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SMOOTH IMAGES OF THE IRRATIONALS

Abstract

In this note we study descriptive set theoretic as well as measure theoretic properties of smooth images of the irrationals.

1 Introduction.

Let $I = [0, 1]$. In [3], we characterized the set of points where the level sets of smooth functions are large. Namely, the following three theorems were proved.

Theorem 1.1. *The following are equivalent.*

- *$M \subseteq I$ is closed, $\mathcal{H}^{\frac{1}{n}}(M) = 0$, and $\beta_n(M) < \infty$.*
- *There is $f \in C^n(I, I)$ such that $M = f(Z_f)$ where $Z_f = \{x : f'(x) = \cdots = f^{(n)}(x) = 0\}$.*

Theorem 1.2. *The following are equivalent.*

- *$M \subseteq I$ is the union of a G_δ set and a countable set, $\mathcal{H}^{\frac{1}{n}}(\overline{M}) = 0$, and $\beta_n(\overline{M}) < \infty$.*
- *There is $f \in C^n(I, I)$ such that $M = \{y : f^{-1}(\{y\})$ is perfect$\}$.*

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Theorem 1.3. *The following are equivalent.*

- *$M \subseteq I$ is an analytic set, $\mathcal{H}^{\frac{1}{n}}(\overline{M}) = 0$, and $\beta_n(\overline{M}) < \infty$.*
- *There is $f \in C^n(I, I)$ such that $M = \{y : f^{-1}(\{y\})$ is uncountable$\}$.*

In the above statements, $\beta_n(\overline{M}) = \sum_{i=1}^{\infty} |J_i|^{\frac{1}{n}}$, where J_i are components of $I \setminus \overline{M}$.

Theorem 1.3 can be regarded in some sense as a parametrization of the Hausdorff dimension of certain analytic sets by smooth functions. Let us consider this idea in detail. Suppose we have a compact set K of Hausdorff dimension s with $1/s$ an integer and $\mathcal{H}^s(K) = 0$. Furthermore, assume that K satisfies the β_n condition. Then, using Theorem 1.3, we can associate in a natural way a C^n function, $n = \frac{1}{s}$, to each analytic subset A of K. Of course, the dimension of A may be smaller than s. To each analytic set A we would like to associate a smooth function where the degree of the smoothness of the function depends on the Hausdorff dimension of A.

Problem 1.4. Parameterize the Hausdorff dimension of analytic sets by smooth functions.

Let us discuss the meaning of word "parameterize" here. Ideally, what we would like is a constructive procedure or an "algorithm" which assigns a function f of n^{th} degree of smoothness to each analytic set A where $1/n$ is the Hausdorff dimension of A. The notion of "smoothness" has to be somewhat flexible here. It is interpreted as some function which has some form of n^{th} derivative. The proof of Theorem 1.3 can be viewed as parametrization of those analytic sets A such that $\dim_H(A) = \dim_H \overline{A} = s$, $\mathcal{H}^s(\overline{A}) = 0$ and $\overline{A}$ satisfies the condition β_n, where $n = 1/s$. More specifically, given such an analytic set A, the proof of Theorem 1.3 produces a C^n function f such that $A = \{y : f^{-1}(y)$ is uncountable$\}$.

Two major obstacles arise when attacking this problem. The first one is how to circumvent having to consider the Hausdorff dimension of the closure of the analytic set A. The second obstacle is how to get rid of the condition β_n. These problems are interrelated. The condition β_n arises from Taylor's theorem on the intervals contiguous to the set of points where the first n derivatives of the functions are zero. Our first attempt was to consider smooth images of $\mathbb{P}$, the set of irrationals on I. This is discussed in Section 2. Several surprising facts emerge. First, the descriptive complexity of smooth images of $\mathbb{P}$ can not be very high. The set must be of type $A \setminus B$ where A is F_σ and B is countable. Second, the ordinary notion of differentiation on $\mathbb{P}$ is not enough. Indeed, we show that given a set N of measure zero which is of type

$A \setminus B$ where A is F_σ and B is countable, there is a function $f : \mathbb{P} \to \mathbb{R}$ such that $f(\mathbb{P}) = M$ and $f'(x) = 0$ for all $x \in \mathbb{P}$. Hence, the higher order of usual differentiation on $\mathbb{P}$ does not imply smaller Hausdorff dimension. However, given any analytic set $M \subseteq I$ of Lebesgue measure zero there are a set $N \subseteq \mathbb{I}$ with $N \cong \mathbb{P}$ and a function f such that $f(N) = M$ and $f'(x) = 0$ for all $x \in N$. We also show that such a set N can not have the property that the Lebesgue measure of $\overline{N} \setminus N$ is zero.

