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SOBOLEV EXTENSION DOMAINS ON METRIC SPACES OF HOMOGENEOUS TYPE

Abstract

Let (X, d, μ) be a metric measure space of homogeneous type with a finite measure. Assume that $\Omega \subset X$ is a bounded domain, which satisfies an $A^*(\varepsilon, \delta)$-condition and $\mu(\partial\Omega) = 0$. We show that there exists a bounded linear extension operator Ext from the Hajłasz space $M^{1,p}(\Omega, d, \mu)$ to $M^{1,p}(X, d, \mu)$, such that $Ext(u)|_\Omega = u$.

1   Introduction

P. Hajłasz defined Sobolev spaces on a metric space, [2]. Let (X, d, μ) be a metric space with a measure μ. A function $u \in L^p(X, \mu)$ belongs to the Sobolev space $M^{1,p}(X, d, \mu)$, $1 < p \leq \infty$, if there exists a non-negative function $g \in L^p(X, \mu)$ such that

$$|u(x) - u(y)| \leq d(x, y)\big(g(x) + g(y)\big), \tag{1.1}$$

for almost every $x, y \in X$. Let d be the Euclidean metric $|\cdot|$ and μ be n-dimensional Lebesgue measure m. If $\Omega \subset \mathbb{R}^n$ is a Sobolev extension domain for some $1 < p \leq \infty$ or if $\Omega = \mathbb{R}^n$, then every $u \in W^{1,p}(\Omega)$ satisfies the inequality (1.1) with $g(x) = M|\nabla u|(x)$. Here M is the Hardy-Littlewood maximal operator. On the other hand if $\Omega \subset \mathbb{R}^n$ is an arbitrary open set, then every $u \in M^{1,p}(\Omega, |\cdot|, m)$ belongs to $W^{1,p}(\Omega)$ and $|\nabla u(x)| \leq 2\sqrt{n}g(x)$

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for almost every $x \in \Omega$ (see [3, Lemma 6, p.227]). Recently V. Gol'dshtein and M. Troyanov gave a new integral characterization for $M^{1,p}$-spaces [1].

Hajłasz and O. Martio proved that if a bounded domain $\Omega \subset \mathbb{R}^n$ satisfies an $A(c)$-condition, then there exists a bounded linear extension operator

$$E^* : M^{1,p}(\Omega, |\cdot|, m) \to W^{1,p}(\mathbb{R}^n),$$

for every $1 < p \leq \infty$, such that $E^* u|_\Omega = u$, [3, Theorem 9, p.240]. The $A(c)$-condition, $0 < c < 1$, asserts that the domain cannot be too thin near the boundary: for every $x \in \partial\Omega$ there exists $y \in \Omega$ such that $B^n(y, cr) \subset B^n(x, r) \cap \Omega$ with $0 < r \leq \operatorname{diam}(\Omega)$. For example uniform domains and John domains, [7], satisfy the $A(c)$-condition. We generalize the result of Hajłasz and Martio to metric measure spaces of homogeneous type.

Assume that μ is a non-negative Borel regular outer measure, which is finite on every bounded set. We also assume that μ satisfies the doubling condition and $\mu(X) < \infty$. We define a new plumpness condition which is weaker than the $A(c)$-condition. We write

$$\Omega(\varepsilon r) = \{z \in \Omega \mid \operatorname{dist}(z, X \setminus \bar{\Omega}) > \varepsilon r\}.$$

A domain $\Omega \subset X$ satisfies an $A^*(\varepsilon, \delta)$-condition, $0 < \varepsilon < 1$ and $0 < \delta \leq 1$, if

$$\mu(B(x, r) \cap \Omega(\varepsilon r)) \geq \delta\mu(B(x, r)),$$

for every $x \in \Omega$ and $0 < r < \operatorname{diam}(\Omega)$.

Assume that $\Omega \subset X$ is a bounded domain, which satisfies the $A^*(\varepsilon, \delta)$-condition, $\bar{\Omega} \neq X$ and $\mu(\partial\Omega) = 0$. Then for every $1 < p \leq \infty$, there exists a bounded linear extension operator

$$\operatorname{Ext} : M^{1,p}(\Omega, d, \mu) \to M^{1,p}(X, d, \mu),$$

such that $\operatorname{Ext}(u)|_\Omega = u$ for every $u \in M^{1,p}(X, d, \mu)$. In the Euclidean case our result follows by the results of Hajłasz and Martio. The proof is based on the proof of Hajłasz and Martio.

2   Notation

Throughout this paper C will denote a general constant which may change even in a single string of estimate.

We let (X, d, μ) denote a metric measure space of homogeneous type. This means that μ is a non-negative, non-trivial Borel regular outer measure on X, finite on every bounded set and there exists a constant $C > 0$ such that for every $x \in X$ and $r > 0$,

$$\mu(B(x, 2r)) \leq C\mu(B(x, r)). \tag{2.1}$$

Here $B(x, r)$ is the open ball with center x and radius r. Condition (2.1) is the doubling condition for the measure μ and the constant C the doubling constant of the measure μ. Note that by the doubling condition, if $B(x, R)$ is a ball in X, $z \in B(x, R)$ and $0 < r \leq R < \infty$ then

$$\frac{\mu(B(z, r))}{\mu(B(x, R))} \geq q\Big(\frac{r}{R}\Big)^Q. \tag{2.2}$$

The constants q and Q depend only on the doubling constant.

The Euclidean metric on $\mathbb{R}^n$ is denoted by $|\cdot|$ and the Lebesgue n-measure by m. We let $\bar{A}$ be the closure of the set A and ∂A the boundary of the set A. By $A \setminus F$ we mean the set $\{x \in A : x \notin F\}$. By χ_A we denote the characteristic function of the set A.

