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# APPROXIMATIONS BY LIPSCHITZ FUNCTIONS GENERATED BY EXTENSIONS

## Abstract

We show that, for each pair of metric spaces that has the Lipschitz extension property, every bounded uniformly continuous function can be approximated by Lipschitz functions. The same statement is valid for functions between a locally convex space and  $\mathbb{R}^n$ . In addition, we show that for a locally bounded, convex function  $F : X \rightarrow \mathbb{R}^n$ , where  $X$  is a separable Fréchet space, the set of points on which the differential of this function exists is dense in  $X$ .

## 1 Introduction

The traditional notion of Lipschitz function is the following one:

**Definition 1.** Let  $X$  and  $Y$  be metric spaces. A function  $f : X \rightarrow Y$  is called a Lipschitz function if there exists a constant  $M \geq 0$  such that

$$d(f(x), f(y)) \leq M \cdot d(x, y)$$

for all  $x, y \in X$ . The smallest number  $M \geq 0$  satisfying the above relation is called the Lipschitz constant of  $f$  and is denoted by  $\text{Lip}(f)$ .

Let us recall the classic result of Rademacher and some extensions to the infinite dimensional case.

**Proposition 1.** (see [8]) *If  $U$  is an open set in  $\mathbb{R}^n$  and  $f : U \rightarrow \mathbb{R}^n$  is a Lipschitz function, then  $f$  is differentiable outside of a Lebesgue null subset of  $U$ .*

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**Proposition 2.** (see [1]) *Let  $X$  be a separable Banach space,  $U$  an open subset of  $X$ , and  $Y$  a Banach space with the Radon-Nikodym property. If  $f : U \rightarrow Y$  is a locally Lipschitz function, then  $f$  is Gâteaux differentiable outside of a Gaussian null subset of  $U$ .*

**Proposition 3.** (see [7]) *Let  $X$  be a Banach space admitting an equivalent norm which is differentiable (Fréchet, Gâteaux, or in some intermediate sense) away from origin. Then, every locally Lipschitz function defined on an open subset  $U$  of  $X$  is differentiable (in the same sense) at every point of some dense subset of  $U$ .*

Therefore, the condition of being Lipschitz may be considered as a weakened version of differentiability. Thus many results about differentiable functions were extended to the Lipschitz functions.

P. Mankiewicz generalized the definition of a Lipschitz function for locally convex spaces. Here we have a particular case of this definition:

**Definition 2.** Let  $X$  be a locally convex space and  $(p_\alpha)_{\alpha \in A}$  a directed family of seminorms generating the topology of  $X$ . A function  $f : S \rightarrow \mathbb{R}^n$ , where  $S$  is a subset of  $X$ , is called Lipschitz if there exist  $\alpha \in A$  and  $L \geq 0$ , such that

$$\|f(x) - f(y)\| \leq L \cdot p_\alpha(x - y)$$

for all  $x, y \in S$ .

Very little seems to be known about this class of functions. The only work known to us which treats it was done by P. Mankiewicz, where some extensions of the classic theorem of Rademacher are obtained and applications of this result to the problem of the topological classification of Fréchet spaces are given.

Let us recall some facts from this paper.

**Definition 3.** Let  $X$  and  $Y$  be locally convex spaces,  $A \subseteq X$ ,  $x \in A$  and  $a \in X$ , such that  $x + \lambda \cdot a \in A$ , for sufficiently small  $\lambda \in \mathbb{R}$ . Then, the function  $F : A \rightarrow Y$  is said to possess a derivative at the point  $x$  in the direction  $a$ , if

$$\lim_{\lambda \rightarrow 0} \frac{F(x + \lambda \cdot a) - F(x)}{\lambda}$$

exists. If it exists, we denote this limit by  $F'_a(x)$ .

