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ON THE SUM OF FUNCTIONS WITH CONDITION (s_3)

Abstract

A function $f : \mathbb{R} \to \mathbb{R}$ satisfies condition (s_3) if for each real $\varepsilon > 0$, for each x and for each set $U \ni x$ belonging to the density topology there is an open interval I such that $A(f) \supset I \cap U \neq \emptyset$ and $f(U \cap I) \subset (f(x) - \varepsilon, f(x) + \varepsilon)$, where $A(f)$ denotes the set of all approximate continuity points of f. In this article it is show that the sum of two functions with the condition (s_3) is the sum of two Darboux functions satisfying this condition (s_3) and that every a.e.-continuous function with some special condition is the sum of two functions with condition (s_3).

Let $\mathbb{R}$ be the set of all reals. Denote by μ Lebesgue measure in $\mathbb{R}$ and by μ_e the outer Lebesgue measure in $\mathbb{R}$.

For a set $A \subset \mathbb{R}$ and a point x we define the upper (lower) outer density $d_u(A, x)$ $(d_l(A, x))$ of the set A at the point x as

$$\limsup_{h \to 0^+} \frac{\mu_e(A \cap [x - h, x + h])}{2h}$$

$$(\liminf_{h \to 0^+} \frac{\mu_e(A \cap [x - h, x + h])}{2h} \quad \text{respectively}).$$

A point x is said to be an outer density point (a density point) of a set A if $d_l(A, x) = 1$ (if there is a measurable set $B \subset A$ such that $d_l(B, x) = 1$).

The family

$$T_d = \{A \subset \mathbb{R} : x \in A \Longrightarrow x \text{ is a density point of } A\}$$

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is a topology called the density topology [1, 4]. The sets $A \in T_d$ are Lebesgue measurable ([1]).

Let T_e denotes the Euclidean topology in $\mathbb{R}$. A function $f : (\mathbb{R}, T_d) \to (\mathbb{R}, T_e)$ continuous at x is called approximately continuity at x ([1]).

For a function $f : \mathbb{R} \to \mathbb{R}$ let $C(f)$ denote the set of all continuity points of f, let $A(f)$ denote the set of all approximate continuity points of f, let $D(f) = \mathbb{R} \setminus C(f)$ denote the set of all discontinuity points of f, and finally let $D_{ap}(f) = \mathbb{R} \setminus A(f)$ denote the set of all approximate discontinuity points of f.

In [2] the following properties are investigated.

A function $f : \mathbb{R} \to \mathbb{R}$ has the property (s_3) [the property (s_1)] at a point x $(f \in s_3(x))$ $[f \in s_1(x)$ respectively] if for each real $\varepsilon > 0$ and for each set $U \in T_d$ containing x there is an open interval I such that $\emptyset \neq I \cap U \subset A(f)$ $[\emptyset \neq I \cap U \subset C(f)$ respectively] and $f(I \cap U) \subset (f(x) - \varepsilon, f(x) + \varepsilon)$.

A function $f : \mathbb{R} \to \mathbb{R}$ has the property (s_4) at a point x $(f \in s_4(x))$ if for each nonempty open set $U \in T_d$ containing x there is an open interval I such that $\emptyset \neq I \cap U \subset A(f)$.

A function f has the property (s_3) (the property (s_1), the property (s_4) respectively) if $f \in s_3(x)$ $(f \in s_1(x), f \in s_4(x)$ respectively) for every point $x \in \mathbb{R}$.

The class of all functions $f : \mathbb{R} \to \mathbb{R}$ with the property (s_3) (with the property (s_1), with the property $s_4(x)$ respectively) we denote by $\mathcal{S}_3$ (by $\mathcal{S}_1$, by $\mathcal{S}_4$ respectively). It is obvious that $\mathcal{S}_1 \subset \mathcal{S}_3 \subset \mathcal{S}_4$. Some examples of functions from $\mathcal{S}_3 \setminus \mathcal{S}_1, \mathcal{S}_4 \setminus \mathcal{S}_3$ are given in [2].

From the definition of the property (s_3) it follows that if $f : \mathbb{R} \to \mathbb{R}$ satisfies condition (s_3), then the set $D_{ap}(f)$ is nowhere dense and of Lebesgue measure zero. But there are functions $f : \mathbb{R} \to \mathbb{R}$ satisfying (s_3) such that $cl(D_{ap}(f))$ is of positive measure.

Example 1. Let $C \subset [0, 1]$ be a Cantor set of positive measure, (I_n) - an enumeration of all components of the set $[0, 1] \setminus C$ such that $I_n \neq I_m$ for $n \neq m$ and let $J_n \subset I_n$ be nondegenerate closed intervals $(n, m = 1, 2, \dots)$. Then the function

$$f(x) = \frac{1}{n} \text{ for } x \in J_n,\ n = 1, 2, \dots, \text{ and } f(x) = 0 \text{ otherwise on } \mathbb{R}$$

has the property (s_3) but for the set $D_{ap}(f) = D(f)$ and containing the endpoints of J_n, $(n \geq 1)$ we have $\mu(\mathrm{cl}(D_{ap}(f))) > 0$.

In [2] it is shown that a function f having property (s_3) is almost everywhere continuous; i.e., $\mu(D(f)) = 0$. Since there are approximately continuous functions f such that $\mu(D(f)) > 0$ ([1]), approximate continuity does not imply the property (s_3).

Part I. In the paper [3] Z. Grande proved that if $f : \mathbb{R} \to \mathbb{R}$ is the sum of two functions $g,\ h \in \mathcal{S}_1$, then there are two Darboux functions ϕ and ψ with property (s_1) such that $f = \phi+\psi$. In this part, by using Grande's method from the proof of theorem 1 in [3], I will prove a similar theorem for the functions with condition (s_3).

It is well known, that the class $\mathcal{D}$ of Darboux functions $f : \mathbb{R} \to \mathbb{R}$ is not closed under certain operations and that every function $f : \mathbb{R} \to \mathbb{R}$ can be represented as the sum of Darboux functions ([1]). Observe too, that the sum of two functions satisfying condition (s_3) can be without this property.

Example 2. The functions

$$f(x) = 0 \text{ for } x \leq 0 \text{ and } f(x) = 1 \text{ for } x > 0,$$

$$g(x) = 1 \text{ for } x < 0 \text{ and } g(x) = 0 \text{ for } x \geq 0$$

are continuous at $x \neq 0$ and unilaterally continuous at $x = 0$. So, they satisfy condition (s_1) (and (s_3) also), but the sum

$$(f+g)(x) = \begin{cases} 0 & \text{for } x = 0 \\ 1 & \text{for } x \neq 0 \end{cases}$$

does not satisfy condition (s_3).

Remark 1. *There are approximately continuous functions $f \in \mathcal{S}_3 \setminus \mathcal{S}_1$.*

For example, there are functions f approximately continuous everywhere and almost everywhere continuous with dense set $D(f)$.

Remark 2. *There are functions $f \in \mathcal{S}_1$ which are not approximately continuous.*

For example, the functions f, g from Example 2 are such.

Theorem 1. *If a function f is the sum of two functions $g, h \in \mathcal{S}_3$, then there are two functions $\phi, \psi \in \mathcal{S}_3 \cap \mathcal{D}$ such that $f = \phi + \psi$.*

Proof. Let $E = \mathrm{cl}(D_{ap}(g) \cup D_{ap}(h))$ and $D = \mathrm{cl}(D(g) \cup D(h))$. It is known that $D_{ap}(g) \subset D(g)$, $D_{ap}(h) \subset D(h)$; so $D \supset E$. Moreover the set E is nowhere dense in $\mathbb{R}$.

If $D = \emptyset$, then we can define $\phi = g$ and $\psi = h$ and the proof is done. So we suppose that $D \neq \emptyset$. We will consider two cases:

$$\mathbf{I}.\mu(D) = 0 \text{ and } \mathbf{II}.\mu(D) > 0.$$

Case **I.** Let $\mu(D)=0$. In this case let $(a^k,b^k)_{k=1}^{\infty}$ be a sequence of all components of the complement $\mathbb{R}\setminus D$ such that $(a^k,b^k)\cap(a^j,b^j)=\emptyset$ for $k\neq j$. If, for a fixed $k\in\mathbb{N}$, where $\mathbb{N}$ denotes the set of all positive integers, the interval (a^k,b^k) is a bounded component of the complement $\mathbb{R}\setminus D$, we find two monotone sequences of points

$$a^k<\cdots<a_{n+1}^k<a_n^k<\cdots a_1^k<b_1^k<\cdots<b_n^k<b_{n+1}^k<\cdots<b^k$$

such that $\lim_{n\to\infty}a_n^k=a^k$ and $\lim_{n\to\infty}b_n^k=b^k$, and

$$\lim_{n\to\infty}\frac{b_{n+1}^k-b_n^k}{b^k-b_n^k}=\lim_{n\to\infty}\frac{a_n^k-a_{n+1}^k}{a_n^k-a^k}=0. \tag{1}$$

In each interval (a_{n+1}^k,a_n^k) $((b_n^k,b_{n+1}^k))$ we find disjoint nondegenerate closed intervals $I_{n,i}^k\subset(a_{n+1}^k,a_n^k)$ $(J_{n,i}^k\subset(b_n^k,b_{n+1}^k))$ for $i=1,2$ such that

$$\frac{l(I_{n,i}^k)}{a_n^k-a_{n+1}^k}<\frac{1}{2^{n+k}},\ \left(\frac{l(J_{n,i}^k)}{b_{n+1}^k-b_n^k}<\frac{1}{2^{n+k}}\right) \tag{2}$$

for $i=1,2$, where $l(H)$ denotes the length of the interval H, and

$$\frac{\mu(\bigcup_{i=1}^2\bigcup_{n=1}^{\infty}(I_{n,i}^k\cup J_{n,i}^k))}{b^k-a^k}<\frac{1}{2^k}. \tag{3}$$

If (a^s,b^s) is an unbounded component of the complement $\mathbb{R}\setminus D$; i.e., $a^s=-\infty$ or $b^s=\infty$, we find two sequences only, $(J_{n,i}^s)$ $(i=1,2)$ or respectively $(I_{n,i}^s)$ $(i=1,2)$, satisfying the above conditions (1), (2).