We say that a condition holds for almost every $x \in A$ if there exists a set $F \subset A$ with $\mu(F) = 0$ such that the condition holds for every $x \in A \setminus F$. We write $u \approx v$ if there exists a constant $C > 0$ so that $C^{-1}u \leq v \leq Cu$. By $\fint_A u\, d\mu = \mu(A)^{-1} \int_A u\, d\mu$ we denote the integral average of the function u over the set A, $\mu(A) > 0$.

The set of p-integrable functions, $1 \leq p \leq \infty$, by measure μ in a set A is denote by $L^p(A, \mu)$. In the Euclidean case we just write $L^p(A, m) = L^p(A)$. We say that a function u belongs to a Sobolev space $M^{1,p}(A, d, \mu)$, $1 < p \leq \infty$, if $u \in L^p(A, \mu)$ and there exists a non-negative $g \in L^p(A, \mu)$ so that

$$|u(x) - u(y)| \leq d(x, y)(g(x) + g(y))$$

for almost every $x, y \in A$. Following [3, p. 228] we call g a generalized gradient of u The Sobolev space $M^{1,p}(A, d, \mu)$ is equipped with the norm

$$\|u\|_{M^{1,p}(A,d,\mu)} = \|u\|_{L^p(A,\mu)} + \inf_g \|g\|_{L^p(A,\mu)},$$

where the infimum is taken over all generalized gradients of the function u. These Sobolev spaces were defined by Hajłasz, [2].

In the Euclidean case the classical Sobolev space on an open set $A \subset \mathbb{R}^n$ is denoted by $W^{1,p}(A)$, $1 \leq p \leq \infty$. It is equipped with the norm

$$\|u\|_{W^{1,p}(A)} = \|u\|_{L^p(A)} + \|\nabla u\|_{L^p(A)}.$$

Here ∇u is the weak gradient of the function u.

We say that a domain $\Omega \subset X$ is an $M^{1,p}$-extension domain, $1 < p \leq \infty$, if there exists a bounded extension operator

$$\text{Ext} : M^{1,p}(\Omega, d, \mu) \to M^{1,p}(X, d, \mu)$$

such that $\text{Ext}(u)|_\Omega = u$ for every $u \in M^{1,p}(X, d, \mu)$. The extension operator is bounded if there exists a constant $C > 0$ such that

$$\| \text{Ext}(u) \|_{M^{1,p}(X,d,\mu)} \leq C \|u\|_{M^{1,p}(\Omega,d,\mu)}$$

for every $u \in M^{1,p}(\Omega, d, \mu)$. In a similar way we define a $W^{1,p}$-extension domain in the Euclidean n-space $\mathbb{R}^n$.

Note that if $\Omega \subset \mathbb{R}^n$ is a $W^{1,p}$-extension domain, $1 < p \leq \infty$, then $M^{1,p}(\Omega, |\cdot|, m) = W^{1,p}(\Omega)$. This means that function spaces $M^{1,p}(\Omega, |\cdot|, m)$ and $W^{1,p}(\Omega)$ are the same as a set and the norms are equivalent. In particular, $M^{1,p}(\mathbb{R}^n, |\cdot|, m) = W^{1,p}(\mathbb{R}^n)$. On the other hand, the reverse does not hold: A. S. Romanov has constructed a bounded domain $\Omega \subset \mathbb{R}^n$, which is not a $W^{1,p}$-extension domain, but $W^{1,p}(\Omega) = M^{1,p}(\Omega, |\cdot|, m)$, [9].

3   A Plumpness Condition

A bounded domain $\Omega \subset \mathbb{R}^n$ satisfies the $A(c)$-condition, $0 < c < 1$, if for every $x \in \partial\Omega$ and every $0 < r < \text{diam}(\Omega)$ there exists $y \in \Omega$ such that $B(y, cr) \subset \Omega \cap B(x, r)$, [3, p.225].

The $A(c)$-condition says that the domain Ω cannot be too "thin" near the boundary. The $A(c)$-condition for a bounded domain is equivalent to an α-plump condition up to a constant. The α-plump condition was defined by Martio and J. Väisälä, [8, p.310]. Another equivalent condition is the corkscrew condition, which is by D. Jerison and C. Kenig [4, 3.1, p.93].

Next we define a new plumpness condition, which in the Euclidean case is weaker than the $A(c)$-condition. Let

$$\Omega(h) = \{x \in \Omega | \, \text{dist}(x, X \setminus \bar{\Omega}) > h\},$$

for every $h > 0$. We say that a bounded domain $\Omega \subset X$ satisfies the $A^*(\varepsilon, \delta)$-condition, $0 < \varepsilon < 1$ and $0 < \delta \leq 1$, if

$$\mu(B(z, r) \cap \Omega(\varepsilon r)) \geq \delta \mu(B(z, r))$$

for every $z \in \Omega$ and every $0 < r < \text{diam}(\Omega)$.