The function  $F : U = \overset{\circ}{U} \subseteq X \rightarrow Y$  is called differentiable at the point  $x \in U$ , if:

- a) there exists  $F'_a(x)$  for all  $a \in X$
  - b) the mapping  $(DF)_x : X \rightarrow Y$ , given by  $(DF)_x(a) = F'_a(x)$ , is linear.
- The mapping  $(DF)_x$  is said to be the differential of  $F$  at the point  $x$ .

Since  $\mathbb{R}^n$ , endowed with the usual norm, is a Gelfand-Fréchet space, Theorem 4.5' from [5] gives us the following result:

**Proposition 4.** *Let  $X$  be a separable Fréchet space and  $\emptyset \neq U = \overset{\circ}{U} \subseteq X$ . If  $F : U \rightarrow \mathbb{R}^n$  is Lipschitz, then the set  $\{x \in U \mid \text{there exists } (DF)_x\}$  is dense in  $U$ .*

From the Weierstrass-Stone theorem, we know that every continuous function  $f : X \rightarrow \mathbb{R}$ , where  $X$  is a compact metric space can be approximated, in the uniform norm by Lipschitz functions.

It is natural to try to find generalizations of this result.

The first step in this direction was made by G. Georganopoulos (see [3]) who proved that a continuous function  $f : X \rightarrow B$ , where  $X$  is a compact metric space and  $B$  a convex subset of a normed space  $Y$  can be approximated in the uniform norm, by Lipschitz functions, defined on  $X$ , with values in  $B$ .

In [4], J. Luukkainen and J. Väisälä proved that every continuous function  $f : X \rightarrow M$ , where  $X$  is a metric space and  $M$  a LIP manifold, can be approximated in the uniform norm by locally Lipschitz functions.

The proofs of these results are based on the existence of a locally Lipschitz partition of unity for metric spaces.

In this paper we present a totally different technique to obtain this kind of approximation results for functions between every pair of metric spaces that satisfies the Lipschitz extension property.

This technique can be applied even when we have a function between a locally convex space and  $\mathbb{R}^n$ .

Based on Proposition 4, we obtain a generalization of the following well known result: for a convex function from an interval of  $\mathbb{R}$  into  $\mathbb{R}$ , the derivative of this function exists everywhere, except on a at most countable set of points.

Namely, we show that if  $X$  is a separable Fréchet space and  $F : X \rightarrow \mathbb{R}$  is a convex, locally bounded function, then

$$\overline{\{x \in X \mid \text{there exists } (DF)_x\}} = X.$$

## 2 The Approximation of Bounded, Uniformly Continuous Functions by Lipschitz Functions

**Theorem 1.** *Let  $(X, d)$  and  $(Y, d')$  be a pair of metric spaces which has the Lipschitz extension property i.e. there is a constant  $C$  which depends only on*

$X$  and  $Y$  such that for every subset  $B$  of  $X$  and every Lipschitz function  $f : B \rightarrow Y$ , there exists a Lipschitz function  $F : X \rightarrow Y$  such that  $F \upharpoonright_B = f$  and such that  $\text{Lip}(F) \leq C \text{Lip}(f)$ . Then, for every bounded uniformly continuous function  $f : X \rightarrow Y$  and every  $\varepsilon > 0$  there exists a Lipschitz function  $F : X \rightarrow Y$  such that  $\sup_{x \in X} d'(f(x), F(x)) < \varepsilon$ .

PROOF. If  $f$  is constant the conclusion is trivial. So we can assume  $f$  to be non-constant. Let us fix an element  $a$  in  $X$ . Therefore  $\sup_{x \in X} d'(f(x), f(a)) \neq 0$ . For  $\varepsilon > 0$ , let us consider  $\varepsilon'$ , such that  $0 < \varepsilon' < \frac{2\varepsilon}{C+1}$ . Since  $f$  is uniformly continuous, there exists  $\delta > 0$  such that  $d(x, y) < \delta \Rightarrow d'(f(x), f(y)) \leq \frac{\varepsilon'}{2}$ .