For a fixed k, for $i=1,2$ and for $n\geq 1$ let $g_{n,i}^k: I_{n,i}^k\to\mathbb{R}$ and $h_{n,i}^k: J_{n,i}^k\to\mathbb{R}$ be continuous functions such that $g_{n,i}^k(x)=0$ if x is an endpoint of $I_{n,i}^k$, $h_{n,i}^k(y)=0$ if y is an endpoint of $J_{n,i}^k$ and

$$(g+g_{n,1}^k)(I_{n,1}^k)\cap(h+h_{n,1}^k)(J_{n,1}^k)\cap(g+h_{n,2}^k)(J_{n,2}^k)\cap(h+g_{n,2}^k)(I_{n,2}^k)\supset[-n,n].$$

If (a^k,b^k) is a bounded component of the complement $\mathbb{R}\setminus D$, then we put (for fixed k)

$$g^k(x)=\begin{cases} g(x)+g_{n,1}^k(x) & \text{for } x\in I_{n,1}^k,\ n\geq 1\\ g(x)+h_{n,2}^k(x) & \text{for } x\in J_{n,2}^k,\ n\geq 1\\ g(x)-h_{n,1}^k(x) & \text{for } x\in J_{n,1}^k,\ n\geq 1\\ g(x)-g_{n,2}^k(x) & \text{for } x\in I_{n,2}^k,\ n\geq 1\\ g(x) & \text{otherwise on } (a^k,b^k)\end{cases}$$

and

$$h^k(x) = \begin{cases} h(x) + h^k_{n,1}(x) & \text{for } x \in J^k_{n,1},\ n \geq 1 \\ h(x) + g^k_{n,2}(x) & \text{for } x \in I^k_{n,2},\ n \geq 1 \\ h(x) - g^k_{n,1}(x) & \text{for } x \in I^k_{n,1},\ n \geq 1 \\ h(x) - h^k_{n,2}(x) & \text{for } x \in J^k_{n,2},\ n \geq 1 \\ h(x) & \text{otherwise on } (a^k, b^k). \end{cases}$$

Similarly we define the functions g^s and h^s on unbounded components (a^s, b^s) of the set $\mathbb{R} \setminus D$.

Putting $\phi(x) = g^k(x),\ \psi(x) = h^k(x)$ on every component (a^k, b^k) of the complement $\mathbb{R} \setminus D$ and $\phi(x) = g(x)$, $\psi(x) = h(x)$ on D we obtain Darboux functions ϕ and ψ continuous on $\mathbb{R} \setminus D$ such that $\phi + \psi = g + h = f$. Since ϕ, ψ are continuous on $\mathbb{R} \setminus D$, for every $x \in \mathbb{R} \setminus D$ we have $\phi \in s_3(x)$ and $\psi \in s_3(x)$. Now, let $x \in D$, let $U \in T_d$ be the set containing x and let $\varepsilon > 0$ be a real. By (1), (2) and (3) the lower density

$$d_l(\mathbb{R} \setminus (\bigcup_{k=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcup_{i=1}^{2} (I^k_{n,i} \cup J^k_{n,i}) \setminus D, x) = 1,$$

Observe that

$$T_d \ni (\mathbb{R} \setminus (\bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} \bigcup_{i=1}^{2} (I^k_{n,i} \cup J^k_{n,i}) \setminus D) \cap U) \cup \{x\} \neq \emptyset$$

and $g(t) = \phi(t), h(t) = \psi(t)$ for $t \in \{x\} \cup (\mathbb{R} \setminus (\bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} \bigcup_{i=1}^{2} (I^k_{n,i} \cup J^k_{n,i}))$. Since $g \in s_3(x)$, there is an open interval

$$I \subset (\mathbb{R} \setminus (\bigcup_{k=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcup_{i=1}^{2} (I^k_{n,i} \cup J^k_{n,i})) \setminus D)$$

such that $I \cap U \neq \emptyset$ and $|g(t) - g(x)| < \varepsilon$ for all $t \in I \cap U$. So,

$$|\phi(t) - \phi(x)| = |g(t) - g(x)| < \varepsilon$$

for all $t \in I \cap U$ and consequently $\phi \in s_3(x)$. Similarly we can prove that $\psi \in s_3(x)$.

Case **II.** Suppose that $\mu(D) > 0$. In this case there are positive numbers

$$c_1 > c_2 > \cdots > c_n > \cdots > 0 \text{ such that } \sum_n c_n < \infty$$

and the sets

$$E_1 = \{x; \operatorname{osc} g(x) \geq c_1\} \cup \{x; \operatorname{osc} h(x) \geq c_1\},$$

$$E_{n+1} = \{x; c_n > \operatorname{osc} g(x) \geq c_{n+1}\} \cup \{x; c_n > \operatorname{osc} h(x) \geq c_{n+1}\}$$

are nonempty for $n \geq 1$.

In the first step of the inductive construction of functions ϕ and ψ we consider the closed set E_1 which is of measure zero evidently. Let $((a^{k,1}, b^{k,1}))_{k=1}^{\infty}$ be a sequence of all components of the complement $\mathbb{R}\setminus E_1$ such that $(a^{k,1}, b^{k,1}) \cap (a^{j,1}, b^{j,1}) = \emptyset$ for $k \neq j$.

If, for a fixed $k \in \mathbb{N}$, the interval $(a^{k,1}, b^{k,1})$ is a bounded component of the complement $\mathbb{R} \setminus E_1$, we find two monotone sequences of points

$$a^{k,1} < \cdots < a^{k,1}_{n+1} < a^{k,1}_n < \cdots a^{k,1}_1 < b^{k,1}_1 < \cdots < b^{k,1}_n < b^{k,1}_{n+1} < \cdots < b^{k,1}$$

such that $\lim_{n\to\infty} a^{k,1}_n = a^{k,1}$ and $\lim_{n\to\infty} b^{k,1}_n = b^{1,k}$, and

$$\lim_{n\to\infty} \frac{b^{k,1}_{n+1} - b^{k,1}_n}{b^{k,1} - b^{k,1}_n} = \lim_{n\to\infty} \frac{a^{k,1}_n - a^{k,1}_{n+1}}{a^{k,1}_n - a^{k,1}} = 0. \tag{1.1}$$

In each interval $(a^{k,1}_{n+1}, a^{k,1}_n)$ $((b^{k,1}_n, b^{k,1}_{n+1}))$ we find disjoint nondegenerate closed intervals (for $i = 1, 2$)

$$I^{k,1}_{n,i} \subset (a^{k,1}_{n+1}, a^{k,1}_n) \setminus E (J^{k,1}_{n,i} \subset (b^{k,1}_n, b^{k,1}_{n+1}) \setminus E)$$

with endpoints from the set $C(g)$ (with endpoints from the set $C(h)$ respectively) such that

$$\frac{l(I^{k,1}_{n,i})}{a^{k,1}_n - a^{k,1}_{n+1}} < \frac{1}{2^{n+k}}, \left(\frac{l(J^{k,1}_{n,i})}{b^{k,1}_{n+1} - b^{k,1}_n} < \frac{1}{2^{n+k}}\right), \tag{1.2}$$

for $i = 1, 2$, and

$$\frac{\mu(\bigcup_{i=1}^{2} \bigcup_{n=1}^{\infty} (I^{k,1}_{n,i} \cup J^{k,1}_{n,i}))}{b^{k,1} - a^{k,1}} < \frac{1}{2^k}. \tag{1.3}$$

