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THE ERDŐS SIMILARITY PROBLEM: A SURVEY

Abstract

Erdős posed the following problem. "Let E be an infinite set of real numbers. Prove that there is a set of real numbers S of positive measure which does not contain a set E' similar (in the sense of elementary geometry) to E." The proof is known for only a few special cases; and not included among these is the geometric sequence $\{2^{-n}\}_{n=1}^{\infty}$. In this paper we examine the known literature, present some new results, and ask a few related questions.

1 Introduction

Erdős posed the following problem [6]. "Let E be an infinite set of real numbers. Prove that there is a set of real numbers S of positive measure which does not contain a set E' similar (in the sense of elementary geometry) to E."

We say that a set E is *Erdős* if there exists a set of positive Lebesgue measure not containing a copy of E; by a *copy of* E we mean a set of the form $x + tE := \{x + te | e \in E\}$ with $x/t \in \mathbb{R}$. For example, each subset of $\mathbb{R}$ that is either unbounded or dense in some interval is Erdős (witnessed by a Cantor set of positive measure). On the other hand, Steinhaus [23] has shown that each set of positive measure contains a copy of each finite set, hence no finite set is Erdős.

Given any set E, if some subset of E is Erdős, then so is E. Hence, much attention has been focused on the case where E is a *zero-sequence*; namely, a strictly monotone sequence of positive numbers converging to zero. If every zero-sequence is Erdős, then the problem is solved. However, this appears to be a difficult problem and hence special cases and reformulations have received considerable attention.

While Erdős mentioned this problem on several occasions [6], [7], [8], [9], his interest apparently went no further than generalizing the Steinhaus result [14].

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On the other hand, in [8] he offers a prize of \$100 for the solution and later in [9] writes "I hope that there are no such sets" when discussing countable sets which might not be Erdős.

Bourgain [2] has suggested that the problem may be motivated by some results of Szemerédi [24], Furstenberg [12] and Furstenberg and Katznelson [13]. Kolountzakis [18] also mentions some work of Lebedev and Olevskiĭ [19] that is related to the problem of finding a copy of a finite set in a set of positive measure.

In what follows, uncited results are the author's work. Some of these are probably known, but the author has found no direct reference.

2 Almost Chronological Survey

Erdős first posed the similarity problem in 1974 at The Fifth Balkan Mathematical Congress [6]. He later repeated the problem in [7], and, speculating that it would be proved, posed the following further problem. "Given a countable set E of $[0,1]$ determine (or estimate) the largest possible measure of a subset S of $[0,1]$ which does not contain a set similar to E." The answer is that any $\mu(S) < 1$ is possible and that $\mu(S) = 1$ is impossible (μ denotes Lebesgue measure). The latter is true since it is easy to show that any set of full measure contains a copy of each countable set (see Theorem 2.3).

Regarding $\mu(S) < 1$, observe that any set S of positive measure (not containing a copy of E) contains a portion $(a,b)\cap S$ having an arbitrarily large measure less than one. Replace S by a suitably scaled and shifted version of $(a,b)\cap S$. Moreover, we may require S to be closed since any set of positive measure contains a closed set of positive measure. Thus, the sets S of interest are closed, nowhere dense, and have positive measure.

On the other hand, the possible Erdős sets E, are closed, nowhere dense, null sets according to the following lemma and the remark following Theorem 2.10.

Lemma 2.1. *A set $E \subset \mathbb{R}$ is Erdős if and only if its closure is Erdős.*

Proof. Suppose that E is Erdős. Then $\overline{E}$, the closure of E, is Erdős follows from the observation that if a set contains no copy of E, then it contains no copy of any larger set.

Suppose that $\overline{E}$ is Erdős and, proceeding toward a contradiction, suppose that E is not. There is a closed set S of positive measure that contains no copy of $\overline{E}$. Since E is not Erdős, $x + tE \subset S$ for some (x,t) with $x/t \in \mathbb{R}$. Since S is closed, $x + t\overline{E} = \overline{x + tE} \subset S$, a contradiction. $\square$

Erdős posed the following related problem to Borwein and Ditor [1]. "Given a measurable set S of real numbers with $\mu(S) > 0$, and a sequence $\{e_n\}_{n=1}^{\infty}$ of real numbers converging to zero, is there always an x such that $x + e_n \in S$ for all n sufficiently large?" They showed that the answer is no with the following theorem.

