cup D_f = (0,1)$? The answer is *yes*; we showed how one can be constructed in [C1].

Is there a strictly increasing singular function in F_1 for which $(0,1)$ equals the union of any three of these sets? The answer, we shall see, is *no*.

Proposition 2. *Let f be a strictly increasing singular function in F_1. Then each of the sets*

$$A_f \cup B_f \cup C_f, \quad A_f \cup B_f \cup D_f, \quad A_f \cup C_f \cup D_f, \quad B_f \cup C_f \cup D_f\,,$$

has a dense complement in $[0,1]$.

PROOF. Let I be a subinterval of $[0,1]$. Because f is a singular function, we deduce that there exist points $a, b \in I$ such that

$$a < b, \quad f'(a) = \infty \quad \text{and} \quad f'(b) = 0\,.$$

Let G denote the graph in $\mathbb{R}^2$

$$\{(x, f(x)) \ : \ a \le x \le b\}\,.$$

Then G is a compact subset of $\mathbb{R}^2$. Let r be the maximum value for which the line (in $\mathbb{R}^2$) $y = x + r$ meets G. Say they meet at the point $(u, f(u))$. By comparing the slope of the line with the slope of the graph, we conclude that $u \ne a$ and $u \ne b$. So $a < u < b$ and $u \in I$. By the same reasoning we find that $D^+f(u) \le 1$ and $D_-f(u) \ge 1$. It follows that $u \notin A_f \cup B_f \cup C_f$. By the analogous argument (with $b < a$ and r minimal) we find a point in I that is not in $A_f \cup B_f \cup D_f$. Of course any point where $f' = \infty$ is not in $A_f \cup C_f \cup D_f$ and any point where $f' = 0$ is not in $B_f \cup C_f \cup D_f$. The conclusion follows. $\square$

Let us recapitulate. For typical $f \in F_1$ and any subinterval I of $[0,1]$ we have:

1) f has derivative ∞ at continuum many points in I,

2) f has a finite derivative at continuum many points in I,

3) there are continuum many points in I at which f has no derivative, finite or infinite,

4) the restriction of f' to the set of all points of differentiability of f, has infinite oscillation at each point.

We conclude by giving an indirect but elementary proof of the well-known result that the set

$$\{x \in (a,b) \;:\; |f'(x)| = \infty\}$$

has measure zero. The arguments will not require the Vitali Covering Theorem nor the differential properties of monotonic functions. We require only the following well-known facts, that we state without proof.

Lemma A. *If $S_1, S_2, S_3, \ldots$ are finitely many subsets of $[a,b]$, then*

$$\sum_i m(S_i) \geq m\big(\cup_i S_i\big)$$

where m denotes Lebesgue outer measure.

Lemma B. *If $S_1, S_2, S_3, \ldots$ is a sequence of subsets of $[a,b]$, then there is an index k such that*

$$m\big(\cup_{i=1}^{k} S_i\big) \geq \frac{1}{2} \cdot m\big(\cup_{i=1}^{\infty} S_i\big)\,.$$

Proof of the result. It suffices to prove the result for bounded functions. Then it will hold for arbitrary functions by truncating such a function at N and $-N$. So let g be a bounded function on $[a,b]$ and let $E \subset [a,b]$ be a set such that $g'(x) = \infty$ at each $x \in E$. The plan is to assume that $m(E) > 0$ and eventually find a contradiction. Fix an integer N so large that on $[a,b]$

$$N > 2 \cdot |g| \tag{1}$$

So

$$E = \bigcup_{j=1}^{\infty} \Big\{ x \in E \;:\; \frac{f(x) - f(u)}{x - u} > \frac{8N}{m(E)} \quad \text{for} \quad 0 < |x - u| < \frac{1}{j} \Big\}\,.$$

By Lemma B, there is an index k for which $m(E_1) > m(E_2)/2$ where

$$E_1 = \Big\{ x \in E \;:\; \frac{g(x) - g(u)}{x - u} > \frac{8N}{m(E)} \quad \text{for} \quad 0 < |x - u| < \frac{1}{k} \Big\}\,.$$