If $(a^{s,1}, b^{s,1})$ is an unbounded component of the complement $\mathbb{R}\setminus E_1$; i.e., $a^{s,1} = -\infty$ or $b^{s,1} = \infty$, we find two sequences only, $(J^{s,1}_{n,i})$ $(i = 1, 2)$ or respectively $(I^{s,1}_{n,i})$ $(i = 1, 2)$, satisfying the above conditions $(1.1), (1.2)$. For a fixed k, for $i = 1, 2$ and for $n \geq 1$ let $g^{k,1}_{n,i} : I^{k,1}_{n,i} \to \mathbb{R}$ and $h^{k,1}_{n,i} : J^{k,1}_{n,i} \to \mathbb{R}$ be continuous functions such that $g^{k,1}_{n,i}(x) = 0$ if x is an endpoint of $I^{k,1}_{n,i}$, $h^{k,1}_{n,i}(y) = 0$ if y is an endpoint of $J^{k,1}_{n,i}$ and

$$(g + g^{k,1}_{n,1})(I^{k,1}_{n,1}) \cap (h + h^{k,1}_{n,1})(J^{k,1}_{n,1}) \cap (g + h^{k,1}_{n,2})(J^{k,1}_{n,2}) \cap (h + g^{k,1}_{n,2})(I^{k,1}_{n,2}) \supset [-n, n].$$

Now, we define the functions ϕ_1 and ψ_1 by

$$\phi_1(x) = \begin{cases} g(x) + g_{n,1}^{k,1}(x) & \text{for } x \in I_{n,1}^{k,1},\ n,k \geq 1 \\ g(x) + h_{n,2}^{k,1}(x) & \text{for } x \in J_{n,2}^{k,1},\ n,k \geq 1 \\ g(x) - h_{n,1}^{k,1}(x) & \text{for } x \in J_{n,1}^{k,1},\ n,k \geq 1 \\ g(x) - g_{n,2}^{k,1}(x) & \text{for } x \in I_{n,2}^{k,1},\ n,k \geq 1 \\ g(x) & \text{otherwise on } \mathbb{R} \end{cases}$$

and

$$\psi_1(x) = \begin{cases} h(x) + h_{n,1}^{k,1}(x) & \text{for } x \in J_{n,1}^{k,1},\ n,k \geq 1 \\ h(x) + g_{n,2}^{k,1}(x) & \text{for } x \in I_{n,2}^{k,1},\ n,k \geq 1 \\ h(x) - g_{n,1}^{k,1}(x) & \text{for } x \in I_{n,1}^{k,1},\ n,k \geq 1 \\ h(x) - h_{n,2}^{k,1}(x) & \text{for } x \in J_{n,2}^{k,1},\ n,k \geq 1 \\ h(x) & \text{otherwise on } \mathbb{R}. \end{cases}$$

The functions ϕ_1 and ψ_1 have the Darboux property and for all $u \in \mathbb{R} \setminus E_1$

$$\operatorname{osc} \phi_1(u) = \operatorname{osc} g(u) \text{ and } \operatorname{osc} \psi_1(u) = \operatorname{osc} h(u).$$

Moreover $\phi_1 + \psi_1 = g + h = f$. Also note that $D_{ap}(\phi_1) = D_{ap}(g)$ and $D_{ap}(\psi_1) = D_{ap}(h)$.

In the second step we consider the closed set $E_1 \cup E_2$ which is of measure zero evidently. Let $((a^{k,2}, b^{k,2}))_{k=1}^{\infty}$ be a sequence of all components of the set $\mathbb{R} \setminus (E_1 \cup E_2)$ such that $(a^{k,2}, b^{k,2}) \cap (a^{j,2}, b^{j,2}) = \emptyset$ and $k \neq j$. If, for a fixed $k \in \mathbb{N}$, $(a^{k,2}, b^{k,2})$ is a bounded component of the complement $\mathbb{R} \setminus (E_1 \cup E_2)$, then we find two monotone sequences of points

$$a^{k,2} < \ldots < a_{n+1}^{k,2} < a_n^{k,2} < \ldots < a_1^{k,2} < b_1^{k,2} < \ldots < b_n^{k,2} < b_{n+1}^{k,2} < \ldots < b^{k,2}$$

such that $\lim_{n\to\infty} a_n^{k,2} = a^{k,2}$, $\lim_{n\to\infty} b_n^{k,2} = b^{k,2}$ and

$$\lim_{n\to\infty} \frac{b_{n+1}^{k,2} - b_n^{k,2}}{b^{k,2} - b_n^{k,2}} = \lim_{n\to\infty} \frac{a_n^{k,2} - a_{n+1}^{k,2}}{a_n^{k,2} - a^{k,2}} = 0 \tag{2.1}$$

In each interval $(a_{n+1}^{k,2}, a_n^{k,2})$, $((b_n^{k,2}, b_{n+1}^{k,2}))$ we find disjoint nondegenerate closed intervals (for $i = 1, 2$)

$$I_{n,i}^{k,2} \subset (a_{n+1}^{k,2}, a_n^{k,2}) \setminus E,\ (J_{n,i}^{k,2} \subset (b_n^{k,2}, b_{n+1}^{k,2}) \setminus E)$$

with endpoints from the set $C(\phi_1)$ (with endpoints from the set $C(\psi_1)$ respectively) such that, for $i = 1, 2$ we have

$$\frac{l(I_{n,i}^{k,2})}{a_n^{k,2} - a_{n+1}^{k,2}} < \frac{1}{2^{n+k}},\ \left(\frac{l(J_{n,i}^{k,2})}{b_{n+1}^{k,2} - b_n^{k,2}} < \frac{1}{2^{n+k}}\right) \tag{2.2}$$

and

$$\frac{\mu(\bigcup_{i=1}^{2}\bigcup_{n=1}^{\infty}(I_{n,i}^{k,2}\cup J_{n,i}^{k,2}))}{b^{k,2}-a^{k,2}} < \frac{1}{2^k} \tag{2.3}$$

and also

$$\operatorname{osc}_{I_{n,i}^{k,2}}\phi_1 < c_2, \operatorname{osc}_{J_{n,i}^{k,2}}\psi_1 < c_2 \tag{2.4}$$

for $n \geq 1$ and $i = 1, 2$.

If $(a^{s,2}, b^{s,2})$ is an unbounded component of the complement $\mathbb{R}\setminus(E_1 \cup E_2)$; i.e., $a^{s,2} = -\infty$ or $b^{s,2} = \infty$, we find two sequences only, $(J_{n,i}^{s,2})_{n=1}^{\infty}$ $(i = 1, 2)$ or $(I_{n,i}^{s,2})_{n=1}^{\infty}$ $(i = 1, 2)$ respectively, satisfying above conditions (2.1), (2.2) and (2.4).

For a fixed $k \in \mathbb{N}$, for $i = 1, 2$ and for a fixed $n \geq 1$ we will construct continuous functions $g_{n,i}^{k,2} : I_{n,i}^{k,2} \to \mathbb{R}$ and $h_{n,i}^{k,2} : J_{n,i}^{k,2} \to \mathbb{R}$. Fix $k, n \in \mathbb{N}$. For $i = 1, 2$ the set $\phi_1(I_{n,i}^{k,2})$ is an interval of the length less than c_2. Let γ_i^2 be the mid point of the interval $\phi_1(I_{n,i}^{k,2})$, for $i = 1, 2$. In the interval $\operatorname{int}(I_{n,i}^{k,2})$, where $\operatorname{int}(H)$ denotes the interior of the set H, choose two points α_i^2 and β_i^2 such that, for $i = 1, 2$, $\phi_1(\alpha_i^2) < \gamma_i^2 < \phi_1(\beta_i^2)$.

The continuous function $g_{n,i}^{k,2} : I_{n,i}^{k,2} \to \mathbb{R}$ (for $i = 1, 2$) we define by

$g_{n,i}^{k,2}(x) = 0$ if x is any endpoint of $I_{n,i}^{k,2}$;

$g_{n,i}^{k,2}(\alpha_i^2) = -c_2$;

$g_{n,i}^{k,2}(\beta_i^2) = c_2$ and

$g_{n,i}^{k,2}$ is linear on the closures of the components of the set $I_{n,i}^{k,2} \setminus \{\alpha_i^2, \beta_i^2\}$.

Similarly we define the continuous functions $h_{n,i}^{k,2} : J_{n,i}^{k,2} \to \mathbb{R}$ $(i = 1, 2)$. The set $\psi_1(J_{n,i}^{k,2})$, for fixed $k, n \in \mathbb{N}$ and $i = 1, 2$, is an interval of the length less than c_2. Let ν_i^2 be the center of the interval $\psi_1(J_{n,i}^{k,2})$ for $i = 1, 2$. In the set $\operatorname{int}(J_{n,i}^{k,2})$ choose two points ξ_i^2 and η_i^2 such that, for $i = 1, 2$, $\psi_1(\xi_i^2) < \nu_i^2 < \psi_1(\eta_i^2)$. Let the continuous function $h_{n,i}^{k,2} : J_{n,i}^{k,2} \to \mathbb{R}$ $(i = 1, 2)$ be such that

$h_{n,i}^{k,2}(x) = 0$ if x is any endpoint of $J_{n,i}^{k,2}$,

$h_{n,i}^{k,2}(\xi_i^2) = -c_2$,

$h_{n,i}^{k,2}(\eta_i^2) = c_2$ and

$h_{n,i}^{k,2}$ is linear on the closures of the components of the set $J_{n,i}^{k,2} \setminus \{\xi_i^2, \eta_i^2\}$.