Theorem 2.2 (Borwein and Ditor [1]).
(i.) There is a measurable set S with $\mu(S) > 0$ and a zero-sequence $\{e_n\}_{n=1}^{\infty}$ such that, for each x, $x + e_n \notin S$ for infinitely many n.
(ii.) If S is a measurable set with $\mu(S) > 0$ and $\{e_n\}_{n=1}^{\infty}$ is a sequence converging to zero, then, for almost all $x \in S$, $x + e_n \in S$ for infinitely many n.

Related to the Erdős question, Miller and Xenikakis obtained the following results for convergent sequences of real numbers.

Theorem 2.3 (H. I. Miller and P. J. Xenikakis [21]). *If $S \subset \mathbb{R}$ is an open set mod $\mathcal{N}$, where $\mathcal{N}$ is the ideal of subsets of measure zero, and $E = \{e_n\}_{n=1}^{\infty}$ is a convergent sequence of reals, then S contains a copy of E.*

Theorem 2.4 (H. I. Miller and P. J. Xenikakis [21]). *If $S \subset \mathbb{R}$ is of the second category in $\mathbb{R}$ and possesses the Baire property and $E = \{e_n\}_{n=1}^{\infty}$ is a convergent sequence of reals, then S contains a copy of E.*

Kolountzakis [18] showed, using a more direct proof, that these two theorems are true for arbitrary bounded countable sets E.

Using transfinite induction and the continuum hypothesis (CH), Miller obtained the following results for uncountable sets.

Theorem 2.5 (H. I. Miller [20]). *If E is an uncountable set of real numbers, then there exists a subset S of $[0, 1]$ such that S has outer Lebesgue measure one and S contains no copy of E.*

Theorem 2.6 (H. I. Miller [20]). *If E is an uncountable set of real numbers, then there exists a subset S of $[0, 1]$ such that S is of the second Baire category in $\mathbb{R}$ and S contains no copy of E.*

Without using CH, J. A. de Reyna obtained the following. (Recall that a set $E \subset \mathbb{R}$ is first category at a point x if there is a neighborhood U of x such that $U \cap E$ is first category; otherwise E is second category at x.)

Theorem 2.7 (J. A. de Reyna [22]). *Let E be a subset of $\mathbb{R}$ with more than two points. There exists a subset of S of the unit interval such that*
(i.) S has outer Lebesgue measure one;
(ii.) S is of the second Baire category at each of its points; and
(iii.) S does not contain a copy of E.

Later, P. Komjáth improved upon the Borwein and Ditor results with the following.

Theorem 2.8 (P. Komjáth [17]). *For any given $\epsilon > 0$ there exists a set $S \subset [0,1]$ of measure $1-\epsilon$ and a sequence $\{e_n\}_{n=1}$ converging to 0 such that, for any given $x \in [0,1]$ and $t \neq 0$, $x + te_n \notin S$ for infinitely many n.*

Theorem 2.9 (P. Komjáth [17]). *For any given zero-sequence e_n and $\epsilon > 0$ there is a set $S \subset [0,1]$, with $\mu(S) > 1-\epsilon$, possessing the property: if $x \in [0,1]$, then $\{n | x + e_n \notin S\}$ is infinite.*

The first Erdős zero-sequence result was proved independently at about the same time by Eigen and Falconer. The result is summarized in the following theorem.

Theorem 2.10 (Eigen [5] and K. J. Falconer [10]). *Let $E = \{e_n\}_{n=1}^{\infty}$ be a zero-sequence such that*

$$\lim_{n\to\infty} \frac{e_{n+1}}{e_n} = 1.$$

Then E is Erdős. Moreover (Falconer), there exists a closed set S with $\mu(S) > 0$ such that for any (x,t) with $x/t \in \mathbb{R}$, $x + te_n \notin S$ for infinitely many n.

This result shows immediately that any set having positive Lebesgue outer measure is Erdős. A simplified Eigen-Falconer construction follows.

Example 2.11. *Given a set E satisfying the hypotheses of Theorem 2.10, there is a closed nowhere dense perfect set S of positive measure containing no copy of E.*

CONSTRUCTION. The set S is constructed in a manner similar to that of the middle thirds Cantor set. At the n-th step, k_n uniformly spaced open intervals are removed from each basic interval formed at the previous step to produce $k_n + 1$ basic intervals of equal size. The fractional amount removed at the n-th step is $\epsilon_n \in (0,1)$ and the ϵ_n sequence is chosen such that $\mu(S) = \Pi_{n=1}^{\infty}(1-\epsilon_n) > 0$. We choose an auxiliary integer sequence $N(n)$ such that $m \geq N(n)$ implies that $e_{m+1}/e_m > 1-\epsilon_n$. Now, choose the integer sequence k_n

recursively such that $n\,\mu(I_n) < e_{N(n)}$, where I_n is a basic interval formed at the n-th step.