Lemma 3.1. *Let $(\mathbb{R}^n, |\cdot|, m)$ be Euclidean n-space. Assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain which satisfies the $A(c)$-condition. Then the domain Ω satisfies the $A^*\left(\frac{c}{4}, \left(\frac{c}{4}\right)^n\right)$-condition.*

PROOF. Fix $r \in (0, \mathrm{diam}(\Omega))$. Let $z \in \Omega$ be arbitrary. If $B^n(z, \frac{r}{2}) \subset \Omega$, then

$$m\big(B^n(z,r) \cap \Omega(\frac{c}{4}r)\big) \geq m\big(B^n(z, (\frac{1}{2} - \frac{c}{4})r)\big)$$
$$\geq \big(\frac{c}{4}\big)^n m\big(B^n(z,r)\big).$$

If $B^n(z, \frac{r}{2}) \not\subset \Omega$, then there exists $x \in \partial\Omega \cap B^n(z, \frac{r}{2})$. By the $A(c)$-condition there exists a point $y \in \Omega$ such that $B^n(y, \frac{c}{2}r) \subset B^n(z, r)$. Hence we obtain

$$m\big(B^n(z,r) \cap \Omega(\frac{c}{4}r)\big) \geq m\big(B^n(y, \frac{c}{4}r)\big)$$
$$\geq \big(\frac{c}{4}\big)^n m\big(B^n(z,r)\big).$$

This completes the proof of Lemma 3.1.   □

It is clear that if a bounded domain in $\mathbb{R}^n$ satisfies the $A(c)$- or $A^*(\varepsilon, \delta)$-condition, then it satisfies the following condition for some constant $C > 0$ depending on c or δ:

$$m(B(z,r) \cap \Omega) \geq Cm(B(z,r)), \tag{3.2}$$

for every $z \in \partial\Omega$ and $0 < r < \mathrm{diam}(\Omega)$.

P. Koskela has proved that if $\Omega \subset \mathbb{R}^n$ is a $W^{1,p}$-extension domain with $n - 1 < p < \infty$, then it satisfies the condition (3.2), [5, Theorem 6.5, p.28]. The condition (3.2) implies that $m(\partial\Omega) = 0$, since $\partial\Omega$ cannot contain points of n-density.

The following example shows that there exists a bounded domain $\Omega \subset \mathbb{R}^2$ which satisfies the $A^*(\varepsilon, \delta)$-condition, but does not satisfy the $A(c)$-condition. In Example 3.4 we construct a bounded domain in $\mathbb{R}^2$ which satisfies the condition (3.2) but does not satisfy the $A^*(\varepsilon, \delta)$-condition.

Example 3.3. Let d be the Euclidean metric $|\cdot|$ and μ the Lebesgue 2-measure m. Let $Q = \{(x_1, x_2) : |x_1 - 0| < 2, |x_2 - 2| < 2\}$ and $E = \bigcup_{n=1}^{\infty} \{\frac{1}{n}(\cos(\frac{k\pi}{n}), \sin(\frac{k\pi}{n})), k = 1, \ldots, n-1\}$. Since Q satisfies the $A^*(\varepsilon, \frac{1-2\varepsilon}{\pi})$-condition for every $0 < \varepsilon < \frac{1}{5}$ and $m(E) = 0$, it is clear that the domain $Q \setminus E$ also satisfies the $A^*(\varepsilon, \frac{1-2\varepsilon}{\pi})$-condition for every $0 < \varepsilon < \frac{1}{5}$. For every $n \geq 1$ the diameter of the largest ball inside $B^2((0,0), \frac{1}{n}) \cap Q \setminus E$ is comparable to $\frac{1}{n^2}$ and hence $Q \setminus E$ does not satisfy the $A(c)$-condition for any $0 < c < 1$.

Example 3.4. Let d be the Euclidean metric $|\cdot|$ and μ the Lebesgue 2-measure m. Let

$$\Omega = (0,1) \times (0,1) \setminus \bigcup_{j=1}^{\infty} \left[\frac{j-1}{j}, \frac{j-1}{j} + \varepsilon_j\right] \times \left[0, \frac{1}{2}\right] \subset \mathbb{R}^2,$$

where $\varepsilon_j = \frac{1}{4^j}$. We denote by $\lfloor x \rfloor$ the largest natural number which is smaller than x. Let $r > 0$. The ball $B^2((1,0), r)$ intersects sets $\left[\frac{j-1}{j}, \frac{j-1}{j} + \varepsilon_j\right] \times \left[0, \frac{1}{2}\right]$ at least when $\frac{j-1}{j} \geq 1 - r$. Thus we assume that $j \geq \lfloor \frac{1}{r} \rfloor$. We obtain

$$m(\Omega \cap B^2((1,0), r)) \geq \frac{1}{4}\pi r^2 - \sum_{j=\lfloor \frac{1}{r} \rfloor}^{\infty} r \frac{1}{4^j}$$

$$= \frac{1}{4}\pi r^2 - r \frac{1}{4^{\lfloor \frac{1}{r} \rfloor}} \sum_{i=1}^{\infty} \frac{1}{4^i}$$

$$= \frac{1}{4}\pi r^2 - r \frac{1}{4^{\lfloor \frac{1}{r} \rfloor}} \frac{1}{3} \geq \frac{1}{8}\pi r^2.$$

Hence the domain Ω satisfies the condition (3.2).

The distance between sets $\left[\frac{j-1}{j}, \frac{j-1}{j} + \varepsilon_j\right] \times \left[0, \frac{1}{2}\right]$, when $j = i, i+1$, is less than $\frac{1}{i^2+i}$. The ball $B^2((1,0), r)$ intersects these sets at least when $\frac{i-1}{i} \geq 1-r$ which implies that $i \geq \lfloor \frac{1}{r} \rfloor$. Assume that $i = \lfloor \frac{1}{r} \rfloor$. Let $0 < \varepsilon < 1$. Then $m(\Omega(\varepsilon r) \cap B^2((1,0), r)) = 0$ when $\frac{1}{i^2+i} \leq \varepsilon \frac{1}{i}$. This happens as soon as we assume that $i \geq \frac{1}{2\varepsilon} - 1$, which implies $r \leq \frac{2\varepsilon}{1-2\varepsilon}$. To see that the domain Ω does not satisfy the $A^*(\varepsilon, \delta)$-condition for any $0 < \varepsilon < 1$ we need only take a sequence of points $(x_i, y_i) \in \Omega$ with $(x_i, y_i) \to (0,1)$ as $i \to \infty$.