Finally, for the second step, we define the functions ϕ_2 and ψ_2 by

$$\phi_2(x) = \begin{cases} \phi_1(x) + g^{k,2}_{n,1}(x) & \text{for } x \in I^{k,2}_{n,1},\ n,k \geq 1 \\ \phi_1(x) - h^{k,2}_{n,1}(x) & \text{for } x \in J^{k,2}_{n,1},\ n,k \geq 1 \\ \phi_1(x) - g^{k,2}_{n,2}(x) & \text{for } x \in I^{k,2}_{n,2},\ n,k \geq 1 \\ \phi_1(x) + h^{k,2}_{n,2}(x) & \text{for } x \in J^{k,2}_{n,2},\ n,k \geq 1 \\ \phi_1(x) & \text{otherwise on } \mathbb{R} \end{cases}$$

and

$$\psi_2(x) = \begin{cases} \psi_1(x) - g^{k,2}_{n,1}(x) & \text{for } x \in I^{k,2}_{n,1},\ n,k \geq 1 \\ \psi_1(x) + h^{k,2}_{n,1}(x) & \text{for } x \in J^{k,2}_{n,1},\ n,k \geq 1 \\ \psi_1(x) + g^{k,2}_{n,2}(x) & \text{for } x \in I^{k,2}_{n,2},\ n,k \geq 1 \\ \psi_1(x) - h^{k,2}_{n,2}(x) & \text{for } x \in J^{k,2}_{n,2},\ n,k \geq 1 \\ \psi_1(x) & \text{otherwise on } \mathbb{R}. \end{cases}$$

Observe that $D_{ap}(\phi_2) = D_{ap}(g)$ and $D_{ap}(\psi_2) = D_{ap}(h)$ and for each point $u \in \mathbb{R} \setminus (E_1 \cup E_2)$ the oscillation $\operatorname{osc}\phi_2(u) = \operatorname{osc} g(u)$ and $\operatorname{osc}\psi_2(u) = \operatorname{osc} h(u)$. Observe too, that for $i = 1, 2$ $\phi_2(I^{k,2}_{n,i}) \supset \phi_1(I^{k,2}_{n,i})$, $\psi_2(J^{k,2}_{n,i}) \supset \psi_1(J^{k,2}_{n,i})$ and for all $x \in \mathbb{R}$, \$$|\phi_2(x) - \phi_1(x)| < 3c_2$ and $|\psi_2(x) - \psi_1(x)| < 3c_2$. Moreover, $\phi_2 + \psi_2 = \phi_1 + \psi_1 = g + h = f$.

In the mth step ($m > 2$), we repeat the construction of the step $(m-1)$, but for the closed set $\bigcup_{j=1}^{m-1} E_j \cup E_m$ of measure zero. Let $m > 2$. In this inductive step, let $((a^{k,m}, b^{k,m}))_{k=1}^{\infty}$ be a sequence of all components of the complement of the set $\mathbb{R} \setminus (\bigcup_{j=1}^{m} E_j)$ such that $(a^{k,m}, b^{k,m}) \cap (a^{j,m}, b^{j,m}) = \emptyset$ and $k \neq j$. If $(a^{k,m}, b^{k,m})$, for fixed $k \in \mathbb{N}$, is a bounded component of the complement $\mathbb{R} \setminus \bigcup_{j=1}^{m} E_j$, we find two sequences of the points

$$a^{k,m} < \ldots < a^{k,m}_{n+1} < a^{k,m}_n < \ldots < a^{k,m}_1 < b^{k,m}_1 < \ldots < b^{k,m}_n < b^{k,m}_{n+1} < \ldots < b^{k,m}$$

such that $\lim_{n\to\infty} a^{k,m}_n = a^{k,m}$, $\lim_{n\to\infty} b^{k,m}_n = b^{k,m}$ and

$$\lim_{n\to\infty} \frac{b^{k,m}_{n+1} - b^{k,m}_n}{b^{k,m} - b^{k,m}_n} = \lim_{n\to\infty} \frac{a^{k,m}_n - a^{k,m}_{n+1}}{a^{k,m}_n - a^{k,m}} = 0. \tag{$m.1$}$$

In each interval $(a^{k,m}_{n+1}, a^{k,m}_n)$ $((b^{k,m}_n, b^{k,m}_{n+1}))$ we find two disjoint nondegenerate closed intervals $I^{k,m}_{n,i} \subset (a^{k,m}_{n+1}, a^{k,m}_n) \setminus E (J^{k,m}_{n,i} \subset (b^{k,m}_n, b^{k,m}_{n+1}) \setminus E)$ (for $i = 1, 2$) with endpoints from the set $C(\phi_{m-1})$ (with endpoints from the set $C(\psi_{m-1})$ respectively) such that, for $i = 1, 2$ we have

$$\frac{l(I^{k,m}_{n,i})}{a^{k,m}_n - a^{k,m}_{n+1}} < \frac{1}{2^{k+n}}, \left(\frac{l(J^{k,m}_{n,i})}{b^{k,m}_{n+1} - b^{k,m}_n} < \frac{1}{2^{n+k}} \right) \tag{$m.2$}$$

and

$$\frac{\mu(\bigcup_{n=1}^{\infty}\bigcup_{i=1}^{2}(I_{n,i}^{k,m}\cup J_{n,i}^{k,m}))}{b^{k,m}-a^{k,m}}<\frac{1}{2^k}, \tag{m.3}$$

and also

$$\operatorname{osc}_{I_{n,i}^{k,m}}\phi_{m-1}<c_m, \operatorname{osc}_{J_{n,i}^{k,m}}\psi_{m-1}<c_m \tag{m.4}$$

for $n \geq 1$ and $i = 1, 2$.

If $(a^{s,m}, b^{s,m})$ is an unbounded component of $\mathbb{R}\setminus\bigcup_{j=1}^{m}E_j$; i.e., $a^{s,m}=-\infty$ or $b^{s,m}=\infty$, then we find two sequences only: $(I_{n,i}^{s,m})_{n=1}^{\infty}$ $(i=1,2)$ or respectively $(J_{n,i}^{s,m})_{n=1}^{\infty}$ $(i=1,2)$ satisfying above conditions $(m.1), (m.2)$ and $(m.4)$.

We will construct continuous functions $g_{n,i}^{k,m}: I_{n,i}^{k,m}\to\mathbb{R}$ and $h_{n,i}^{k,m}: J_{n,i}^{k,m}\to\mathbb{R}$ for $i=1,2$ and $n=1,2,\ldots$, and $k=1,2\ldots$. Fix $k,n\in\mathbb{N}$. For $i=1,2$ the image $\phi_{m-1}(I_{n,i}^{k,m})$ is the interval of the length less than c_m. Let the point γ_i^m be the center of the interval $\phi_{m-1}(I_{n,i}^{k,m})$. In the interval $\operatorname{int}(I_{n,i}^{k,m})$ choose two numbers α_i^m and β_i^m such that, for $i=1,2$, $\phi_{m-1}(\alpha_i^m)<\gamma_i^m<\phi_{m-1}(\beta_i^m)$.

Next, for $i=1,2$, the continuous functions $g_{n,i}^{k,m}: I_{n,i}^{k,m}\to\mathbb{R}$ we define by

$g_{n,i}^{k,m}(x)=0$ if x is any endpoint of $I_{n,i}^{k,m}$,

$g_{n,i}^{k,m}(\alpha_i^m)=-c_m$,

$g_{n,i}^{k,m}(\beta_i^m)=c_m$ and

$g_{n,i}^{k,m}$ is linear on the closures of the components of $I_{n,i}^{k,m}\setminus\{\alpha_i^m,\beta_i^m\}$.

The construction of the continuous function $h_{n,i}^{k,m}: J_{n,i}^{k,m}\to\mathbb{R}$ (for $i=1,2$) is similar. For fixed $k,n\in\mathbb{N}$ and for $i=1,2$ the image $\psi_{m-1}(J_{n,i}^{k,m})$ is the interval of length less than c_m. Let ν_i^m be the mid point of the interval $\psi_{m-1}(J_{n,i}^{k,m})$. In the set $\operatorname{int}(J_{n,i}^{k,m})$ choose two points ξ_i^m and η_i^m such that, for $i=1,2$, $\psi_{m-1}(\xi_i^m)<\nu_i^m<\psi_{m-1}(\eta_i^m)$.

Let the continuous functions $h_{n,i}^{k,m}: J_{n,i}^{k,m}\to\mathbb{R}(i=1,2)$ be such that

$h_{n,i}^{k,m}(x)=0$ if x is any endpoint of $J_{n,i}^{k,m}$,

$h_{n,i}^{k,m}(\xi_i^m)=-c_m$,

$h_{n,i}^{k,m}(\eta_i^m)=c_m$ and

$h_{n,i}^{k,m}$ is linear on the closures of the components of $J_{n,i}^{k,m}\setminus\{\xi_i^m,\eta_i^m\}$.