Considering $x+tE$, where $x \in S$ and $t > 0$ ($t < 0$ is similar), $x+tE \not\subset S$ is immediate if x is the right end point of some basic interval in the construction. Otherwise, choose $n \in \mathbb{N}$ such that $1/n < t$ and let I_n be the basic interval formed at step n in the construction such that $x \in I_n$. Let O_n be the open interval removed at step n adjacent to I_n on the right. Let $m \in \mathbb{N}$ be minimal such that $x + te_m \in I_n$. We will show that $t(e_{m-1} - e_m) < \mu(O_n)$, which implies $e_{m-1} \in O_n$ and so $e_{m-1} \notin S$.

Observe that $e_m/n < te_m \leq \mu(I_n)$ implies $m > N(n)$, hence $(te_m)/(te_{m-1}) > 1-\epsilon_n$. We have that $\mu(O_n)/(\mu(O_n)+te_m) \geq \mu(O_n)/(\mu(O_n)+\mu(I_n)) > \epsilon_n$ which implies that $\mu(O_n) > te_m\epsilon_n/(1-\epsilon_n)$. After some algebra we obtain $t(e_{m-1} - e_m) < \mu(O_n)$ as desired. □

The similarity problem has been reformulated in two different ways. The first is due to Bourgain who used mainly probabilistic methods to prove the following. (In Bourgain's terminology, a set has *property E* if it is not Erdős.)

Theorem 2.12 (J. Bourgain [2])**.** *The following conditions are equivalent for a*
bounded set E of $\mathbb{R}^d$, $d \in \mathbb{N}$:
(i.) There is a constant C such that

$$\int \inf_{1<t<2} \sup_{e\in E^*} |f(x+te)|\,dx \leq C \int |f(x)|\,dx$$

whenever E^* is a finite subset of E and f is a continuous function on $\mathbb{R}^d$ with compact support.
(ii.) E is not Erdős.

Using this result, Bourgain was able to show the following.

Corollary 2.13 (J. Bourgain [2])**.** *If E_1, E_2, and E_3 are infinite sets of real numbers, then $E_1 + E_2 + E_3 := \{e_1 + e_2 + e_3 | e_i \in E_i, i = 1, 2, 3\}$ is Erdős.*

We can use this corollary to show that the middle-thirds Cantor set is Erdős as follows. A symmetric perfect set is the set of all finite or infinite subsums of the series $\sum_{n=1}^{\infty} \lambda_n$, where $\{\lambda_n\}_{n=1}^{\infty}$ is a sequence of positive numbers such that $\sum_{n=1}^{\infty} \lambda_n = 1$ and

$$\lambda_n > \sum_{m=n+1}^{\infty} \lambda_m \text{ for } n \in \mathbb{N} := \{1, 2, \dots\}.$$

(Set $\lambda_n = 2 \cdot 3^{-n}$ to obtain the middle-thirds Cantor set.)

The next result follows by letting E_1, E_2, and E_3 be infinite, mutually disjoint subsequences of $\{\lambda_n\}_{n=1}^{\infty}$.

Corollary 2.14. *Any symmetric perfect set is Erdős.*

Jasinski views the Erdős Problem as a "tiling puzzle".

Theorem 2.15 (J. Jasinski [16])**.** *Let $E \subset (0,1)$ be a set. The following conditions are equivalent.*
(i.) $\exists S \subset \mathbb{R}, \mu(S) > 0,$ such that $(x + tE) \cap (\mathbb{R} \setminus S) \neq \emptyset \;\; \forall (x,t)$ with $x/t \in \mathbb{R}$.
(ii.) $\exists S \subset (0,1), \mu(S) > 0,$ such that $\forall 0 < t < 1, (\mathbb{R} \setminus S) + tE = \mathbb{R}$.
(iii.) $\exists$ open $O \subset \mathbb{R}, \mu((0,1) \setminus O) > 0,$ such that $\forall 0 < t < 1, O + tE = \mathbb{R}$. (*)

The tiling idea is to place countably many tiles (open intervals) O_i in $(0,1)$. These tiles do not cover $(0,1)$; rather the addition of tE "expands" them so that the union covers $\mathbb{R}$; namely, $\mathbb{R} \subset O + tE$, where $O = (-\infty, 0) \cup (\bigcup_{i=1}^{\infty} O_i) \cup (1, +\infty)$. To use this idea for a given zero-sequence $\{e_n\}_{n=1}^{\infty}$, define $k(\delta) = \min\{k | \forall n \geq k, (e_n - e_{n+1}) \leq \delta\}$ for $k \in \mathbb{N}$ and $\delta > 0$.