4   Main Theorem

We recall the following well known Whitney type covering lemma, [6, Lemma 2.9].

Lemma 4.1. *Suppose that E is an open set of finite measure in (X, d, μ), $E \neq X$. Let $r(x) = \frac{1}{10}d(x, X \setminus E)$. Then there exist $N \geq 1$ and a sequence $\{x_i\}_{i=1}^{\infty}$ such that by denoting $r(x_i) = r_i$, the following properties hold:*

(1) $E = \bigcup_{i=1}^{\infty} B(x_i, r_i)$.

(2) $B(x_i, 5r_i) \subset E$ for every $i = 1, 2, \ldots$.

(3) The balls $B(x_i, \frac{1}{4}r_i)$ are pairwise disjoint.

(4) *For every $i = 1, 2 \ldots$, if $x \in B(x_i, 5r_i)$, then $5r_i \le d(x, X \setminus E) \le 15r_i$.*

(5) *For every $i = 1, 2, \ldots$ there exists $y_i \in X \setminus E$ such that $d(x_i, y_i) \le 15r_i$.*

(6) $\sum_{i=1}^{\infty} \chi_{B(x_i, 5r_i)}(x) \le N$ *for every $x \in X$.*

It is easy to prove that the balls in the above Whitney decomposition have the following properties:

(7) *There exists a constant $C > 0$ such that if $B(x_i, r_i) \cap B(x_j, r_j) \ne \emptyset$, then $C^{-1} r_i \le r_j \le C r_i$.*

(8) *Suppose that B is a Whitney ball, that is a ball from the Whitney decomposition of E. Then there are at most C Whitney balls which intersect the ball B. The constant C depends only on the doubling constant.*

By inequality (2.2) the following condition holds:

(9) *There exists a constant $C > 0$ such that if $B(x_i, r_i) \cap B(x_j, r_j) \ne \emptyset$, then $C^{-1} \mu(B(x_i, r_i)) \le \mu(B(x_j, r_j)) \le C \mu(B(x_i, r_i))$. The constant C depends only on the doubling constant.*

The following lemma is a simple consequence of the inequality (2.2).

Lemma 4.2. *Let $\lambda > 0$ be fixed. Let $B(x_1, r)$ and $B(x_2, r)$ be two balls in (X, d, μ), with $d(x_1, x_2) \le \lambda r$. There exists a constant $C > 0$ which depends only on λ and the doubling constant, such that*

$$C^{-1} \mu(B(x_2, r)) \le \mu(B(x_1, r)) \le C \mu(B(x_2, r)).$$

The construction of the extension operator. Assume that Ω is a bounded domain in a homogeneous type metric space (X, d, μ) with $\mu(X) < \infty$ and $\Omega \ne X$. Assume that Ω satisfies the $A^*(\varepsilon, \delta)$-condition and $\mu(\partial \Omega) = 0$. Let $\mathcal{W}_0$ be the Whitney decomposition of $X \setminus \bar{\Omega}$ given by Lemma 4.1. Let $\mathcal{W}$ be the collection of balls $B \in \mathcal{W}_0$ with $\mathrm{diam}(B) \le \mathrm{diam}(\Omega)$. We write $\mathcal{W} = \{B_i : i = 1, 2, \ldots\}$.

For every Whitney ball $B_i = B(x_i) = B(x_i, r_i) \in \mathcal{W}$ we pick $x_i^* \in \Omega$ such that $\frac{15}{10} \mathrm{dist}(x_i, \Omega) \ge \mathrm{dist}(x_i, x_i^*)$. We set

$$U(x_i^*) = B(x_i^*, r_i) \cap \Omega(\varepsilon r_i).$$

Here $\Omega(\varepsilon r_i) = \{x \in \Omega | \mathrm{dist}(x, X \setminus \bar{\Omega}) > \varepsilon r_i\}$. By the $A^*(\varepsilon, \delta)$-condition and Lemma 4.2 there exists a constant $C > 0$ depending only on the doubling constant and δ so that

$$C^{-1} \mu(B(x_i)) \le \mu(U(x_i^*)) \le C \mu(B(x_i))$$

for every $i = 1, 2, 3 \ldots$. We set $\tilde{u}(x_i) = \fint_{U(x_i^*)} u(z) \, d\mu(z)$, and hence $\tilde{u}$ is defined on all the center points of the Whitney balls from the collection $\mathcal{W}$. Next we extend $\tilde{u}$ onto $\bigcup_{i=1}^{\infty} B_i$. Let ϕ_i, $i = 1, 2, 3, \ldots$, be a partition of unity corresponding to the collection $\mathcal{W}$ [6, Lemma 2.16, p. 278]: $\operatorname{spt} \phi_i = \overline{\{x \in X : \phi_i(x) \neq 0\}} \subset B(x_i, 2r_i)$, $0 \leq \phi_i \leq 1$, every ϕ_i is Lipschitz continuous with a Lipschitz constant $\dfrac{C}{r_i}$, where the constant C depends only on the doubling constant, and $\sum_{i=1}^{\infty} \phi_i(x) = 1$ for all $x \in \bigcup_{i=1}^{\infty} B_i$. Now we set

$$\tilde{u}(x) = \sum_{i=1}^{\infty} \phi_i(x) \tilde{u}(x_i). \tag{4.3}$$

It is easy to see that $\tilde{u}$ is defined and continuous at every point in $\bigcup_{i=1}^{\infty} B_i$. We set

$$Eu(x) = \begin{cases} u(x), & \text{if } x \in \Omega, \\ \tilde{u}(x), & \text{if } x \in \bigcup_{i=1}^{\infty} B_i, \\ 0, & \text{if } x \in X \setminus \left(\bigcup_{i=1}^{\infty} B_i \cup \bar{\Omega} \right). \end{cases}$$

Since $\mu(\partial \Omega) = 0$, Eu is defined almost everywhere in X.