Let us consider  $\eta > 0$  such that

$$\eta < \min \left\{ \delta, \frac{\varepsilon' \cdot \delta}{4 \cdot \sup_{x \in X} d'(f(x), f(a))} \right\}.$$

Eventually working with a smaller  $\eta$ , we can suppose that the set

$$\mathcal{M} = \{A \subseteq X \mid \text{for each } x, y \in A, x \neq y, \text{ we have } d(x, y) \geq \eta\}$$

is not empty. Ordering  $\mathcal{M}$  by inclusion, we obtain an inductive ordered set, hence, taking into account Zorn's lemma, there exists a maximal element  $S$  of  $\mathcal{M}$ . If  $x, y \in S$  and  $d(x, y) \geq \delta$ , then

$$\begin{aligned} d'(f(x), f(y)) &\leq d'(f(x), f(a)) + d'(f(a), f(y)) \leq 2 \cdot \sup_{z \in X} d'(f(z), f(a)) \\ &= 2 \cdot \frac{\sup_{z \in X} d'(f(z), f(a))}{\delta} \cdot \delta \\ &\leq 2 \cdot \frac{\sup_{z \in X} d'(f(z), f(a))}{\delta} \cdot d(x, y). \end{aligned}$$

If  $x, y \in S$  and  $0 < d(x, y) < \delta$ , then

$$d'(f(x), f(y)) \leq \frac{\varepsilon'}{2} = \frac{\varepsilon'}{2 \cdot \eta} \cdot \eta \leq \frac{\varepsilon'}{2 \cdot \eta} \cdot d(x, y).$$

Therefore  $d'(f(x), f(y)) \leq \max \left\{ 2 \cdot \frac{\sup_{z \in X} d'(f(z), f(a))}{\delta}, \frac{\varepsilon'}{2 \cdot \eta} \right\} \cdot d(x, y)$ , for all  $x, y \in S$ , so  $f \upharpoonright_S : S \rightarrow Y$  is a Lipschitz function.

We can consider, according to our hypothesis, a Lipschitz function  $F : X \rightarrow Y$  such that  $F \upharpoonright_S = f \upharpoonright_S$  and

$$d'(F(x), F(y)) \leq C \max \left\{ 2 \cdot \frac{\sup_{z \in X} d'(f(z), f(a))}{\delta}, \frac{\varepsilon'}{2 \cdot \eta} \right\} \cdot d(x, y),$$

for all  $x, y \in X$ .

For  $x \in X \setminus S$ , there exists  $x_0 \in S$ , such that  $d(x, x_0) < \eta$  because otherwise  $S \cup \{x\} \in \mathcal{M}$ , which contradicts the fact that  $S$  is a maximal element of  $\mathcal{M}$ . For  $x \in S$ , there exists  $x_0 = x \in S$ , such that  $d(x, x_0) = 0 < \eta$ . Hence, for each  $x \in X$ , there exists  $x_0 \in S$ , such that  $d(x, x_0) < \eta < \delta$ . Then, for all  $x \in X$ , we have

$$\begin{aligned} d'(f(x), F(x)) &\leq d'(f(x), F(x_0)) + d'(F(x_0), F(x)) \\ &= d'(f(x), f(x_0)) + d'(F(x_0), F(x)) \\ &\leq \frac{\varepsilon'}{2} + C \max \left\{ 2 \cdot \frac{\sup_{x \in X} d'(f(x), f(a))}{\delta}, \frac{\varepsilon'}{2 \cdot \eta} \right\} \cdot \eta \\ &\leq \frac{\varepsilon'(C+1)}{2} < \varepsilon. \end{aligned}$$

Hence  $d'(f(x), F(x)) \leq \varepsilon$  for all  $x \in X$ , and  $F : X \rightarrow Y$  is a Lipschitz function.  $\square$

Now let us recall some results which provide pairs of metric spaces that have the Lipschitz extension property.