The results of Section 2 lead us to consider alternate notion of smoothness in order to tackle the "parametrization" problem. The appropriate notion of smoothness here is to consider Peano derivatives. In Section 3, we state some results concerning smooth images of the irrationals where smoothness is in terms of Peano derivatives. These results bring us closer to a solution of the "parametrization" problem, but the general problem is still open.

2 Pointwise Lipschitz Images of the Irrationals.

We establish some notation and terminology. We use $\mathbb{P}$ and $\mathbb{Q}$ to denote the set of irrationals and the set of rationals in the interval $(0, 1)$, respectively. If $A, B \subseteq \mathbb{R}$, then $A \cong B$ means that A is homeomorphic to B. If $M \subseteq \mathbb{R}$, then we let $C^1_0(M)$ denote the set of functions defined on M which have derivative zero everywhere, and we let $\mathrm{Lip}_p(M)$ denote the set of pointwise Lipschitz functions defined on M; i.e., for each $x \in M$, there exits $L_x \in \mathbb{R}$ and $\eta_x > 0$ such that for all $0 \leq |x - y| < \eta_x$ and $y \in M$, we have that $|f(x) - f(y)| \leq L_x|x - y|$. Let us recall some standard facts.

Lemma 2.1. *Let $f : X \to \mathbb{R}$, where X is any subset of a metric space, be a Lipschitz function with Lipschitz constant L. Then there exists a Lipschitz function $\tilde{f} : \overline{X} \to \mathbb{R}$ with the same Lipschitz constant L and $\tilde{f}(x) = f(x)$ for all $x \in X$.*

Lemma 2.2. *([1]) Let A and B be two countable dense subsets of the open interval J. Then there exists a C^1 homeomorphism h from J onto J such that $h(A) = B$ and $h'(x) > 0$, for every $x \in J$.*

Lemma 2.3. *Let $f : \mathbb{P} \to \mathbb{R}$ be pointwise Lipschitz. Then $f(\mathbb{P})$ is of the type $F \setminus D$ where F is an F_σ set and D is countable.*

PROOF. For each $n \in \mathbb{N}$, let

$$\mathbb{P}_n = \{x \in \mathbb{P} : \forall y \in \mathbb{P} \ \&\ |x - y| < \frac{1}{n},\ |f(x) - f(y)| \leq n \cdot |x - y|\}.$$

It is clear that $\mathbb{P}_n$ is closed relative to $\mathbb{P}$ and $\mathbb{P} = \bigcup_{n=1}^{\infty} \mathbb{P}_n$. Let $0 \leq i \leq n - 1$. Then, on $\mathbb{P}_{n,i} = P_n \cap [\frac{i}{n}, \frac{i+1}{n}]$, f is Lipschitz with the Lipschitz constant n.

Let $f_{n,i}$ be the restriction of f to $\mathbb{P}_{n,i}$ and let $\tilde{f}_{n,i}$ be the Lipschitz extension of $f_{n,i}$ to $\overline{\mathbb{P}_{n,i}}$, the closure of $\mathbb{P}_{n,i}$ relative to $[0,1]$. Clearly, $\tilde{f}_{n,i}(\overline{\mathbb{P}_{n,i}})$ is compact. Moreover,

$$f(\mathbb{P}) = \bigcup_{n=1}^{\infty} \bigcup_{i=1}^{n-1} f(\mathbb{P}_{n,i}) = \bigcup_{n=1}^{\infty} \bigcup_{i=1}^{n-1} f_{n,i}(\mathbb{P}_{n,i}) = \bigcup_{n=1}^{\infty} \bigcup_{i=1}^{n-1} \tilde{f}_{n,i}(\overline{\mathbb{P}_{n,i}} \setminus \mathbb{Q}).$$

Hence the assertion follows. □

From the proposition above it follows that we cannot obtain all the analytic sets as C_0^1 images of the irrationals, but the converse of Lemma 2.3 holds. Let us start with a simple lemma.