Theorem 2.16 (J. Jasinski [16])**.** *Let $E = \{e_n\}_{n=1}^{\infty} \subset (0,1)$ be a zero-sequence. If there exist a sequence $\{\delta_n\}_{n=1}^{\infty}$ such that*

$$\sum_{n=1}^{\infty} \delta_n < 1 \text{ and } \sum_{n=1}^{\infty} e_{k(\delta_n)} = +\infty,$$

then there exists a double sequence of open intervals $\{O_{n,m}\}_{n,m=1}^{\infty}$ such that $O = (-\infty, 0) \cup (\bigcup_{n,m=1}^{\infty} O_{n,m}) \cup (1, +\infty)$ has property (*). Hence E is Erdős.

In the same work, Jasinski showed that Theorem 2.16 does not apply to zero-sequences $E = \{e_n\}_{n=1}^{\infty}$ such that $\limsup_{(n \to \infty)} e_{n+1}/e_n < u < 1$. On the other hand, we have the following.

Corollary 2.17 (J. Jasinski [16])**.** *If $E = \{e_n\}_{n=1}^{\infty}$ and $\{e_n - e_{n+1}\}_{n=1}^{\infty}$ are zero-sequences and $\sum_{n=1}^{\infty} e_n = +\infty$, then E is Erdős.*

That the Eigen-Falconer result and Corollary 2.17 are independent can be seen, on the one hand, by considering the sequence $\{1/n^2\}_{n=1}^{\infty}$, and on the other hand by considering the following example.

Example 2.18. *There is a zero-sequence $\{e_n\}_{n=1}^{\infty}$ such that Corollary 2.17 applies and $\liminf_{(n \to \infty)} e_{n+1}/e_n = 0$.*

Construction. Let $b_1 > 0$ be arbitrary and $b_1 > b_2 > \cdots > b_N = 0$ be a uniform partition of $[0, b_1]$ such that $b_1 + \cdots + b_N \geq 1$. Now, for a modified partition by replacing b_N by a small positive number such that $b_N/b_{N-1} < b_1$.

Construct the desired zero-sequence E step-wise, where the elements added at each step are the partition elements $b_1, \cdots, b_N$: let the b_1 for step n be the b_N from the previous step. The resulting sequence $\{e_n\}_{n=1}^{\infty}$ is such that $e_n - e_{n+1}$ is non-increasing, $\sum_{n=1}^{\infty} e_n = \infty$ and $\liminf_{(n\to\infty)} e_{n+1}/e_n = 0$; an obvious modification gives $e_n - e_{n+1}$ strictly decreasing. □

We can improve a little on Jasinski's idea by defining $\Delta e_n = \max\{e_i - e_{i+1} | i \geq n\}$ for a given zero-sequence $\{e_n\}_{n=1}^{\infty}$. The improvement is evident by observing that the Eigen-Falconer result is now a corollary of the following theorem.

Theorem 2.19. *Let $E = \{e_n\}_{n=1}^{\infty}$ be a zero-sequence. If*

$$\liminf_{n\to\infty} \frac{\Delta e_n}{e_n} = 0,$$

then E is Erdős.

Proof. Since $\liminf_{(n\to\infty)} \frac{\Delta e_n}{e_n} = 0$, there exists a subsequence n_k such that

$$\sum_{k=1}^{\infty} \frac{1}{1 + \frac{k+1}{k+3}\frac{e_{n_k}}{\Delta e_{n_k}}} < \frac{1}{2}.$$

For each $k \in \mathbb{N}$, let O_k consist of $(0, \Delta e_{n_k}/(k+1))$, together with countably many disjoint open intervals of the same length, with consecutive intervals separated by a distance of $e_{n_k}/(k+3)$, and extending to infinity in both directions.