**Proposition 5.** (see[10]) Let  $X = Y = \{x \in \mathbb{R}^n \mid \|x\| = 1\}$  and  $f : B \rightarrow Y$  be a Lipschitz function, where  $B$  is a subset of  $X$ . Then there exists a Lipschitz function  $F : X \rightarrow Y$  such that  $F \upharpoonright_B = f$ . Moreover, for  $\text{Lip}(f) \leq 1$ ,

$$\text{Lip}(F) = \text{Lip}(f).$$

**Proposition 6.** (see[10]) Let  $X$  and  $Y$  be Hilbert spaces and  $f : B \rightarrow Y$  be a Lipschitz function where  $B$  is a subset of  $X$ . Then there exists a Lipschitz function  $F : X \rightarrow Y$  such that  $F \upharpoonright_B = f$ . Moreover  $\text{Lip}(F) = \text{Lip}(f)$ .

**Proposition 7.** (see[9]) Let  $K$  be a compact extremely disconnected Hausdorff space,  $X = Y = \mathcal{C}(K)$  and  $f : B \rightarrow Y$  a Lipschitz function, where  $B$  is a subset of  $X$ . Then there exists a Lipschitz function  $F : X \rightarrow Y$  such that  $F \upharpoonright_B = f$ . Moreover  $\text{Lip}(F) = \text{Lip}(f)$ .

**Proposition 8.** (see[2]) Let  $\Omega^n$  be the space of all nonempty compact, convex subsets of  $\mathbb{R}^n$  provided with the Hausdorff metric  $d_H$ , let  $X$  be a Hilbert space and  $f : B \rightarrow \Omega^n$  be a Lipschitz function, where  $B$  is a subset of  $X$ . Then there exists a Lipschitz function  $F : X \rightarrow \Omega^n$  such that  $F \upharpoonright_B = f$ . Moreover

$$\text{Lip}(F) \leq 2n \sqrt{\frac{28}{3}} \text{Lip}(f).$$

The technique used in Theorem 1 provides the following result.

**Theorem 2.** . *Let  $X$  be a locally convex space,  $(p_\alpha)_{\alpha \in A}$  a directed family of seminorms generating the topology of  $X$ , and  $f : X \rightarrow \mathbb{R}^n$  a bounded uniformly continuous function. Then, for every  $\varepsilon > 0$ , there exists a Lipschitz function,  $F : X \rightarrow \mathbb{R}^n$  such that  $\sup_{x \in X} \|f(x) - F(x)\| < \varepsilon$ .*

The corresponding extension result that is used here is the following assertion.

**Proposition 9.** *Let  $X$  be a locally convex space,  $(p_\alpha)_{\alpha \in A}$  a directed family of seminorms generating the topology of  $X$ , and  $f : B \rightarrow \mathbb{R}^n$  a Lipschitz function, where  $B$  is a subset of  $X$ . Then there exists a Lipschitz function  $F : X \rightarrow \mathbb{R}^n$  such that  $F|_B = f$ . Moreover if  $\alpha \in A$  and  $f$  satisfy*

$$\|f(x) - f(y)\| \leq Lp_\alpha(x - y),$$

for all  $x, y \in B$ , then the Lipschitz extension  $F$  can be chosen to satisfy

$$\|F(x) - F(y)\| \leq \sqrt{n}Lp_\alpha(x - y),$$

for all  $x, y \in X$ .

The proof of this result is omitted, being basically the proof of the famous result given in [6].

From Theorem 2 we obtain, by Proposition 4, the following.