Lemma 2.4. *If $\{M_k\}_{k\in\mathbb{N}}$ is a sequence of sets, each one of which is the image of $\mathbb{P}$ under a pointwise Lipschitz map, then so is $\cup_{k=1}^{\infty} M_k$.*

Proof. Let $I_0 = (0, 1/2)$, and for $n > 0$, let $I_n = (1 - 1/2^n, 1 - 1/2^{n+1})$. Clearly, M_n is the pointwise Lipschitz image of $I_n \cap \mathbb{P}$ under some map f_n. Let f be the union of f_n's. Then, f is the desired function. □

Lemma 2.5. *Let J be an open interval and D be a countable subset of $\mathbb{R}$. Then, there is a pointwise Lipschitz map $f : \mathbb{P} \to J$ such that $f(\mathbb{P}) = J \setminus D$.*

Proof. Let f be a linear increasing function from $(0, 1)$ onto J. Let B be a countable set such that $D \cap J \subseteq B$ and B is dense in J. Let $A = f(\mathbb{Q})$. By Lemma 2.2, there is a homeomorphism $h : J \to J$ such that $h(A) = B$ and $h'(x) > 0$ for all $x \in J$. Then, $h \circ f$ shows that $J \setminus B$ is the pointwise Lipschitz image of $\mathbb{P}$. Since singletons are the pointwise Lipschitz images of $\mathbb{P}$, we have by Lemma 2.4 that $J \setminus D = J \setminus B \cup (B \setminus D)$ is the pointwise Lipschitz image of $\mathbb{P}$. □

Lemma 2.6. *Suppose that $M \subseteq (0, 1)$ is a nowhere dense perfect set and $D \subseteq (0, 1)$ is countable. Then, there exists a pointwise Lipschitz map $f : \mathbb{P} \to \mathbb{R}$ such that $f(\mathbb{P}) = M \setminus D$.*

Proof. Let $B \subset (0, 1)$ be a countable set such that $D \subseteq B$, B contains the endpoints of components of $(0, 1) \setminus M$ which are in $(0, 1)$, and B is dense in $(0, 1)$. We will show that $M \setminus B$ is the pointwise Lipschitz image of $\mathbb{P}$. Since $M \setminus D = M \setminus B \cup (B \setminus D)$, as in the proof of Lemma 2.5, it will follow that $M \setminus D$ is the pointwise Lipschitz image of $\mathbb{P}$. To this end, we may obtain by Lemma 2.2 a C^1 homeomorphism $h : (0, 1) \to (0, 1)$ such that $h(\mathbb{Q}) = B$ and $h'(x) > 0$ for all $x \in (0, 1)$. Let $N = h^{-1}(M)$. Let (u_i, v_i) be components of $(0, 1) \setminus N$. We note that $u_i, v_i \in \mathbb{Q} \cup \{0, 1\}$ and that $h(u_i), h(v_i)$ are endpoints of M provided that $u_i, v_i \notin \{0,1\}$. For each i, let $\{x^i_k\}_{k\in\mathbb{Z}}$ be such that for all i, we have that

- $x^i_0 = \frac{u_i+v_i}{2}$,
- $x^i_k \in \mathbb{Q}$,
- $x^i_j < x^i_k$ whenever $j < k$, and
- $\lim_{k\to\infty} x^i_k = v_i$ and $\lim_{k\to-\infty} x^i_k = u_i$.