Finally, in the inductive step $m > 2$, we define the functions ϕ_m and ψ_m by

$$\phi_m(x) = \begin{cases} \phi_{m-1}(x) + g^{k,m}_{n,1}(x) & \text{for } x \in I^{k,m}_{n,1},\ n,k \geq 1 \\ \phi_{m-1}(x) - h^{k,m}_{n,1}(x) & \text{for } x \in J^{k,m}_{n,1},\ n,k \geq 1 \\ \phi_{m-1}(x) - g^{k,m}_{n,2}(x) & \text{for } x \in I^{k,m}_{n,2},\ n,k \geq 1 \\ \phi_{m-1}(x) + h^{k,m}_{n,2}(x) & \text{for } x \in J^{k,m}_{n,2},\ n,k \geq 1 \\ \phi_{m-1}(x) & \text{otherwise on } \mathbb{R}, \end{cases}$$

$$\psi_m(x) = \begin{cases} \psi_{m-1}(x) - g^{k,m}_{n,1}(x) & \text{for } x \in I^{k,m}_{n,1},\ n,k \geq 1 \\ \psi_{m-1}(x) + h^{k,m}_{n,1}(x) & \text{for } x \in J^{k,m}_{n,1},\ n,k \geq 1 \\ \psi_{m-1}(x) + g^{k,m}_{n,2}(x) & \text{for } x \in I^{k,m}_{n,2},\ n,k \geq 1 \\ \psi_{m-1}(x) - h^{k,m}_{n,2}(x) & \text{for } x \in J^{k,m}_{n,2},\ n,k \geq 1 \\ \psi_{m-1}(x) & \text{otherwise on } \mathbb{R}. \end{cases}$$

Observe that $D_{ap}(\phi_m) = D_{ap}(g)$ and $D_{ap}(\psi_m) = D_{ap}(h)$ and for each point $u \in \mathbb{R} \setminus \bigcup_{j=1}^{m} E_j$ the oscillation $\operatorname{osc} \phi_m(u) = \operatorname{osc} g(u)$ and $\operatorname{osc} \psi_m(u) = \operatorname{osc} h(u)$. Observe too, that for $i = 1, 2$

$$\phi_m(I^{k,m}_{n,i}) \supset \phi_{m-1}(I^{k,m}_{n,i}), \psi_m(J^{k,m}_{n,i}) \supset \psi_{m-1}(J^{k,m}_{n,i})$$

and for all $x \in \mathbb{R}$, $|\phi_m(x) - \phi_{m-1}(x)| < 3c_m$ and $|\psi_m(x) - \psi_{m-1}(x)| < 3c_m$. Moreover $\phi_m + \psi_m = \phi_{m-1} + \psi_{m-1} = \ldots = \phi_1 + \psi_1 = g + h = f$. The sequences $(\phi_m)_{m=1}^{\infty}$ and $(\psi_m)_{m=1}^{\infty}$ uniformly converge to functions ϕ and ψ respectively. Observe that $\phi + \psi = \lim_{m\to\infty}(\phi_m + \psi_m) = g + h = f$. The functions ϕ and ψ, as the uniform limits, are continuous in each point of the set $\mathbb{R} \setminus D$. Thus they satisfy condition (s_3) at all points of the complement $\mathbb{R} \setminus D$.

We will prove that ϕ and ψ satisfy also the property (s_3) at all points of the set D. For this fix a point $x \in D$, a real $\varepsilon > 0$ and a set $U \in T_d$ such that $x \in U$. Let j be the integer such that $|\phi_j - \phi| < \frac{\varepsilon}{3}$. Since the function g has the property (s_3) and $d_u(\{u; \phi_j(u) \neq g(u)\}, x) = 0$, there is an open interval $I \subset \{u; \phi_j(u) = g(u)\}$ such that

$$\emptyset \neq I \cap U \subset A(\phi) \text{ and } g(I \cap U) = \phi_j(I \cap U) \subset \Big(g(x) - \frac{\varepsilon}{3}, g(x) + \frac{\varepsilon}{3}\Big).$$

Consequently, for $u \in I \cap U$ we have

$$|\phi(u) - \phi(x)| \leq |\phi(u) - \phi_j(u)| + |\phi_j(u) - \phi_j(x)| + |\phi_j(x) - \phi(x)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

So the function $\phi \in s_3(x)$ for all $x \in D$. In the same way we can check that $\psi \in s_3(x)$ for these points. Thus $\phi, \psi \in \mathcal{S}_3$.

Now we will prove that ϕ has the Darboux property. Suppose, to the contrary, that it has not the Darboux property. Then there are points a, b with $a < b$ and $\phi(a) \neq \phi(b)$ and a real $c \in K = (\min(\phi(a), \phi(b)), \max(\phi(a), \phi(b))$ such that $\phi^{-1}(c) \cap [a, b] = \emptyset$. If there is a point $x \in E_1 \cap [a, b]$, then there is a nondegenerate closed interval $I \subset [a, b]$ such that $\phi(I) \supseteq \phi_1(I) \supset K \ni c$, a contradiction. Fix a point

$$z \in [a, b] \cap \mathrm{cl}(\{u; \phi(u) < c\}) \cap \mathrm{cl}(\{u; \phi(u) > c\}).$$

Observe that $z \in D$ and there is an integer $m > 1$ such that $z \in E_m$. Thus $\operatorname{osc} \phi_{m-1}(z) < c_m$ and there is an open interval $V \ni z$ such that $\operatorname{osc}_V \phi_{m-1} < c_m$. So we have either $\phi(z) = \phi_{m-1}(z) < c$ or $\phi(z) = \phi_{m-1}(z) > c$. Suppose that $\phi_{m-1}(z) < c$. Then there is a point $v \in [a, b] \cap V$ such that $\phi_{m-1}(v) > c$. Since $v \in V$, we have $\phi_{m-1}(v) - \phi_{m-1}(z) < c_m$ and consequently $c - \phi_{m-1}(z) < c_m$. From the construction of ϕ_m it follows that there is a nondegenerate closed interval $I \in [a, b] \cap V$ such that $\phi(I) \supseteq \phi_m(I) \supset [\phi_{m-1}(z), \phi_{m-1}(v)] \ni c$, a contradiction. If $\phi_{m-1}(z) > c$ the reasoning is similar. So $\phi \in \mathcal{D}$. The same we can show that the function ψ has the Darboux property. $\square$

Part II. In this part I will show that every a.e. continuous function with some special condition is the sum of two functions with condition (s_3).

Remark 3. *If $f \in S_4$ is almost everywhere continuous and approximately continuous at least unilaterally at the point x, then $f \in s_3(x)$.*

PROOF. Let $U \in T_d$ be the set containing x and let $\varepsilon > 0$. There is a point $t \in U \cap C(f)$ such that $|f(t) - f(x)| < \frac{\varepsilon}{2}$. Since $t \in C(f)$, there is an open interval I_1 such that $t \in I_1$ and $|f(u) - f(t)| < \frac{\varepsilon}{2}$ for all $u \in I_1$. Now, observe that for all $u \in I_1$ we have

$$|f(u) - f(x)| \leq |f(u) - f(t)| + |f(t) - f(x)| < \varepsilon. \tag{4}$$

Since $\emptyset \neq I_1 \cap U \in T_d$ and $f \in S_4$, there is an open interval $I_2 \subset I_1$ such that $\emptyset \neq I_2 \cap U \subset A(f)$. So, by (4) the function $f \in s_3(x)$. $\square$

Obviously the sum of two functions almost everywhere continuous belonging to S_4 is also an almost everywhere continuous function belonging to S_4. But the uniform limit of functions from the class S_4 need not be a function from S_4.

Example 3. Let $\{w_1, w_2, \ldots\}$ be a decreasing sequences of all rationals from the interval $[0, 1]$ and let $f_n : [0, 1] \to [0, 1]$ for $n = 1, 2, \ldots,$ be defined by

$$f_n(x) = \begin{cases} \frac{1}{n} & \text{for } x \in \{w_1, w_2, \ldots w_n\} \\ 1 & \text{for } x \in [0, 1] \setminus \{w_1, w_2, \ldots w_n\}. \end{cases}$$

Then, for all n, the function $f_n \in S_4$, the sequence (f_n) uniformly converges, but $\lim_{n\to\infty} f_n \notin S_4$.

Now, by using Grande's methods from the proof of theorem 2 in [3], I will prove the fundamental theorems of this part. We need the following lemmas below. Lemma 2 is a modification of Lemma 1.