**Corollary 1.** *If  $X$  is a separable Fréchet space, then for each Lipschitz function  $F : X \rightarrow \mathbb{R}^n$  we have  $\{x \in X \mid \text{there exists } (DF)_x\} = X$ . So, in this case, each bounded uniformly continuous function from  $X$  to  $\mathbb{R}^n$  can be approximated, in the uniform norm, by functions having the property that the set of points on which the differential exists is dense in  $X$ .*

**Corollary 2.** *Assume Banach space  $X$  has a  $\beta$ -smooth renorm. Then every uniformly continuous function on  $X$  can be uniformly approximated by Lipschitz functions which are densely  $\beta$ -differentiable.*

### 3 Convex Functions Are Differentiable on a Dense Set

**Theorem 3.** *Let  $X$  be a separable Fréchet space,  $(p_n)_{n \geq 1}$  a family of seminorms generating the topology of  $X$ , and  $F : X \rightarrow \mathbb{R}$  a locally bounded, convex function. Then  $\{x \in X \mid \text{there exists } (DF)_x\} = X$ .*

PROOF. According to our hypothesis, for each  $x \in X$  there exist  $n_0 \geq 1$ ,  $\varepsilon_0 > 0$  and  $M_x$  such that  $|F(u)| \leq M_x$ , for all  $u \in V_x = x + \{w \mid p_{n_0}(w) < 2\varepsilon_0\}$ . For arbitrary  $u, v \in U_x = x + \{w \mid p_{n_0}(w) < \varepsilon_0\}$  and  $\varepsilon > 0$ , let us consider

$$t = v + \frac{\varepsilon_0}{p_{n_0}(v-u) + \varepsilon}(v-u).$$

Then, we have

$$\begin{aligned} p_{n_0}(t-x) &\leq p_{n_0}\left((v-x) + \frac{\varepsilon_0}{p_{n_0}(v-u) + \varepsilon}(v-u)\right) \leq \\ &\leq p_{n_0}(v-x) + \varepsilon_0 \frac{p_{n_0}(v-u)}{p_{n_0}(v-u) + \varepsilon} \leq p_{n_0}(v-x) + \varepsilon_0 < 2\varepsilon_0. \end{aligned}$$

Hence  $t \in V_x$ , so

$$|F(t)| \leq M_x. \quad (1)$$

Let us note that  $u \in U_x \subseteq V_x$ , so

$$|F(u)| \leq M_x. \quad (2)$$

Since

$$v = \frac{\varepsilon_0}{p_{n_0}(v-u) + \varepsilon_0 + \varepsilon}u + \frac{p_{n_0}(v-u) + \varepsilon}{p_{n_0}(v-u) + \varepsilon_0 + \varepsilon}t$$

and  $F$  is convex, we obtain

$$F(v) - F(u) \leq \frac{p_{n_0}(v-u) + \varepsilon}{p_{n_0}(v-u) + \varepsilon_0 + \varepsilon}(F(t) - F(u)). \quad (3)$$

From (1), (2) and (3) we get

$$F(v) - F(u) \leq \frac{2M_x}{\varepsilon_0}[p_{n_0}(v-u) + \varepsilon].$$

Because  $\varepsilon > 0$  was arbitrary, we have

$$F(v) - F(u) \leq \frac{2M_x}{\varepsilon_0}p_{n_0}(v-u)$$

for all  $u, v \in U_x$ . Changing  $u$  and  $v$ , we get

$$F(u) - F(v) \leq \frac{2M_x}{\varepsilon_0}p_{n_0}(u-v)$$

for all  $u, v \in U_x$ . Therefore

$$|F(v) - F(u)| \leq \frac{2M_x}{\varepsilon_0}p_{n_0}(v-u)$$

for all  $u, v \in U_x$ . Hence  $F \upharpoonright_{U_x}$  is Lipschitz.

Taking into account Proposition 4, we come to the following conclusion: for each  $x \in X$  there exists an open set  $U_x$  of  $X$ , containing  $x$ , such that

$$\overline{\{y \in U_x \mid \text{there exists } (DF)_y\}} \supseteq U_x.$$

Consequently,

$$\overline{\{x \in X \mid \text{there exists } (DF)_x\}} = X.$$

□

**Remark 1.** Actually, in the above theorem, it suffices to assume  $F$  bounded above only, a traditional assumption.

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