Fix (u_i, v_i). Suppose $u_i \neq 0$. For each $k \leq -1$, let $p^i_k \in N \setminus \mathbb{Q}$ be such that $|p^i_k - u_i| < |u_i - x^i_{k-1}|$. This is possible since N is perfect. If $u_i = 0$, then $v_i \neq 1$ and we let $p^i_k = p^i_{|k|}$. Similarly, if $v_i \neq 1$, then, for each $k \geq 0$, let $p^i_k \in N \setminus \mathbb{Q}$ be such that $|p^i_k - v_i| < |v_i - x^i_{k+1}|$. If $v_i = 1$, then $u_i \neq 0$ and we let $p^i_0 = p^i_{-1}$ and $p^i_k = p^i_{-k}$ for all $k \geq 1$. We do this for each (u_i, v_i).

Define f on $\mathbb{P}$ as follows. If $x \in \mathbb{P} \setminus \bigcup_i (u_i, v_i)$, let $f(x) = h(x)$. If $x \in \mathbb{P} \bigcap (x^i_k, x^i_{k+1})$, let $f(x) = h(p^i_k)$. Our first observation is that $f(\mathbb{P}) = M \setminus B$. Let us now show that f is pointwise Lipschitz.

Case 1. $x \in \mathbb{P} \setminus N$. In this case f is clearly pointwise Lipschitz at x as it is constant in a neighborhood of x.

Case 2. $x \in \mathbb{P} \cap N$. We will show that f is pointwise Lipschitz at x from the left. The argument from the right will be analogous. Let L be a Lipschitz constant for h at x and $\epsilon > 0$ be such that if $|h(x) - h(y)| < L|x - y|$ for all $|y - x| < \epsilon$. Let $y \in \mathbb{P}$, $y < x$ and y be sufficiently close to x. If $y \in \mathbb{P} \cap N$, then $|f(x) - f(y)| = |h(x) - h(y)| < L|x - y|$. We next estimate $|f(x) - f(y)|$ when $y \notin \mathbb{P} \cap N$. Let $y \in (x^i_k, x^i_{k+1})$. We may assume that $|u_i - x| < \epsilon/2$. We first consider the case $k \geq 0$.

$$\begin{aligned}|f(x) - f(y)| &= |h(x) - h(p^i_k)| \leq L|x - p^i_k| \leq L(|x - v_i| + |v_i - p^i_k|)\\ &\leq L(|x - v_i| + |v_i - x^i_{k+1}|) \leq L(|x - y| + |x - y|) = 2 \cdot L|x - y|\end{aligned}$$

If $k \leq -1$, then

$$\begin{aligned}|f(x) - f(y)| &= |h(x) - h(p^i_k)| \leq L|x - p^i_k| \leq L(|x - y| + |y - p^i_k|)\\ &\leq L\left(|x - y| + 2 \cdot \left|\frac{u_i - v_i}{2}\right|\right) \leq 3 \cdot L|x - y|.\end{aligned}$$

□

Theorem 2.7. *Let $M \subseteq [0,1]$. The following are equivalent:*

1. *M is of the type $F \setminus D$ where F is an F_σ set and D is countable.*

2. *There exists a pointwise Lipschitz function $f : \mathbb{P} \to M$ such that $f(P) = M$.*

PROOF. (1) $\Rightarrow$ (2). Every F_σ subset of I can be written as countable union of sets each one of which is either an open interval, a nowhere dense perfect set or a singleton set. Now, applying Lemmas 2.5, 2.6, 2.4, we have that (1) $\implies$ (2). That (2) $\Rightarrow$ (1) follows from Lemma 2.3. □

Theorem 2.8. *Let $M \subseteq [0,1]$. The following are equivalent:*

1. *M is of the type $F \setminus D$ where F is an F_σ set and D is countable, and $\lambda(F) = 0$.*

2. *There exists a function $f : \mathbb{P} \to M$ in $C_0^1(P)$ such that $f(P) = M$.*