Let $B(z, \operatorname{diam}(\Omega))$ be a ball containing Ω. Let φ be a Lipschitz continuous function with a constant $\frac{1}{\operatorname{diam}(\Omega)}$ such that $0 \leq \varphi \leq 1$, φ is 1 in $B(z, \operatorname{diam}(\Omega))$ and 0 outside the ball $B(z, 2 \operatorname{diam}(\Omega))$. Note that $B(z, 2 \operatorname{diam}(\Omega)) \subset \bigcup_{i=1}^{\infty} B_i \cup \bar{\Omega}$. We set $\operatorname{Ext}(u)(x) = \varphi(x) Eu(x)$. We write for every $B_i \in \mathcal{W}$

$$\fint_{U^*(B_i)} |u(z)| \, d\mu(z) = \max_{B_i \cap B_j \neq \emptyset} \fint_{U(x_j^*)} |u(z)| \, d\mu(z),$$

where $U(x_j^*) = B(x_j^*, r_j) \cap \Omega(\varepsilon r_j)$ and $U^*(B_i)$ is $U(x_j^*)$ for some x_j^* with $B_i \cap B_j \neq \emptyset$. The maximum exists, since for every $B_i \in \mathcal{W}$ there is only a finite number of balls $B_j \in \mathcal{W}$ with $B_i \cap B_j \neq \emptyset$. For every $x \in X \setminus \bar{\Omega}$ we write

$$\fint_{U^*(x)} g(z) \, d\mu(z) = \max_{x \in B_i, B_i \cap B_j \neq \emptyset} \fint_{U(x_j^*)} g(z) \, d\mu(z),$$

where $U^*(x)$ is $U(x_j^*)$ for some x_j^* with $x \in B_i$ and $B_i \cap B_j \neq \emptyset$.

Lemma 4.4. *There exists natural numbers N_1 and N_2 depending on the doubling constant and ε such that for every $z \in \Omega$*

$$\sum_{i=1}^{\infty} \chi_{U(x_i^*)}(z) \leq N_1 \tag{4.5}$$

and

$$\sum_{i=1}^{\infty} \chi_{U^*(B_i)}(z) \leq N_2. \tag{4.6}$$

PROOF. Fix $z \in \Omega$ and assume that z belongs to the sets $U(x_j^*)$, $j = 1, 2, \ldots$. By the $A^*(\varepsilon, \delta)$-condition and the construction of the Whitney decomposition there exists a constant $C > 0$ depending on ε such that for every $j = 1, 2, \ldots$

$$C^{-1}d(z, X \setminus \bar{\Omega}) \leq r_j \leq Cd(z, X \setminus \bar{\Omega}).$$

On the other hand $d(x_j, z) < 7r_j$. The balls $\frac{1}{4}B_j$ are disjoint. Hence by inequality (2.2) we obtain that the point z cannot belong to more than

$$N_1 = \frac{\mu(B(z, 8Cd(z, X \setminus \bar{\Omega})))}{\mu(B(y, \frac{1}{4}C^{-1}d(z, X \setminus \bar{\Omega})))} \leq q^{-1}(32C^2)^Q$$

sets $U(x_j^*)$, here $y \in B(z, 7Cd(z, X \setminus \bar{\Omega}))$. Inequality (4.5) is proved.

In the Whitney decomposition every ball B_i intersects only a uniformly bounded number of balls B_j, $j = 1, \ldots, N$. We obtain $N_2 = N \cdot N_1$. This completes the proof of Lemma 4.4. $\square$

Theorem 4.7. *Let (X, d, μ) be a metric space of homogeneous type with $\mu(X) < \infty$. Assume that $\Omega \subset X$ is a bounded domain, $\Omega \neq X$, which satisfies the $A^*(\varepsilon, \delta)$-condition and $\mu(\partial\Omega) = 0$. Then for every $1 < p \leq \infty$, the domain Ω is an $M^{1,p}$-extension domain; there exists a bounded linear extension operator*

$$\text{Ext} : M^{1,p}(\Omega, d, \mu) \to M^{1,p}(X, d, \mu)$$

such that $\text{Ext}(u)|_\Omega = u$ for every $u \in M^{1,p}(\Omega, d, \mu)$.

Remark 4.8. Note that we do not need the assumption $\mu(X) < \infty$ in the proof of Theorem 4.7. We need this assumption only in the construction of the Whitney type covering lemma. Our proof works also if we replace balls by cubes. Thus in the Euclidean case we can use the standard Whitney decomposition, see [10, Theorem 1, p.167], and assume that $X = \mathbb{R}^n$. The proof of Theorem 4.7 is based on the proof of Hajłasz and Martio [3, Theorem 9, p.240].

PROOF OF THEOREM 4.7. We prove Theorem 4.7 in three steps. First we prove that $\tilde{u}$, see the definition (4.3), is p-integrable. Then we prove that Eu has a generalized gradient and finally we prove that the generalized gradient is p-integrable.