Let $k \in \mathbb{N}$ and $c > 0$ be such that $1/(k+1) > c \geq 1/(k+2)$. Observe that

$$(0, \frac{\Delta e_{n_k}}{k+1} + \frac{e_{n_k}}{k+3}] \subset O_k + cE,$$

since $1/(k+1) > c$; $\Delta e_{n_k} \geq e_i - e_{i+1}$, for $i \geq n_k$, and $c \geq 1/(k+2)$, imply that

$$\frac{\Delta e_{n_k}}{k+1} + ce_{i+1} > ce_i \quad \text{and} \quad \frac{\Delta e_{n_k}}{k+1} + ce_{n_k} > \frac{\Delta e_{n_k}}{k+1} + \frac{e_{n_k}}{k+3}.$$

Let $O = (-\infty, 0) \cup (\bigcup_{k=1}^{\infty} O_k) \cup (1, +\infty)$. Since

$$\mu(O_k \cap [0,1]) < 2\left(\frac{\frac{\Delta e_{n_k}}{k+1}}{\frac{\Delta e_{n_k}}{k+1} + \frac{e_{n_k}}{k+3}}\right),$$

$\mu(O \cap [0,1]) < 1$ (the factor 2 accounts crudely for the fact that 1 is not necessarily a left end point of some interval in O_k). Then property (*) holds for $0 < c < 1/2$.

The proof of Theorem 2.15 shows that the set $[0,1] \setminus O$ just constructed may be used to form a subset of $[0,1]$ having positive measure and containing no copy of E. This implies that E is Erdős and completes the proof. □

Let $E \subset \mathbb{R}$ and $S \subset [0,1]$. Denote, by $\mathcal{B}(E,S)$, the set of points (x,t) with $x/t \in \mathbb{R}$ such that $x + tE \subset S$. In order that a set E be Erdős, there must exist a set S of positive measure such that $\mathcal{B}(E,S) = \emptyset$. Using mainly probabilistic methods, Kolountzakis [18] showed that for any infinite E there is a set S for which $\mu^2(\mathcal{B}(E,S)) = 0$ (μ^2 denotes Lebesgue plane measure). This result follows from the following main theorem.

Theorem 2.20 (Kolountzakis [18]). *Let $E \subset \mathbb{R}$ be an infinite set. Then there exists $S \subset [0,1]$, with $\mu(S)$ arbitrarily close to 1, such that*

$$\mu(\{t | \exists x \text{ such that } (x,t) \in \mathcal{B}(E,S)\}) = 0.$$

The following theorem of Kolountzakis implies the Eigen-Falconer result as well as providing a proof that $\{2^{-n^\alpha}\}_{n=1}^\infty + \{2^{-n^\alpha}\}_{n=1}^\infty$ is Erdős for $0 < \alpha < 2$. (In Kolountzakis' terminology, a set is *universal* if it is not Erdős.)

Theorem 2.21 (Kolountzakis [18]). *Let $E \subset \mathbb{R}$ be an infinite set which contains, for arbitrarily large N, a subset $\{e_1, \dots, e_N\}$ with $e_1 > \cdots > e_N > 0$ and*

$$-\ln \delta_N = o(N),$$

where $\delta_N = \min_{(1 \le m < N)}(e_m - e_{m+1})/e_1$. Then E is Erdős.

Since this theorem is only of interest for bounded sets (unbounded sets are already Erdős), it may be restated in the following more convenient way.

Theorem 2.22. *Any bounded, infinite $E \subset \mathbb{R}$ such that*

$$\liminf_{|E^*| \to \infty} \frac{-\ln \delta(E^*)}{|E^*|} = 0$$

is Erdős, where E^* is a finite subset of E of cardinality $|E^*|$ and $\delta(E^*)$ denotes the length of the shortest component of $(\min E^*, \max E^*) \setminus E^*$.

These theorems are limited by the following.

Lemma 2.23. *Let* $E = \{e_n\}_{n=1}^{\infty}$ *be a zero-sequence such that* $\limsup_{(n\to\infty)} e_{n+1}/e_n < u < 1$. *Then*

$$\liminf_{|E^*|\to\infty} \frac{-\ln\delta(E^*)}{|E^*|} > 0.$$

Proof. Without loss of generality suppose that $e_{n+1}/e_n \le u$ for all n. Choose an integer $N \ge 2$ and $E^* = \{\hat{e}_1, \ldots, \hat{e}_N\} \subset E$ such that $\hat{e}_1 > \cdots > \hat{e}_N > 0$. Let $1 \le m < N$ be such that $\delta(E^*) = \hat{e}_m - \hat{e}_{m+1}$.

Observe that $\hat{e}_m \le \hat{e}_1 u^{m-1} \le e_1 u^{m-1}$, and that $N - m \le \hat{e}_m/(\hat{e}_m - \hat{e}_{m+1})$. Then $\delta(E^*) = \hat{e}_m - \hat{e}_{m+1} \le \hat{e}_m/(N-m) \le e_1 u^{m-1}/(N-m) \le e_1 u^{N-1} u^{-(N-m)}$.