PROOF. (1) $\Rightarrow$ (2). As $\lambda(F \setminus D) = 0$, by ([2]: Theorem 7.6), there exists a one-to-one continuous function $f_1 : [0,1] \to [0,1]$ such that ${f_1}'(x) = 0$ for every $x \in f_1^{-1}(F \setminus D)$. The set $\tilde{M} = {f_1}^{-1}(F \setminus D)$ is still of type $\tilde{F} \setminus \tilde{D}$ where $\tilde{F}$ is an F_σ set and $\tilde{D}$ is countable. By Theorem 2.7, there exists a pointwise Lipschitz function $f_2 : \mathbb{P} \to \tilde{M}$ such that $f_2(\mathbb{P}) = \tilde{M}$. Hence, $f = f_1 \circ f_2$ is the function in $C_0^1(P)$ we are looking for.
(2) $\Rightarrow$ (1). Let $M = f(\mathbb{P})$ where $f : \mathbb{P} \to M$ in $C_0^1(\mathbb{P})$. Then, by Lemma 2.3, M is of the type $F \setminus D$ where F is an F_σ set and D is countable. To see that $\lambda(M) = 0$, for each $\epsilon > 0$, we find a countable decomposition $\{P_{n,i}\}$ of $\mathbb{P}$ such that f is Lipschitz on each $P_{n,i}$ with Lipschitz constant ϵ. This fact together with standard theorems implies that $\lambda(M) = 0$. For each $n \in \mathbb{N}$, let

$$P_n = \{x \in \mathbb{P} : \forall y \in \mathbb{P} \ \& \ |x - y| < \frac{1}{n},\ |f(x) - f(y)| \leq \epsilon \cdot |x - y|\}.$$

It is clear that P_n is closed relative to $\mathbb{P}$ and $\mathbb{P} = \bigcup_{n=1}^{\infty} P_n$. For each $n \in \mathbb{N}$ and $1 \leq i \leq n$, let $P_{n,i} = P_n \cap (\frac{i-1}{n}, \frac{i}{n})$. Then, $P_{n,i}$ is the desired countable collection on which f is Lipschitz with the Lipschitz constant ϵ. □

Theorem 2.9. *Let $N \subseteq [0,1]$ such that $\lambda(\overline{N} \setminus N) = 0$. If $f : N \to \mathbb{R}$ is a pointwise Lipschitz map, then $f(N)$ is of the type $F \setminus Z$ where F is an F_σ set and Z has Lebesgue measure zero.*

PROOF. For each $n \in \mathbb{N}$, let

$$N_n = \{x \in N : \forall y \in N \ \& \ |x - y| < \frac{1}{n},\ |f(x) - f(y)| \leq n \cdot |x - y|\}.$$

It is clear that N_n is closed relative to N and $N = \bigcup_{n=1}^{\infty} N_n$. Let $0 \le i \le n-1$. Then, on $N_{n,i} = N_n \cap [\frac{i}{n}, \frac{i+1}{n}]$, f is Lipschitz with the Lipschitz constant n. Let $f_{n,i}$ be the restriction of f to $N_{n,i}$ and let $\tilde{f}_{n,i}$ be the Lipschitz extension of $f_{n,i}$ to $\overline{N_{n,i}}$, the closure of $N_{n,i}$ relative to $[0,1]$. Clearly, $\tilde{f}_{n,}(\overline{N_{n,i}})$ is compact. Moreover

$$f(N) = \bigcup_{n=1}^{\infty} \bigcup_{i=1}^{n-1} f(N_{n,i}) = \bigcup_{n=1}^{\infty} \bigcup_{i=1}^{n-1} f_{n,i}(N_{n,i}) = \bigcup_{n=1}^{\infty} \bigcup_{i=1}^{n-1} \tilde{f}_{n,i}(\overline{N_{n,i}} \setminus (\overline{N} \setminus N)).$$

Hence the assertion follows. □

Theorem 2.10. *Let $A \subseteq [0,1]$ be an analytic set with $\lambda(A) = 0$. Then there exists a set N homeomorphic to $\mathbb{P}$ and a pointwise Lipschitz map $f : [0,1] \to [0,1]$ such that $f(N) = A$ and $f'(x) = 0$ for every $x \in N$.*