Step 1. We show that $\tilde{u} \in L^p(\bigcup_{i=1}^{\infty} B_i, \mu)$ and $\|\tilde{u}\|_{L^p(\bigcup_{i=1}^{\infty} B_i, \mu)} \le C\|u\|_{L^p(\Omega,\mu)}$.
Assume first that $1 < p < \infty$. Let $B \in \mathcal{W}$. By Hölder's inequality we obtain
for every $x \in B$

$$
\begin{aligned}
|\tilde{u}(x)| &\le \fint_{U^*(B)} |u(z)|\, d\mu(z) \\
&\le \mu(U^*(B))^{-1}\Big(\int_{U^*(B))} d\mu\Big)^{\frac{p-1}{p}}\Big(\int_{U^*(B)} |u|^p\, d\mu\Big)^{\frac{1}{p}}.
\end{aligned}
\tag{4.9}
$$

By the $A^*(\varepsilon, \delta)$-condition and Lemma 4.2, $\mu(B) \approx \mu(U^*(B))$. Hence we obtain

$$
\begin{aligned}
\int_B |\tilde{u}|^p\, d\mu &\le \int_B \mu(U^*(B))^{-1} \int_{U^*(B)} |u(z)|^p\, d\mu(z)\, d\mu \\
&\le C \int_{U^*(B)} |u|^p\, d\mu.
\end{aligned}
$$

Here the constant C depends on the doubling constant and δ. The condition
(4.6) implies that

$$
\begin{aligned}
\int_{\bigcup_{i=1}^{\infty} B_i} |\tilde{u}|^p d\mu &\le \sum_{i=1}^{\infty} \int_{B_i} |\tilde{u}|^p\, d\mu \le \sum_{i=1}^{\infty} C \int_{U^*(B_i)} |u|^p\, d\mu \\
&\le CN_2 \int_\Omega |u|^p\, d\mu.
\end{aligned}
$$

If $p = \infty$, we obtain, as in the estimate (4.9), that

$$
\sup_{x \in B} |\tilde{u}(x)| \le \fint_{U^*(B)} |u(z)|\, d\mu(z) \le \operatorname*{ess\,sup}_{z \in \Omega} |u(z)|,
$$

for every $B \in \mathcal{W}$, and hence $\|\tilde{u}\|_{L^\infty(\bigcup_{i=1}^{\infty} B_i, \mu)} \le \|u\|_{L^\infty(\Omega,\mu)}$. This completes
the step 1.

Step 2. We show that the function Eu satisfies the inequality

$$
|Eu(x_1) - Eu(x_2)| \le Cd(x_1, x_2)(G(x_1) + G(x_2)) \tag{4.10}
$$

for almost every $x_1, x_2 \in \bigcup_{i=1}^{\infty} B_i \cup \bar{\Omega}$, where

$$
G(x) = \begin{cases} \fint_{U^*(x)} g\, d\mu, & \text{if } x \in \bigcup_{i=1}^{\infty} B_i \\ g(x), & \text{if } x \in \Omega \end{cases} \tag{4.11}
$$

and g is the generalized gradient of u.

Let x_1 and x_2 be the centers of the balls $B_1, B_2 \in \mathcal{W}$. Let $B_1', B_2' \in \mathcal{W}$ be such that

$$|\tilde{u}(x_1') - \tilde{u}(x_2')| = \max_{(j,i)} |\tilde{u}(x_j) - \tilde{u}(x_i)|.$$

Here the maximum is taken over those (j, i) for which $B_j, B_i \in \mathcal{W}$, $B_j \cap B_1 \neq \emptyset$ and $B_i \cap B_2 \neq \emptyset$. We obtain

$$|\tilde{u}(x_1) - \tilde{u}(x_2)| = \Big| \fint_{U((x_1')^*)} u \, d\mu - \fint_{U((x_2')^*)} u \, d\mu \Big|.$$

The definition of $M^{1,p}(\Omega, d, \mu)$ yields

$$|\tilde{u}(x_1) - \tilde{u}(x_2)| \leq d((x_1')^*, (x_2')^*)\Big(\fint_{U^*(x_1)} g \, d\mu + \fint_{U^*(x_2)} g \, d\mu \Big).$$

Since $d((x_1')^*, (x_2')^*) \leq Cd(x_1', x_2') \leq Cd(x_1, x_2)$, we obtain the inequality (4.10).

Assume $y_1 \in B_1 \in \mathcal{W}$, $y_2 \in B_2 \in \mathcal{W}$ and

$$\mathrm{dist}(y_1, y_2) \leq \frac{1}{2} \max\{\mathrm{diam}(B_1), \mathrm{diam}(B_2)\}.$$

This implies $r_1 \approx r_2$. Let $B_1', B_2' \in \mathcal{W}$ be such that

$$|\tilde{u}(x_1') - \tilde{u}(x_2')| = \max_{(j,i)} |\tilde{u}(x_j) - \tilde{u}(x_i)|.$$

Here the maximum is taken over those (j, i) for which $B_j, B_i \in \mathcal{W}$, $B_j \cap B_1 \neq \emptyset$ and $B_i \cap B_2 \neq \emptyset$. We obtain

$$
\begin{aligned}
|\tilde{u}(y_1) - \tilde{u}(y_2)| &\leq \Big| \sum_i \phi_i(y_1)\tilde{u}(x_i) - \sum_i \phi_i(y_2)\tilde{u}(x_i) \Big| \\
&= \Big| \sum_i \phi_i(y_1)\tilde{u}(x_i) - \sum_i \phi_i(y_1)\tilde{u}(x_1) \\
&\quad - \sum_i \phi_i(y_2)\tilde{u}(x_i) + \sum_i \phi_i(y_2)\tilde{u}(x_1) \Big| \\
&\leq \sum_i |\tilde{u}(x_i) - \tilde{u}(x_1)||\phi_i(y_1) - \phi_i(y_2)| \\
&\leq |\tilde{u}(x_1') - \tilde{u}(x_2')| \sum_i |\phi_i(y_1) - \phi_i(y_2)|.
\end{aligned}
$$