Lemma 1. *(see [3]) If $A \subset \mathbb{R}$ is a nonempty compact set of Lebesgue measure zero, $U \supset A$ is an open set and $E \subset U \setminus A$ is a dense set in U, then there is a family $K_{i,j} \subset U \setminus A$, $i, j = 1, 2, \ldots$, of pairwise disjoint nondegenerate closed intervals with the endpoints belonging to E such that for each positive integer i and each point $x \in A$ the upper density*

$$d_u(\bigcup_{j=1}^{\infty} K_{i,j}, x) = 1 \tag{5}$$

and for each positive real ε the set of all pairs (i, j) for which $\operatorname{dist}(K_{i,j}, A) = \inf\{|x - y|; x \in K_{i,j}, y \in A\} \geq \varepsilon$ *is empty or finite.*

Lemma 2. *Let $U \subset \mathbb{R}$ be an open set. If $A \subset U$ is nonempty compact set of Lebesgue measure μ zero and there is an open set $V \subset U \setminus A$ such that $\mu(U \setminus V) = 0$ and $E \subset V$ is dense in V, then there is a family of pairwise disjoint nondegenerate closed intervals $K_{i,j} \subset V$, $i, j = 1, 2, \ldots$ with the endpoints belonging to E such that for each positive integer i and each point $x \in A$ condition* (5) *holds, and for each real $\varepsilon > 0$ the set of all pairs (i, j) for which* $\operatorname{dist}(K_{i,j}, A) \geq \varepsilon$ *is empty or finite.*

Proof. Observe that in the proof of Lemma 1 (see [3]) we can choose pairwise disjoint nondegenerate closed intervals $K_{i,j} \subset V \subset U \setminus A$ satisfying condition (5) or, if $K_{i,j}$ $(i, j = 1, 2 \ldots)$ is the family of pairwise disjoint nondegenerate closed intervals satisfying the conclusion of Lemma 1, consider the family $K_{i,j} \cap V$ $(i, j = 1, 2, \ldots)$ where $V \subset U \setminus A$ is an open set. In the set $K_{i,j} \cap V$ $(i, j = 1, 2, \ldots)$ we can choose a family $L^l_{i,j}$ $(l = 1, 2 \ldots, k(i, j))$ of pairwise disjoint nondegenerate closed intervals with the endpoints belonging to E such that $\mu(K_{i,j} \setminus \bigcup_{l=1}^{k(i,j)} L^l_{i,j}) = 0$ for $i, j = 1, 2, \ldots, l \leq k(i, j)$. Then, for each point $x \in A$ the family $L^l_{i,j}$ $(i, j = 1, 2, \ldots, l \leq k(i, j))$ satisfies the conclusion of Lemma 2. $\square$

Theorem 2. *If $f : \mathbb{R} \to \mathbb{R}$ is the sum of two functions $g, h \in S_3$, then f is almost everywhere continuous and satisfies:*

(a) the set $D_{ap}(f)$ is nowhere dense,

(b) *for every nonempty set $U \in T_d$ contained in* $\mathrm{cl}(D_{ap}(f))$ *the set* $U \cap D_{ap}(f)$ *is nowhere dense in* U.

PROOF. Since h, g are almost everywhere continuous, $f = g + h$ is the same. We have observed above that the sets $D_{ap}(g)$ and $D_{ap}(h)$ are nowhere dense. So, $D_{ap}(f) \subset D_{ap}(g) \cup D_{ap}(h)$ is also nowhere dense. Now we prove that $D_{ap}(f)$ satisfies condition (b). If $\mu(\mathrm{cl}(D_{ap}(f))) = 0$, then f satisfies condition (b). So, we assume that $\mu(\mathrm{cl}(D_{ap}(f))) > 0$ and fix a nonempty set $U \in T_d$ and an open interval I such that $I \cap U \neq \emptyset$. Since g has property (s_3) and $I \cap U$ is a nonempty set belonging to T_d, there is an open interval $I_1 \subset I$ such that $\emptyset \neq I_1 \cap U \subset A(g)$. Similarly, by property (s_3) of h, there is an open interval $I_2 \subset I_1$ such that $\emptyset \neq I_2 \cap U \subset A(h)$. But $f = g + h$; so $I_2 \cap U \subset A(f)$. $\square$

Theorem 3. *Suppose that* $f : \mathbb{R} \to \mathbb{R}$ *is the function almost everywhere continuous and* $\mu(\mathrm{cl}(D_{ap}(f))) = 0$. *Then there are functions* $g, h \in S_3$ *such that* $f = g + h$.

PROOF. First suppose that the set $D_{ap}(f)$ is bounded. Then $\mathrm{cl}(D_{ap}(f))$ is a compact set. If $\mu(\mathrm{cl}(D_{ap}(f)) = 0$, then by Lemma 1 there is a family $K_{i,j}(i, j = 1, 2, \ldots)$ of pairwise disjoint nondegenerate closed intervals

$$K_{i,j} \subset \mathbb{R} \setminus \mathrm{cl}(D_{ap}(f)), i, j \geq 1$$

with the endpoints belonging to $C(f)$ such that for each real $\varepsilon > 0$ the set of all pairs (i, j) for which $\mathrm{dist}(K_{i,j}, \mathrm{cl}(D_{ap}(f))) \geq \varepsilon$ is empty or finite and such that for each positive integer i and each point $x \in \mathrm{cl}(D_{ap}(f))$ the upper density $d_u(\bigcup_{j=1}^{\infty} K_{i,j}, x) = 1$. Let (w_i) be a sequence of all rationals and let

$$g(x) = \begin{cases} w_i & \text{for } x \in K_{2i-1,j}, i, j \geq 1 \\ f(x) - w_i & \text{for } x \in K_{2i,j}, i, j \geq 1 \\ f(x) & \text{otherwise on } \mathbb{R} \end{cases}$$

and

$$h(x) = \begin{cases} f(x) - w_i & \text{for } x \in K_{2i-1,j}, i, j \geq 1 \\ w_i & \text{for } x \in K_{2i,j}, i, j \geq 1 \\ 0 & \text{otherwise on } \mathbb{R}. \end{cases}$$

Evidently, $g + h = f$. Moreover the functions $g, h \in S_4$ and g, h are almost everywhere continuous. If $x \in \mathbb{R} \setminus \mathrm{cl}(D_{ap}(f))$, then g, h are approximately continuous at least unilaterally at x. So, by Remark 1, the functions $g, h \in s_3(x)$.

If $x \in \mathrm{cl}(D_{ap}(f))$, $x \in U \in T_d$ and $\varepsilon > 0$, then there is an index k such that $|f(x) - w_k| < \varepsilon$. Since $d_u(\bigcup_{j=1}^{\infty} K_{2k-1,j}, x) = 1$, there is an index m such that $\emptyset \neq \mathrm{int}(K_{2k-1,m}) \cap U \subset A(f)$. For $u \in \mathrm{int}(K_{2k-1,m} \cap U)$ we have $|g(u) - g(x)| = |w_k - f(x)| < \varepsilon$. Thus $g \in s_3(x)$. Similarly we can verify that $h \in s_3(x)$ for $x \in \mathrm{cl}(D_{ap}(f))$.

Now, suppose that $D_{ap}(f)$ is unbounded. Let $(a_k)(k = 0, \pm 1, \pm 2, \ldots)$ be a sequence of points of $\mathrm{int}(C(f))$ which converges to $-\infty$ as $k \to -\infty$ and to $+\infty$ as $k \to +\infty$. Then, for $k = 0, \pm 1, \pm 2, \ldots$ the set $D_{ap}(f) \cap (a_k, a_{k+1})$ is bounded and $\mathrm{cl}(D_{ap}(f) \cap (a_k, a_{k+1}))$ is a compact set. On each interval $[a_k, a_{k+1})$ $k = 0 \pm 1, \pm 2, \ldots$ we can define the functions $g_k, h_k \in S_3$ such that $f = h_k + g_k$ for $k = 0, -1, 1, -2, 2, \ldots$. For this we repeat the construction of the functions h_k, g_k on (a_k, a_{k+1}), for a fixed k, which was presented for the case of the set $D_{ap}(f)$ bounded in $\mathbb{R}$ but now, for fixed a k, in each interval (a_k, a_{k+1}), $U = U_k \subset (a_k, a_{k+1})$ and $\mathrm{cl}(D_{ap}(f) \cap (a_k, a_{k+1})) \subset U_k$. Finally, we define $g, h : \mathbb{R} \to \mathbb{R}$ by $h(x) = h_k(x), g(x) = g_k(x)$ for $x \in [a_k, a_{k+1})$ and $k = 0 \pm 1, \pm 2, \ldots$. Then, in this case, $h, g \in S_3$ and $f = h + g$. $\square$

Theorem 4. *Let $f \in S_4$ be an almost everywhere continuous function satisfying conditions $(a), (b)$ from Theorem 1 and the condition*

(c) $D_{ap}(f)$ is an $F_\sigma -$ set.

Then there are functions $g, h \in S_3$ such that $f = g + h$.