PROOF. By Theorem 7.6 in [2], there exists an absolutely contiuous, strictly increasing function $h : [0,1] \to [0,1]$ such that $h'(x) = 0$ for every x in $h^{-1}(A)$ and $\lambda(h^{-1}(A)) = 0$. Let $B = h^{-1}(A)$. By Proposition 2.10 in [4], there is a G_δ set N and Lipschitz map $g : [0,1] \to [0,1]$ such that $g(N) = B$. Moreover, this set can be chosen to be homeomorphic to $\mathbb{P}$. Then, $f = h \circ g$ is the desired function. □

Theorem 2.11. *Let $A \subseteq [0,1]$ be an analytic set. Then, there is a set N homeomorphic to $\mathbb{P}$ and a pointwise Lipschitz mapping $f : N \to A$ such that $f(N) = A$.*

PROOF. Let F be an F_σ subset of A such that $\lambda(A \setminus F) = 0$. Now applying Theorem 2.10 to $A \setminus F$ and Theorem 2.7 to F, we get our desired result. □

Remark 2.12. *Theorem 2.9 implies that in order for an analytic set to be a pointwise Lipschitz image of a G_δ set, the G_δ set has to be special. For instance, consider a G_δ set $M \subset [0,1]$ such that $\lambda(M) = 0$ and $\overline{M} = [0,1]$. Then, M can not be pointwise Lipschitz image of a set N such that $\overline{N} \setminus N$ has Lebesgue measure zero.*

The remark above leads us to the following problem.

Problem 2.13. Given two homeomorphic sets $A, B \subseteq \mathbb{R}$, when is there a Lipschitz map from A onto B?

A solution to this problem even for sets homeomorphic to the Cantor set would be of interest.

3 Smooth Images of the Irrationals.

In this section, we study smooth images of the irrationals, where smoothness refers to the order of Peano derivatives. It seems that the Peano derivative is the right derivative for our context. See [6] for a recent survey on Peano derivatives. Let us recall the definition of the Peano derivative first.

Let A be a subset of $\mathbb{R}$ which has no isolated point and $f : A \to \mathbb{R}$ be a function. We say that f has k-th Peano derivative at $x \in A$ if there are numbers $f_1(x)$, $f_2(x)$, …, $f_k(x)$ such that

$$f(x+h) = f(x) + hf_1(x) + \frac{h^2}{2} f_2(x) + \ldots + \frac{h^k}{k!} f_k(x) + o(h^k)$$

as $h \to 0$. The number $f_k(x)$ is called *the k-th Peano derivative of f at x*. We are primarily interested in functions which have first n Peano derivatives zero at every point of its domain. We denote this class by D_p^n.

Fix $n \in \mathbb{N}$ and $\epsilon \geq 0$. We let $\mathcal{A}_n(\epsilon)$ be the set of all analytic subsets $M \subseteq \mathbb{R}$ which satisfies the following properties:

- $\mathcal{H}^{1/n}(M) = 0$.
- There is a sequence of compact sets $K_1, K_2, \ldots$ with $M \subseteq \cup_{i=1}^{\infty} K_i$ such that for all i we have that
  - $K_i \subseteq K_{i+1}$,
  - $\mathcal{H}^{1/n}(K_i) \leq \epsilon$, and
  - K_i satisfies the condition β_n; i.e., $\sum_{k=1}^{\infty} |J_k|^{\frac{1}{n}} < \infty$ where J_k's are the components of $[\min K_i, \max K_i] \setminus K_i$.

For notational convenience, we let $\mathcal{A}_n = \mathcal{A}_n(0)$ and $\mathcal{A}_n^+ = \bigcap_{\epsilon > 0} \mathcal{A}_n(\epsilon)$. Note that $M \in \mathcal{A}_n$ if $\mathcal{H}^{1/n}(\overline{M}) = 0$ and $\overline{M}$ satisfies condition β_n.

Lemma 3.1. *Let $f \in D_p^n$ whose range is set B. Then, $\mathcal{H}^{\frac{1}{n}}(B) = 0$.*

PROOF. The proof of this theorem follows by a standard covering argument. For example, see Theorem 3.4.3 in [5]. □

Lemma 3.2. *Let $L \subseteq \mathbb{R}$ be compact and $f : L \to \mathbb{R}$ be such that*

$$|f(x) - f(y)| \leq \epsilon |x-y|^n$$

for all $x, y \in L$. Then, $\mathcal{H}^{1/n}(f(L)) < \epsilon \mathcal{H}^1(L)$ and $f(L)$ satisfies condition β_n.