Also, $\hat{e}_m(1-u) \le \hat{e}_m - \hat{e}_{m+1}$, hence $(1-u) \le 1/(N-m)$, from which it follows that $u^{-(N-m)} \le u^{-1/(1-u)}$. Thus $\delta(E^*) \le e_1 u^{N-1} u^{-1/(1-u)}$, and

$$\frac{-\ln\delta(E^*)}{N} \ge \frac{-\ln e_1}{N} - \frac{N-1}{N}\ln u + \frac{\ln u}{N(1-u)}.$$

The result follows from $-\ln u > 0$. □

The similarity problem was reformulated again by Humke and Laczkovich into a combinatorial problem.

Theorem 2.24 (Humke and Laczkovich [15])**.** *Let* $E \subset [0,1]$ *be such that* $\inf(E) = 0$ *and* $\sup(E) = 1$. *Then the following are equivalent.*
(i.) E is Erdős;
(ii.) $\lim_{(n\to\infty)} \frac{\Lambda_n}{n} = 0$*;*
(iii.) $\liminf_{(n\to\infty)} \frac{\Lambda_n}{n} = 0$*;*
where Λ_n *is the cardinality of the smallest set* $B \subset \mathbb{N}_n := \{1, 2, ..., n\}$ *that intersects each set of the form*

$$E_{x,y} = \{x + [ey] | e \in E\},$$

where $x, y, x+y \in \mathbb{N}_n$*;* $y \ge [n/2]$*; and* $[ey]$ *denotes the integer part of* ey.

From this theorem they obtain the following special case. (It also follows from a slight modification of the proof of Theorem 2.19.)

Theorem 2.25 (Humke and Laczkovich [15])**.** *Any bounded* $E \subset \mathbb{R}$ *such that*

$$\inf_{u<v} \frac{E(u,v)}{v-u} = 0$$

is Erdős, where $E(u,v)$ *denotes the length of the longest component of* $(u,v)\backslash E$.

Corollary 2.26 (Humke and Laczkovich [15]). *If $E = \{e_n\}_{n=1}^{\infty}$ and $\{e_n - e_{n+1}\}_{n=1}^{\infty}$ are zero-sequences and $\limsup_{(n\to\infty)} e_{n+1}/e_n = 1$ then E is Erdős.*

The following two examples demonstrate that Theorem 2.25 and Theorem 2.21 are independent.

Example 2.27. *There is a zero-sequence E such that $\inf_{(u<v)} \frac{E(u,v)}{v-u} = 0$ and $\liminf_{(|E^*|\to\infty)} \frac{-\ln\delta(E^*)}{|E^*|} > 0$.*

Construction. Let $C_0 = [0,1]$. Let C_n, $n \geq 1$, consist of $n+1$ equal length disjoint closed intervals, uniformly spaced in $[0,1]$, such that $0, 1 \in C_n$ and $\mu(C_n) = 1 - n2^{-n^2}$. Let ∂C_n denote boundary of C_n. Observe that ∂C_n consists of $2(n+1)$ points and that the largest (respectively, smallest) component of $[0,1]\backslash\partial C_n$ has length $\frac{1}{n+1} - \frac{n}{n+1}2^{-n^2}$ (respectively, 2^{-n^2}). Thus, with $u = 0$ and $v = 1$, $(\partial C_n)(u,v)/(v-u) \leq \frac{1}{n+1}$.

Let $E_n = 2^{-n}(\partial C_n + 1)$, ∂C_n scaled and shifted right by 2^{-n}, and $E = \bigcup_{n=0}^{\infty} E_n$. Observe that $|\bigcup_{m=0}^{n} E_m| \leq \sum_{m=0}^{n} 2(m+1) = (n+1)(n+2)$.

To see that $\liminf_{(|E^*|\to\infty)} \frac{-\ln\delta(E^*)}{|E^*|} > 0$, let E^* be a finite subset of E and N be maximal such that $E^* \cap E_{N+1} \neq \emptyset$. Then $\delta(E^*) \leq 2^{-N^2}$ and

$$\frac{-\ln\delta(E^*)}{|E^*|} \geq \frac{N^2 \ln 2}{(N+2)(N+3)},$$

showing that the liminf is non-zero.