Proof. If $\mu(\mathrm{cl}(D_{ap}(f))) = 0$, the conclusion of the theorem follows from Theorem 3. So, let $\mu(\mathrm{cl}(D_{ap}(f))) > 0$. At first suppose that $D_{ap}(f)$ is bounded. Since $D_{ap}(f)$ is an F_σ-set, there is an increasing sequence of closed sets $F_1 \subset F_2 \subset \ldots$ such that $D_{ap}(f) = \bigcup_{i=1}^{\infty} F_i$. Let $(a_n)_n$ be a sequence of positive real numbers such that $a_n \searrow 0$ and $\sum_{n=1}^{\infty} a_n < \infty$. For $n = 1, 2, \ldots$ let

$$A_n = \{x; \operatorname{osc} f(x) \geq a_n\}.$$

The sets A_n $(n = 1, 2, \ldots)$ are closed sets of measure μ zero and $D(f) = \bigcup_{i=1}^{\infty} A_i$. Without loss of the generality we can assume that for $i = 1, 2, \ldots$ the set $F_i \cap A_i \neq \emptyset$, because if not, we can consider a subsequence of $(a_n)_n$. Let $H_i = F_i \cap A_i$ for $i = 1, 2, \ldots$. The sets H_i $(i = 1, 2, \ldots)$ are closed sets of μ measure zero and form an increasing sequence of subsets. We can assume that for each $i = 1, 2, \ldots$ $H_{i+1} \setminus H_i \neq \emptyset$, because if not, we can consider some subsequence of the sequence (H_i). Obviously $H_i \subset A_i$ for $i = 1, 2, \ldots$. By Lemma 2 there is a family of pairwise disjoint closed intervals $K_{1,i,j} \subset \mathbb{R} \setminus H_1$, $i, j = 1, 2, \ldots$, with endpoints belonging to $C(f)$ such that for each $i = 1, 2, \ldots$ and for each $x \in H_1$ the upper density $d_u(\bigcup_{j=1}^{\infty} K_{1,i,j}, x) = 1$ and for each real $\varepsilon > 0$ the set of pairs (i, j) for which $\mathrm{dist}(K_{1,i,j}, H_1) \geq \varepsilon$ is empty or finite. In the interiors $\operatorname{int}(K_{1,i,j})$ we find closed intervals $I_{1,i,j} \subset \operatorname{int}(K_{1,i,j})$ such that for each point $x \in A_1$ and for each integer $i = 1, 2, \ldots$ the upper density $d_u(\bigcup_{j=1}^{\infty} I_{1,i,j}, x) = 1$. Let $(w_{1,i})_i$ be a sequence of all rationals and let $g_1, h_1 : \mathbb{R} \to \mathbb{R}$ be defined by

$$g_1(x) = \begin{cases} w_{1,i} & \text{for } x \in I_{1,2i,j},\ i,j = 1,2,\ldots \\ f(x) & \text{for } x \in \mathbb{R} \setminus \bigcup_{i,j=1}^{\infty} \operatorname{int}(K_{1,2i,j}) \\ & \text{linear on the components of the sets} \\ & K_{1,2i,j} \setminus \operatorname{int}(I_{1,2i,j}),\ i,j = 1,2,\ldots \end{cases}$$

and $h_1(x) = f(x) - g_1(x)$ for $x \in \mathbb{R}$. As in the proof of Theorem 1 we can prove that $g_1, h_1 \in S_3(x)$ for $x \in H_1$ and

$$A(f) \subset A(g_1) \cap A(h_1), C(f) \subset C(g_1) \cap C(h_1).$$

In the second step we consider the set $A_2 \setminus A_1 = A_2 \cap (\mathbb{R} \setminus A_1)$. There are pairwise disjoint open intervals $P_{2,k} \subset \mathbb{R} \setminus A_1, k \geq 1$, with the centers belonging to $C(f)$ such that every set $A_2 \cap P_{2,k}$ is nonempty and compact and $A_2 \setminus A_1 = \bigcup_k (A_2 \cap P_{2,k})$. A construction of such intervals $P_{2,k}$ may be the following. We find a bounded open set $G \supset A_2$ and divide each component of the the set $G \setminus A_1$ by points belonging to $C(f)$ into open intervals. As $P_{2,k}$ we take all from the above intervals which have common points with A_2.

If $x \in (A_2 \cap \operatorname{int}(K_{1,2i,j})) \setminus A_1$ for some pair (i, j), then g_1 is continuous at x, and consequently $\operatorname{osc} g_1(x) = 0$ and $\operatorname{osc} h_1(x) = \operatorname{osc} f(x) < a_1$. If

$$x \in A_2 \setminus A_1 \setminus \bigcup_{i,j \geq 1} K_{1,2i,j},$$

then $g_1(t) = f(t)$ and $h_1(t) = 0$ on an open interval containing x and contained in $\mathbb{R} \setminus A_1$. So $\operatorname{osc} g_1(x) = \operatorname{osc} f(x) < a_1$ and $\operatorname{osc} h_1(x) = 0$. Similarly we show that $\max(\operatorname{osc} g_1(x), \operatorname{osc} h_1(x)) < a_1$ if $x \in A_2 \setminus A_1$ is an endpoint of some $K_{1,2i,j}$. So for each integer k and each point $x \in A_2 \cap P_{2,k}$ there is an open interval $J_{2,k}(x) \subset P_{2,k}$ containing x such that on the interval $J_{2,k}(x)$ the oscillation $\operatorname{osc}_{J_{2,k}(x)} g_1 < a_1$ and $\operatorname{osc}_{J_{2,k}(x)} h_1 < a_1$. Since the set $A_2 \cap P_{2,k}$ is compact, there are points $x_1, x_2, \ldots, x_{j(k)}$ such that

$$A_2 \cap P_{2,k} \subset J_{2,k}(x_1) \cup \ldots \cup J_{2,k}(x_{j(k)}).$$

Without loss of the generality we can assume that the above intervals $J_{2,k}(x_j)$, $j \leq j(k)$, are pairwise disjoint. For each pair of positive integers (i, j) such that $A_2 \cap K_{1,i,j} \neq \emptyset$ we find an open set $U(K_{1,i,j}) \subset \operatorname{int}(K_{1,i,j})$ such that

$A_2 \cap K_{1,i,j} \subset U(K_{1,i,j})$ and $\frac{\mu(\mathrm{cl}(U(K_{1,i,j})))}{\mu(K_{1,i,j})} < \frac{1}{4^{1+i+j}}$. If for some integers i_1, j_1, j_2 the intersection $A_2 \cap \mathrm{int}(K_{1,i_1,j_1}) \cap J_{2,k}(x_{j_2}) \neq \emptyset$ then, by Lemma 2, we find pairwise disjoint nondegenerate closed intervals

$$K_{2,i,j}(K_{1,i_1,j_1}, J_{2,k}(x_{j_2})) \subset U(K_{1,i_1,j_1}) \cap J_{2,k}(x_{j_2}) \setminus H_2$$

with the endpoints belonging to $C(f)$ such that for every positive integer i and every point $x \in H_2 \cap J_{2,k}(x_{j_2}) \cap K_{1,i_1,j_1}$ the upper density

$$d_u(\bigcup_{j=1}^{\infty} K_{2,i,j}(K_{1,i_1,j_1}, J_{2,k}(x_{j_2})), x) = 1$$

and for every real $\varepsilon > 0$ the set of all pairs (i,j) for which

$$\mathrm{dist}(H_2 \cap J_{2,k}(x_{j_2}) \cap K_{1,i_1,j_1}, K_{2,i,j}(K_{1,i_1,j_1}, J_{2,k}(x_{j_2}))) > \varepsilon$$

is empty or finite.

In every interval $\mathrm{int}(K_{2,i,j}(K_{1,i_1,j_1}, J_{2,k}(x_{j_2})))$ we find a closed interval $I_{2,i,j}(K_{2,i,j}(K_{1,i_1,j_1}, J_{2,k}(x_{j_2}))$ such that for every integer i and for every point $x \in H_2 \cap J_{2,k}(x_{j_2}) \cap K_{1,i_1,j_1}$ the upper density

$$d_u(\bigcup_{j=1}^{\infty} I_{2,i,j}(K_{2,i,j}(K_{1,i_1,j_1}, J_{2,k}(x_{j_2}))), x) = 1. \tag{6}$$

For each positive integer $j \leq j(k)$ let $(w_i(x_j))$ be an enumeration of all rationals of the interval $(y_j - \frac{a_1}{2}, y_j + \frac{a_1}{2})$, where y_j is the center of the interval $[\inf_{H_2 \cap J_{2,k}(x_j)} g_1, \sup_{H_2 \cap J_{2,k}(x_j)} g_1]$, and let $(u_i(x_j))$ be an enumeration of all rationals of the interval $(z_j - \frac{a_1}{2}, z_j + \frac{a_1}{2})$, where z_j is the midpoint of the interval $[\inf_{H_2 \cap J_{2,k}(x_j)} h_1, \sup_{H_2 \cap J_{2,k}(x_j)} h_1]$. Put

$$g_2(x) = w_i(x_{j_2}) \text{ and } h_2(x) = f(x) - g_2(x)$$

$$\text{for } x \in I_{2,2i,j}(K_{1,i_1,j_1}, J_{2,k}(x_{j_2})),\ j_2 \leq j(k),\ i,j = 1,2,\ldots,$$

$$h_2(x) = u_i(x_{j_2}) \text{ and } g_2(x) = f(x) - h_2(x)$$

$$\text{for } x \in I_{2,2i-1,j}(K_{1,i_1,j_1} J_{2,k}(x_{j_2})),\ j_2 \leq j(k),\ i,j = 1,2,\ldots,$$

$$g_2(x) = g_1(x) \text{ and } h_2(x) = h_1(x)$$

$$\text{for } x \in K_{1,i_1,j_1} \setminus \bigcup_{j_2 \leq j(k)} \bigcup_{i,j=1}^{\infty} K_{2,i,j}(K_{1,i_1,j_1}, J_{2,k}(x_{j_2})),$$