Proof. Again, proof of $\mathcal{H}^{1/n}(f(L)) < \epsilon\mathcal{H}^1(L)$ follows from a standard covering argument. That $f(L)$ satisfies condition β_n follows by an argument similar to the proof of Lemma 2.7 in [3]. $\square$

Theorem 3.3. *Let $B = f(A)$ for some $f \in D_p^n$ with $dom(f) = A$, $A \subseteq [0,1]$. Then, $B \in \mathcal{A}_n^+$.*

Proof. Fix $\epsilon > 0$. We need to show that $B \in \mathcal{A}_n(\epsilon)$. By Lemma 3.1 we have that $\mathcal{H}^{\frac{1}{n}}(B) = 0$. For each $j \in \mathbb{N}$, let

$$L_j = \{x \in A : |f(x) - f(y)| \leq \epsilon|x-y|^n \ \forall y \in A,\ |x-y| < \frac{1}{j}\}.$$

Note that $\overline{L}_j$ is the finite union of disjoint compact sets on each piece of which the hypothesis of Lemma 3.2 are satisfied. Hence, by Lemma 3.2 it follows that $K_j = f(\overline{L}_j)$ is a compact set with $\mathcal{H}^{\frac{1}{n}}(K_j) \leq \epsilon$ and K_j satisfies condition β_n. Now, $\{K_j\}$ is the required sequence to complete the proof of this theorem. $\square$

Theorem 3.4. *Let $B \in \mathcal{A}_n$. Then there is $A \subseteq [0,1]$ homeomorphic to $\mathbb{P}$ and $f : A \to \mathbb{R}$ in D_p^n such that $f(A) = B$.*

Proof. Let $\{K_i\}$ be a sequence of compact sets which witnesses the fact that $B \in \mathcal{A}_n$. By Theorem 2.28 in [3], we may choose a compact set L_i with Lebesgue measure zero and $g_i \in D_p^n$ such that $g_i(L_i) = K_i$. Moreover, we can make $L_i \cap L_j = \emptyset$ if $i \neq j$. Let $L = \bigcup_i L_i$ and $g = \bigcup_i g_i$. Then, $g \in D_p^n$ and $B \subseteq g(L)$. Let $B' = g^{-1}(B)$. Then B' is an analytic set with Lebesgue measure zero. By Theorem 2.10, there is a set $A \subseteq [0,1]$ homeomorphic to $\mathbb{P}$ and a pointwise Lipschitz map $h : A \to \mathbb{R}$ such that $h(A) = B'$. Letting $f = g \circ h$, we get the desired result. $\square$

We next address what can be said about $f(\mathbb{P})$ when $f \in D_p^n$. Of course, it follows from Lemma 2.3 that $f(\mathbb{P})$ has the form $F \setminus D$, where F is F_σ and D is countable. It also follows from Theorem 3.3 that $f(\mathbb{P}) \in \mathcal{A}_n^+$. By an argument similar to the proof of Theorem 3.4, it can be shown that if B is a set of the from $F \setminus D$, where F is an F_σ set and D is countable and $B \in \mathcal{A}_n$, then there is a function $f \in D_p^n$ whose domain is actually $\mathbb{P}$ such that $f(\mathbb{P}) = B$. Indeed, all one has to do is apply Theorem 2.8 instead of Theorem 2.10.

We end this note with some problems.

Problem 3.5. Does $\mathcal{A}_n$ equal A_n^+?

Problem 3.6. Theorem 3.3 gives a necessary condition for a set to be the image of a function in D_p^n and Theorem 3.4 gives a sufficient condition for a set to be the image of a function in D_p^n. Is there a natural condition which is both necessary and sufficient?

Problem 3.7. More specifically, characterize $f(\mathbb{P})$ when $f \in D_p^n$.

Of course, if the answer to Problem 3.5 is yes, then the answer to Problem 3.6 is yes.

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