On the other hand, to see that $\inf_{(u<v)} \frac{E(u,v)}{v-u} = 0$, consider the sets $(u,v) = (2^{-n}, 2^{-n+1})$, $n \geq 1$. Then

$$\frac{E(2^{-n}, 2^{-n+1})}{(2^{-n+1} - 2^{-n})} \leq \frac{2^{-n}\frac{1}{n+1}}{(2^{-n+1} - 2^{-n})} = \frac{1}{n+1},$$

showing that the inf is zero. □

Example 2.28. *There is a zero-sequence E such that $\inf_{(u<v)} \frac{E(u,v)}{v-u} > 0$ and $\inf_{(|E^*|\to\infty)} \frac{-\ln\delta(E^*)}{|E^*|} = 0$.*

Construction. Recall the construction of the ordinary middle-thirds Cantor set. At step 0 we have $C_0 = [0,1]$, which is composed of a single basic interval. At step n, $n \geq 1$, we remove the open middle third from each of the 2^{n-1} basic intervals of length 3^{-n+1} formed at step $n-1$. Denote by C_n the union of these 2^n disjoint closed intervals of length 3^{-n}. The middle thirds Cantor set is given by $\bigcap_{n=0}^{\infty} C_n$. Let ∂C_n denote boundary of C_n. Observe that ∂C_n consists of 2^{n+1} points and that the largest (respectively, smallest) component of $[0,1]\backslash\partial C_n$ has length 3^{-1} (respectively, 3^{-n}). For any two points $u, v \in \partial C_n$, $u < v$, there is a last step $m \geq n$ in the construction of C_n such that u and v are elements of the same basic interval. Then $v - u \leq 3^{-m}$ and the largest gap in ∂C_n between u and v has length at lease 3^{-m-1}; hence,

$$\frac{C_n(u,v)}{v-u} \geq \frac{3^{-m-1}}{3^{-m}} = \frac{1}{3}.$$

Let $E_n = 2^{-n}(\partial C_n + 1)$, ∂C_n scaled and shifted right by 2^{-n}, and $E = \bigcup_{n=0}^{\infty} E_n$.

To see that $\liminf_{(|E^*|\to\infty)} \frac{-\ln\delta(E^*)}{|E^*|} = 0$, consider the sequence of sets $E^* = E_n$, $n \geq 0$. Then $\delta(E_n) = 2^{-n}3^{-n}$, and $\frac{-\ln\delta(E_n)}{|E_n|} = \frac{n\ln 2 + n\ln 3}{2^{n+1}}$, showing that the liminf is zero.

On the other hand, to see that $\inf_{(u<v)} \frac{E(u,v)}{v-u} > 0$, let $M, N \in \mathbb{N}$ be minimal such that $u \in E_M$ and $v \in E_N$. If $u, v \in E_M$ or $u, v \in E_N$, then $\frac{E(u,v)}{v-u} \geq \frac{1}{3}$. Otherwise if $M = N + 1$, then $u < 2^N < v$ and suppose that $v - 2^N \geq 2^N - u$ (the other case is similar). Then

$$\frac{E(u,v)}{v-u} \geq \frac{E(2^N, v)}{2(v-2^N)} \geq \frac{1}{2\cdot 3}.$$

Finally, if $M > N + 1$, then

$$\frac{E(u,v)}{v-u} \geq \frac{E(2^{N+1}, 2^N)}{2^{N-1}} = \frac{E(2^{N+1}, 2^N)}{4(2^N - 2^{N+1})} \geq \frac{1}{4\cdot 3}.$$

Hence $\inf_{(u<v)} \frac{E(u,v)}{v-u} > 0$. □

The most recent results on the Erdős problem appear in unpublished work of M. Chlebík [3]. We say that a set E is *c-Erdős* if there exists a set $S \subset \mathbb{R}$ satisfying $\mu(\mathbb{R} \setminus S) < \infty$ and not containing a copy of E. (Chlebík uses the condition that for all $\epsilon > 0$, $\mu(\mathbb{R} \setminus S) < \epsilon$. The two conditions are equivalent since we may replace S satisfying $\mu(\mathbb{R} \setminus S) < \infty$ by a version of S scaled by $\alpha < \epsilon/\mu(\mathbb{R} \setminus S)$.)

Considering bounded sets E, we have the following lemma.