Each function ϕ_i is a Lipschitz continuous function with a constant $\dfrac{c}{r_i}$. By Lemma 4.1 (6) each $z \in X$ belongs at most N cubes $B_i \in \mathcal{W}$. We write $r = \min\{r_i : B_i \in \mathcal{W}, B_1 \cap B_i \neq \emptyset \text{ or } B_2 \cap B_i \neq \emptyset\}$. There exists a constant $C > 0$ such that $r \approx r_1 \geq C \operatorname{dist}(x_1', x_2')$. This yields

$$
\begin{aligned}
|\tilde{u}(y_1) - \tilde{u}(y_2)| &\leq 2N|\tilde{u}(x_1') - \tilde{u}(x_2')|\frac{C}{r}\operatorname{dist}(y_1, y_2) \\
&\leq Cd(y_1, y_2)\frac{\operatorname{dist}(x_1', x_2')}{r}\Big(\fint_{U((x_1')^*)} g\, d\mu + \fint_{U((x_2')^*)} g\, d\mu \Big) \\
&\leq Cd(y_1, y_2)\Big(\fint_{U^*(y_1)} g\, d\mu + \fint_{U^*(y_2)} g\, d\mu \Big).
\end{aligned}
$$

Assume $y_1 \in B_1 \in \mathcal{W}$, $y_2 \in B_2 \in \mathcal{W}$ and

$$
\operatorname{dist}(y_1, y_2) > \frac{1}{2}\max\{\operatorname{diam}(B_1), \operatorname{diam}(B_2)\}.
$$

We obtain

$$
|\tilde{u}(y_1) - \tilde{u}(y_2)| \leq |\tilde{u}(y_1) - \tilde{u}(x_1)| + |\tilde{u}(x_1) - \tilde{u}(x_2)| + |\tilde{u}(x_2) - \tilde{u}(y_2)|
$$

and this case follows by the previous case.

Assume that $y \in B_1 \in \mathcal{W}$, and $z \in \Omega$. Let $B' \in \mathcal{W}$ such that

$$
|\tilde{u}(x') - u(z)| = \max_{B_i \in \mathcal{W}, B_i \cap B_1 \neq \emptyset} |\tilde{u}(x_i) - u(z)|.
$$

We obtain

$$
\begin{aligned}
|\tilde{u}(y) - u(z)| &\leq |\tilde{u}(x') - u(z)| \\
&\leq \Big| \fint_{U((x')^*)} u\, d\mu - u(z) \Big| \\
&\leq \fint_{U((x')^*)} |u(w) - u(z)|\, d\mu(w) \\
&\leq \fint_{U((x')^*)} d(w, z)(g(w) + g(z))\, d\mu(w) \\
&\leq Cd(x', z)\Big(\fint_{U^*(y)} g(w)\, d\mu(w) + g(z) \Big).
\end{aligned}
$$

Since $d(x', z) \leq d(x', y) + d(y, z) \leq Cd(y, z)$ we obtain the inequality (4.10). This completes the step 2.

Step 3. We show that $\|G\|_{L^p(\bigcup_{i=1}^{\infty} B_i)} \leq C\|g\|_{L^p(\Omega)}$. Here g is the generalized gradient of the function u and G is the generalized gradient of the function Eu, see the inequality (4.10) and the definition (4.11).

Let $1 < p < \infty$. We use Hölder's inequality to obtain

$$\int_{\bigcup_{i=1}^{\infty} B_i} |G(z)|^p \, d\mu(z) \leq \sum_{i=1}^{\infty} \int_{B_i} |G(z)|^p \, d\mu(z)$$

$$= \sum_{i=1}^{\infty} \int_{B_i} \Big(\fint_{U^*(z)} g(x) \, d\mu(x) \Big)^p \, d\mu(z)$$

$$\leq \sum_{i=1}^{\infty} \int_{B_i} \Big(\mu(U^*(z))^{-1} \Big(\int_{U^*(z)} d\mu \Big)^{\frac{p-1}{p}} \Big(\int_{U^*(z)} g(x)^p \, d\mu(x) \Big)^{\frac{1}{p}} \Big)^p \, d\mu(z)$$

$$= \sum_{i=1}^{\infty} \int_{B_i} \mu(U^*(z))^{-1} \int_{U^*(z)} g(x)^p \, d\mu(x) \, d\mu(z).$$

We write

$$I(i) = \{j \in \mathbb{N} : B_j \in \mathcal{W} \text{ and there exists } B \in \mathcal{W} \text{ such that}$$
$$B_i \cap B \neq \emptyset \text{ and } B \cap B_j \neq \emptyset\}.$$

The fact that $\mu(B_i) \approx \mu(U(x_j^*))$ for every $j \in I(i)$ yields

$$\int_{\bigcup_{i=1}^{\infty} B_i} |G(z)|^p \, d\mu(z) \leq \sum_{i=1}^{\infty} \int_{B_i} \sum_{j \in I(i)} \mu(U(x_j^*))^{-1} \int_{U(x_j^*)} g(x)^p \, d\mu(x) \, d\mu(z)$$

$$\leq C \sum_{i=1}^{\infty} \sum_{j \in I(i)} \int_{U(x_j^*)} g(x)^p \, d\mu(x).$$

Every Whitney ball $B(x_i)$ intersects at most N Whitney balls. By the condition (4.5) every point $x \in \Omega$ belongs to at most N_1 sets $U(x_i^*)$, $i = 1, 2, \ldots$. This implies $\sum_{i=1}^{\infty} \sum_{j \in I(i)} \chi_{U(x_j^*)}(z) \leq N^2 \cdot N_1$ for every $z \in \Omega$. We obtain

$$\int_{\bigcup_{i=1}^{\infty} B_i} G^p \, d\mu \leq C \int_{\Omega} g^p \, d\mu.$$

If $p = \infty$, we obtain

$$\sup_{x \in \bigcup_{i=1}^{\infty} B_i} G(x) = \sup_{x \in \bigcup_{i=1}^{\infty} B_i} \fint_{U^*(x)} g \, d\mu \leq \operatorname*{ess\,sup}_{x \in \Omega} g.$$

This completes the step 3.