and assume that the function g_2 is linear and $h_2 = f - g_2$ on the components of the sets $K_{2,i,j}(K_{1,i_1,j_1}, J_{2,k}(x_{j_2})) \setminus I_{2,i,j}(K_{1,i_1,j_1}, J_{2,k}(x_{j_2}))$. In the same way, modifying the values of g_1 and h_1 on respectively constructed closed intervals, we define the functions g_2 and h_2 on components $L_{2,m}$ of the set $P_{2,k} \setminus H_1 \setminus \bigcup_{i,j=1}^{\infty} K_{1,i,j}$ for which $L_{2,m} \cap A_2 \neq \emptyset$. Put $g_2(x) = g_1(x)$ and $h_2(x) = h_1(x)$ otherwise on $\mathbb{R}$. Observe that if the function f is continuous at a point x, then from the constructions of g_1 and g_2 it follows that $x \in \mathbb{R} \setminus A_2$, and g_1 and g_2 are continuous at x. Consequently, the functions h_1 and h_2 as the differences of functions continuous at x, are also continuous at this point. So, $C(f) \subset C(g_2) \cap C(h_2)$. Similarly $A(f) \subset A(g_2) \cap A(h_2)$. Moreover it is evident that $|g_2 - g_1| \leq a_1$, $|h_2 - h_1| \leq a_1$ and $g_2 + h_2 = f$. We will show that $g_2, h_2 \in s_3(x)$ for $x \in H_2$. For this fix a point $x \in H_2$, a set $U \ni x$ belonging to T_d and a real $\varepsilon > 0$. If $x \in H_1$, then we find a rational $w_{1,k}$ with $|g_1(x) - w_{1,k}| < \varepsilon$. Since $d_u(\bigcup_{j=1}^{\infty} I_{1,2k,j}, x) = 1$ and $\frac{\mu(\mathrm{cl}(U(K_{1,2k,j})))}{\mu(K_{1,2k,j})} < \frac{1}{4^{1+2k+j}}$, we obtain $d_u((g_1)^{-1}(w_{1,k}) \cap \bigcup_{j=1}^{\infty} I_{1,2k,j}, x) = 1$ and consequently there is an integer m and an open interval $I \subset I_{1,2k,m} \setminus \mathrm{cl}(U(K_{1,2k,m}))$ such that $\emptyset \neq I \cap U$. But $g_2(u) = w_{1,k}$ for $u \in I \cap U$, so $I \cap U \subset C(g_2) \subset A(g_2)$. Moreover for $u \in I \cap U$ we have $|g_2(u) - g_2(x)| = |w_{1,k} - g_2(x)| < \varepsilon$. So $g_2 \in s_3(x)$ for $x \in H_1$. Similarly we show that $h_2 \in s_3(x)$ for $x \in H_1$.

Using (6) by similar reasoning we can show that $g_2, h_2 \in s_3(x)$ for $x \in H_2 \setminus H_1$. Let $(K_{2,i,j})$ be a double sequence of all closed intervals on which we have modified the functions g_1 and h_1 to obtain g_2 and h_2. Similarly, in the n^{th} step, we change the functions g_{n-1} and h_{n-1} on respectively taken closed intervals $K_{n,2i,j}$ and $K_{n,2i-1,j}$ and define functions g_n and h_n such that g_n (and respectively h_n) has constant rational values on respective closed intervals $I_{n,2i,j} \subset \mathrm{int}(K_{n,2i,j})$ (resp. on $I_{n,2i-1,j}$), $C(f) \subset C(g_n) \cap C(h_n)$, $A(f) \subset A(g_n) \cap A(h_n)$, $g_n, h_n \in s_3(x)$ for $x \in H_n$, $|g_n - g_{n-1}| \leq a_{n-1}$, $|h_n - h_{n-1}| \leq a_{n-1}$ and $g_n + h_n = f$. Moreover, we suppose that for every triple (k, i_1, j_1), where $k < n$ and $i_1, j_1 = 1, 2, \ldots$,

$$\frac{\mu(K_{k,i_1,j_1} \setminus \bigcup_{i,j=1}^{\infty} K_{n,i,j})}{\mu(K_{k,i_1,j_1})} > 1 - \frac{1}{4^{n+i+j}}. \tag{7}$$

Let $g = \lim_{n\to\infty} g_n$ and $h = \lim_{n\to\infty} h_n$. Observe that the above limits are uniform. Evidently, $g + h = f$. Since $f \in S_4$ and $A(f) \subset A(g) \cap A(h)$, the functions g, h have the property (s_4).

We will prove that the functions g, h have the property (s_3). For this, fix a real $\varepsilon > 0$, a point $x \in \mathbb{R}$ and a set $U \in T_d$ containing x. If $x \in C(f)$, then g is continuous at x and there is a real $\eta > 0$ such that $|g(t) - g(x)| < \varepsilon$ for $t \in (x - \eta, x + \eta)$. But g has the property (s_4); so there are an open interval $J \subset (x - \eta, x + \eta)$ such that $A(g) \supset J \cap U \neq \emptyset$. Since $|g(t) - g(x)| < \varepsilon$ for

$t \in J \cap U$, we obtain $g \in s_3(x)$. Similarly we can prove that $h \in s_3(x)$.

If $x \in A(f)$,, then $x \in A(g) \cap A(h)$ and because the functions $g, h \in S_4$ and they are approximately continuous at x, from Remark 3 it follows that $g, h \in s_3(x)$.

Suppose that $x \in D_{ap}(g) \cap D_{ap}(h)$. Then there is an positive integer n such that $x \in H_n \setminus H_{n-1}$ (where $H_0 = \emptyset$). Let $k > n$ be a positive integer such that $\sum_{i=k+1}^{\infty} a_i < \frac{\varepsilon}{3}$. There is a rational value w of the function g_n such that $|g_n(x) - w| < \frac{\varepsilon}{3}$ and $d_u((g_n)^{-1}(w), x) = 1$. By condition (7) the upper density

$$d_u((g_n)^{-1}(w) \setminus \bigcup_{m>n} \bigcup_{l,j=1}^{\infty} K_{m,l,j}, x) = 1.$$

So

$$d_u((g_n)^{-1}(w) \setminus \bigcup_{m=n+1}^{k-1} \bigcup_{l,j=1}^{\infty} K_{m,l,j}, x) = 1,$$

and by the construction of g_n and $K_{m,l,j}$ also

$$d_u(\mathrm{int}(g_n)^{-1}(w) \setminus \bigcup_{m=n+1}^{k-1} \bigcup_{l,j=1}^{\infty} K_{m,l,j}), x) = 1.$$

Since $x \in U \in T_d$, we have

$$d_u(U \cap \mathrm{int}(g_n)^{-1}(w) \setminus \bigcup_{m=n+1}^{k-1} \bigcup_{l,j=1}^{\infty} K_{m,l,j}), x) = 1.$$

Consequently, there is an open interval

$$I \subset \mathrm{int}(g_n)^{-1}(w) \setminus \bigcup_{m=n+1}^{k-1} \bigcup_{l,j=1}^{\infty} K_{m,l,j}) \setminus A_k$$

such that $I \cap U \neq \emptyset$. Evidently, $\emptyset \neq I \cap U \subset A(f) \subset A(g)$. For $t \in I \cap U$ we obtain $g_n(t) = g_k(t)$ and

$$|g(t) - g(x)| = |g(t) - g_k(t) + w - g_n(x)| \le \sum_{i=k+1}^{\infty} a_i + \frac{\varepsilon}{3} < \frac{2\varepsilon}{3} < \varepsilon.$$

So $g \in s_3(x)$. The proof that $h \in s_3(x)$ is analogous.

Up to now we have supposed that the set $D(f)$ is bounded. Now we consider the general case. Since $D(f)$ is a first category set, there are points $x_k \in$

$\mathbb{R}\setminus D(f)$, $k = 0, 1, -1, 2, -2, \ldots$ such that $\lim_{k\to-\infty} x_k = -\infty$, $\lim_{k\to\infty} x_k = \infty$ and $x_k < x_{k+1}$ for all integers k. Then $\mathbb{R} = \bigcup_{k=-\infty}^{\infty}[x_k, x_{k+1}]$. Every restricted function $f_k = f/[x_k, x_{k+1}]$ is the sum of two functions $g_k, h_k : [x_k, x_{k+1}] \to \mathbb{R}$ having the property (s_3) and continuous at the points x_k and x_{k+1}. Let

$$g(x) = \begin{cases} g_k(x) - (a_1 + \cdots + a_k) & \text{for } x \in [x_k, x_{k+1}],\ k \geq 1 \\ g_0(x) & \text{for } x \in [0, 1] \\ g_k(x) + (a_0 + a_{-1} + \cdots + a_{k+1}) & \text{for } x \in [x_k, x_{k+1}],\ k \leq -1, \end{cases}$$

where $a_k = g_k(k) - g_{k-1}(k)$ for $k = 0 \pm 1, \pm 2, \ldots$ and $h(x) = f(x) - g(x)$ for $x \in \mathbb{R}$. Observe that the functions g and h have property (s_3) and $f = g+h$. $\square$

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