Lemma 2.29. *Each bounded $E \subset \mathbb{R}$ is Erdős if and only if it is c-Erdős.*

PROOF. We need only consider the case when E is Erdős. Let $\epsilon > 0$ and $S \subset \mathbb{R}$ have positive measure and contain no copy of E. Since S has positive measure we may choose a sequence $\{S'_n\}_{n=1}^{\infty}$ of suitably scaled and shifted portions of S such that $\mu([-n,n] \setminus S'_n) < \epsilon 2^{-n}$ for each n. Define $S_n = (-\infty, -n) \cup S'_n \cup (n, +\infty)$. Replace S with the set $\bigcap_{n=1}^{\infty} S_n$ and observe that $\mu(\mathbb{R} \setminus S) < \epsilon$. □

The situation is different for unbounded E; each such set is Erdős though not necessarily c-Erdős. For example, $\mathbb{N}$, the set of positive integers, is not c-Erdős while $\{n^{\alpha}\}_{n=1}^{\infty}$ is c-Erdős whenever $\alpha \in (0,1)$ [3].

The following result extends Theorem 2.25 to the case of unbounded E.

Theorem 2.30 (Chlebík [3])**.** *Let $E \subset \mathbb{R}$ have the following density property*

$$\inf_{u<v} \frac{E(u,v)}{v-u}(|u|+|v|+1) = 0. \tag{$P1$}$$

Then E is c-Erdős.

Corollary 2.31 (Chlebík [3])**.** *Let $E = \{e_n\}_{n=1}^{\infty}$ be an increasing sequence of positive numbers tending to $+\infty$ such that for every $\epsilon > 0$ there exist integers $1 \le m < n$ such that $e_i - e_{i-1} < \epsilon \left(1 - \frac{e_m}{e_n}\right)$ for any $i = m+1, m+2, \cdots, n$. (This property is implied by $e_n \to +\infty$ and $e_{n+1} - e_n \to 0$ as $n \to \infty$.)*

Then E is c-Erdős.

Results which limit the size of the set $\mathcal{B}(E,S)$ for certain $E \subset \mathbb{R}$ are given in the next three theorems.

Theorem 2.32 (Chlebík [3])**.** *Let ν be a σ-finite Borel measure on $\mathcal{C} := \{(x,t) | x/t \in \mathbb{R}\}$ and $E \subset \mathbb{R}$ have the following density property*

$$\sup \left\{ \frac{\operatorname{card}(E \cap [-n,n])}{n} \,\middle|\, n \in \mathbb{N} \right\} = +\infty. \tag{$P2$}$$

Then for every $\epsilon > 0$ there exists an open set $G \subset \mathbb{R}$ with $\mu(G) < \epsilon$ such that $(x+tE) \cap G$ is infinite for ν-a.e. $(x,t) \in \mathcal{C}$.

Theorem 2.33 (Chlebík [3])**.** *Let $E \subset \mathbb{R}$ possess property $(P2)$ and $B \subset \mathcal{C}$ be a set of σ-finite 1-dimensional Hausdorff measure. Then for every $\epsilon > 0$ there exists an open set $G \subset \mathbb{R}$ with $\mu(G) < \epsilon$ such that $(x+tE) \cap G$ is infinite for each $(x,t) \in B$.*

Theorem 2.34 (Chlebík [3]). *Let $E \subset \mathbb{R}$ have the following density property*

$$\sup\{\text{card}(E \cap [\alpha, \alpha+1] | \alpha \in \mathbb{R}\} = +\infty. \tag{$P3$}$$

Let $B = \mathbb{R} \times C \subset \mathcal{C}$, where C is countable. Then for every $\epsilon > 0$ there exists an open set $G \subset \mathbb{R}$ with $\mu(G) < \epsilon$ such that $(x + tE) \cap G$ is infinite for each $(x, t) \in B$.

3 In Conclusion

It is interesting to note that none of these results can be applied to the geometric sequence $\{2^{-n}\}_{n=1}^{\infty}$ or, indeed, to any zero-sequence $\{e_n\}_{n=1}^{\infty}$ such that $\limsup_{(n\to\infty)} e_{n+1}/e_n < 1$.

The difficult nature of this problem suggests that one might look to other related problems for hints as to how to proceed. Toward that end, consider the following question related to the Steinhaus result that each set of positive measure contains a copy of each finite set is the following.

Question 3.1. *Is it true that if a measurable set contains a copy of each finite set, then the set has positive measure?*

The result of Miller that any set of full measure contains a copy of each countable set suggests another question.

Question 3.2. *Is it true that for every uncountably infinite set, E, of real numbers there exists $S \subset [0, 1]$ of full measure that does not contain a subset similar to E?*

An affirmative answer to the following question would show that any uncountable Borel set is Erdős.

Question 3.3. *Does Corollary 2.14 extend to arbitrary nowhere dense perfect sets?*

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