By the Steps 1, 2 and 3 we obtain $Eu \in M^{1,p}(\bigcup_{i=1}^{\infty} B_i \cup \bar{\Omega}, d, \mu)$ and $\|Eu\|_{M^{1,p}(\bigcup_{i=1}^{\infty} B_i \cup \bar{\Omega}, d, \mu)} \leq \|u\|_{M^{1,p}(\Omega, d, \mu)}$. We set $G = 0$ in $X \setminus (\bigcup_{i=1}^{\infty} B_i \cup \bar{\Omega})$. If $x, y \in B(z, 2\operatorname{diam}(\Omega))$ we obtain

$$
\begin{aligned}
|\operatorname{Ext}(u)(x) - \operatorname{Ext}(u)(y)| \\
= |Eu(x)\varphi(x) - Eu(y)\varphi(y)| \\
= |Eu(x)\varphi(x) - Eu(y)\varphi(y) + Eu(y)\varphi(x) - Eu(y)\varphi(x)| \\
\leq |Eu(x) - Eu(y)| + |Eu(y)||\varphi(x) - \varphi(y)| \\
\leq C \operatorname{diam}(\Omega)^{-1} d(x, y) \big(G(x) + |Eu(x)| + G(y) + |Eu(y)|\big).
\end{aligned}
$$

If $x \notin B(z, 2\operatorname{diam}(\Omega))$ and $y \in B(z, 2\operatorname{diam}(\Omega))$ we obtain

$$
\begin{aligned}
|\operatorname{Ext}(u)(x) - \operatorname{Ext}(u)(y)| = |Eu(y)\varphi(y)| \\
= |Eu(y)\varphi(x) - Eu(y)\varphi(y)| \\
\leq C d(x, y)|Eu(y)|.
\end{aligned}
$$

If $x, y \notin B(z, 2\operatorname{diam}(\Omega))$ we obtain

$$
|\operatorname{Ext}(u)(x) - \operatorname{Ext}(u)(y)| = 0
$$

By the Steps 1 and 3 we obtain

$$
\begin{aligned}
\|\operatorname{Ext}(u)\|_{M^{1,p}(X, d, \mu)} \leq \|Eu\|_{L^p(B(z, 2\operatorname{diam}(\Omega)), \mu)} \\
+ \|G + |Eu|\|_{L^p(B(z, 2\operatorname{diam}(\Omega)), \mu)} \\
\leq C\|u\|_{M^{1,p}(\Omega, d, \mu)}.
\end{aligned}
$$

This completes the proof of Theorem 4.7. $\square$

Remark 4.12. In the Euclidean case Theorem 4.7 follows by the theorem of Hajłasz and Martio [3, Theorem 9, p.240]. Assume that X is the n-dimensional Euclidean space $\mathbb{R}^n$, $n \geq 2$, d is the Euclidean metric $|\cdot|$ and μ is the Lebesgue n-measure m. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain which satisfies the $A^*(\varepsilon, \delta)$-condition. We write $\Omega^* = \bar{\Omega} \setminus \overline{(\mathbb{R}^n \setminus \bar{\Omega})}$. It is clear that $\overline{\Omega^*} \subset \bar{\Omega}$. On the other hand we obtain $\Omega = \bar{\Omega} \setminus \partial\Omega = \bar{\Omega} \setminus \overline{(\mathbb{R}^n \setminus \Omega)} \subset \Omega^*$ This implies $\Omega \subset \Omega^* \subset \overline{\Omega^*} \subset \bar{\Omega}$ and hence $\overline{\Omega^*} = \bar{\Omega}$. Since $m(\partial\Omega) = 0$ we obtain $m(\Omega^* \setminus \Omega) = 0$. This yields

$$
M^{1,p}(\Omega, |\cdot|, m) = M^{1,p}(\Omega^*, |\cdot|, m).
$$

Now Theorem 4.7 follows by [3, Theorem 9, p.240] as soon as we prove that Ω^* satisfies the $A(c)$-condition for some $0 < c < 1$. Since $\operatorname{dist}(x, \mathbb{R}^n \setminus \overline{\Omega^*}) =$

$\operatorname{dist}(x, \mathbb{R}^n \setminus \bar{\Omega})$ for every $x \in \Omega^*$ we obtain $\Omega(t) \subset \Omega^*(t)$ for all $t > 0$. Let $x \in \Omega^*$. For every $\sigma > 0$ we find $y_\sigma \in \Omega$ such that $|x - y_\sigma| \le \sigma$ and hence $m(B^n(x, r + \sigma) \cap \Omega^*(\varepsilon r)) \ge m(B^n(y_\sigma, r) \cap \Omega(\varepsilon r)) \ge \delta m(B^n(y_\sigma, r))$. Thus the domain Ω^* satisfies the $A^*(\varepsilon, \delta)$-condition. Let $y \in \partial\Omega^*$ and $r > 0$. We choose $x \in \Omega^*$ such that $\operatorname{dist}(y, x) \le \frac{1}{3}r$. Since Ω^* satisfies the $A^*(\varepsilon, \delta)$-condition there exists a point $z \in \Omega^* \cap B^n(x, \frac{1}{3}r)$ such that $\varepsilon \frac{1}{3}r < \operatorname{dist}(z, \mathbb{R}^n \setminus \overline{\Omega^*})$. Since $\operatorname{dist}(z, \mathbb{R}^n \setminus \overline{\Omega^*}) = \operatorname{dist}(z, \partial\Omega^*)$ we obtain $B^n(z, \frac{\varepsilon}{3}r) \subset \Omega^* \cap B^n(y, r)$ and hence Ω^* satisfies the $A(\frac{\varepsilon}{3})$-condition.

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