Choose points $u_0, u_1, u_2, \ldots, u_p$ such that

$$a = u_0 < u_1 < u_2 < \ldots < u_p = b \text{ and } u_i = u_{i-1} < \frac{1}{k} \text{ for } i = 1, 2, \ldots, p\,.$$

For each index i for which the interval $[u_{i-1}, u_i]$ meets E_1, choose a point $x_i \in [u_{i-1}, u_i] \cap E_1$ such that $2(u_i - x_i)$ exceeds the diameter of the set $[u_{i-1}, u_i] \cap E_1$. Then

$$u_i - x_i > \frac{1}{2} \cdot m\big([u_{i-1}, u_i] \cap E_1\big)\,.$$

We sum over the indices i for which $[u_{i-1}, u_i] \cap E_1$ is nonvoid and obtain (by Lemma A)

$$\sum_i (u_i - x_i) > \sum_i \frac{1}{2} \cdot m\big([u_{i-1}, u_i] \cap E_1\big) > \frac{1}{2} \cdot m(E_1)\,.$$

But $m(E_1) > \frac{1}{2} \cdot m(E)$, so

$$\sum_i (u_i - x_i) > \frac{1}{4} \cdot m(E)\,. \tag{2}$$

By the definition of E_1,

$$\sum_i \big(g(u_i) - g(x_i)\big) > \sum_i \frac{8N}{m(E)} \cdot (u_i - c_i) = \frac{8N}{m(E)} \cdot \sum_i (u_i - x_i)$$

and by (2),

$$\sum_i \big(g(u_i) - g(x_i)\big) > \frac{8N}{m(E)} \cdot \frac{m(E)}{4} = 2N\,. \tag{3}$$

Note also that no two of the intervals $[x_i, u_i]$ overlap.

From Lemma A we deduce that one of the sets $[u_{i-1}, u_i] \cap E_1$ does not have measure zero; call this set E_2. Thus there is a subinterval $[c, d]$ of $[a, b]$, containing this subset E_2 of E_1, such that

$$d - c < \frac{1}{k}, \quad m(E_2) > 0, \quad \text{and} \quad g'(x) = \infty \text{ for each } x \in E_2\,.$$

We repeat the preceding arguments with $[c, d]$ in place of $[a, b]$ and E_2 in place of E, to find mutually nonoverlapping subintervals

$$[y_1, v_1],\ [y_2, v_2],\ [y_3, v_3],\ \ldots,\ [y_t, v_t]$$

of $[c,d]$, such that $y_j \in E_2$ for all j and

$$\sum_{j=1}^{t} \big(g(v_j) - g(y_j)\big) = 2N\,. \tag{4}$$

We index these intervals so that $y_1 \le y_2 \le y_3 \le \ldots \le y_t$. Because the intervals do not overlap, we have in fact

$$y_1 < v_1 \le y_2 < v_2 \le y_3 < v_3 \le \ldots \le y_t < v_t\,. \tag{5}$$

But each $y_j \in E_1$ also, and we deduce from the definition of E_1 that $v_j - y_j$ and $g(v_j) - g(y_j)$ are both positive, and $y_j - v_{j-1}$ and $g(y_j) - g(v_{j-1})$ are both nonnegative. It follows from (5) that

$$g(y_1) < g(v_1) \le g(y_2) < g(v_2) \le g(y_3) < g(v_3) \le \ldots \le g(y_t) < g(v_t)\,. \tag{6}$$

From (4) and (6) we obtain

$$g(v_t) - g(y_1) \ge \sum_{j=1}^{t} \big(g(v_j) - g(y_j)\big) > 2N\,. \tag{7}$$

By (1) we have $g(v_t) - g(y_1) < N$. Combining this with (7), we have

$$N > 2N\,. \tag{8}$$

Finally, $0 > N$ contrary to (1). This contradiction completes the proof. □

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