Real Analysis Exchange
Vol. 29(2), 2003/2004, pp. 543–555

Luisa Di Piazza, Department of Mathematics and Applications, University of Palermo, via Archirafi 34, 90123 Palermo, Italy.
email: dipiazza@math.unipa.it

KURZWEIL-HENSTOCK TYPE INTEGRATION ON BANACH SPACES

Abstract

In this paper properties of Kurzweil-Henstock and Kurzweil-Henstock-Pettis integrals for vector valued functions are studied. In particular, the absolute integrability for Kurzweil-Henstock integrable functions is characterized and a Kurzweil-Henstock version of the Vitali Theorem for Pettis integrable functions is given.

1 Introduction

In this paper we continue the investigation of properties concerning Kurzweil-Henstock type integrals for vector valued functions, started in [4] and [6]. In particular we consider the Kurzweil-Henstock integral and the Kurzweil-Henstock-Pettis integral. The last one is a generalization of the Pettis integral obtained by replacing the Lebesgue integrability of the functions by the Kurzweil-Henstock integrability. In general it integrates a family of functions larger than the Kurzweil-Henstock integrable one (see Remark 1). Moreover concerning its relations with the Kurzweil-Henstock integral, the Kurzweil-Henstock-Pettis integral sometimes does not share with the Pettis integral properties analogous to ones relating to the relations between Pettis and McShane integrals. Indeed for strongly measurable functions or when the Banach space is separable, the Pettis and the McShane integrals coincide (see [13] and [9]). Analogous properties fail for the Kurzweil-Henstock-Pettis and the Kurzweil-Henstock integrals (see example in Remark 1).

In Section 3 we characterize the Kurzweil-Henstock integrable functions by the notion of equiintegrability. In Section 4 we study the absolute integrability of the Kurzweil-Henstock integrable functions and, if the unit ball of the dual

Key Words: Kurzweil-Henstock integral; Pettis integral; equiintegrability
Mathematical Reviews subject classification: 2000 Mathematics Subject Classification: Primary 28B05; Secondary 26A39
Received by the editors February 5, 2003
Communicated by: P. S. Bullen

of the range space is weak* separable, we find a characterization in terms of the bounded variation of the indefinite Pettis integral (see Theorem 3). In the last Section 5 we give some convergence theorems for both, the Kurzweil-Henstock and the Kurzweil-Henstock-Pettis integrals, based on the notion of equiintegrability.

Although all our results have n-dimensional analogues, here we work for simplicity in the case of functions defined on $[0,1]$ only.

2 Notations and Preliminaries

Let $[0,1]$ be the unit interval of the real line equipped with the usual topology and the Lebesgue measure. $\mathcal{M}$ denotes the family of all Lebesgue measurable subset of $[0,1]$. If $E \in \mathcal{M}$, then $|E|$ denotes the Lebesgue measure of E. A *tagged partition in* $[0,1]$, or simply a *partition in* $[0,1]$ is a finite collection of pairs $\mathcal{P} = \{(I_1,t_1),\ldots,(I_p,t_p)\}$, where $I_1,\ldots,I_p$ are nonoverlapping subintervals of $[0,1]$ and t_i is a point of $[0,1]$, $i = 1,\ldots,p$. If $t_i \in I_i$, $i = 1,\ldots,p$ we call $\mathcal{P}$ a *Perron partition*. Given a subset E of $[0,1]$, we say that the partition $\mathcal{P}$ is *anchored on* E if $t_i \in E$ for each $i = 1,...,p$. If $\cup_{i=1}^{p} I_i = [0,1]$ we say that $\mathcal{P}$ is a *partition of* $[0,1]$. A *gauge* on $E \subset [0,1]$ is a positive function on E. For a given gauge δ, we say that a partition $\{(I_1,t_1),\ldots,(I_p,t_p)\}$ is δ-*fine* if $I_i \subset (t_i - \delta(x_i), t_i + \delta(x_i))$, $i = 1,\ldots,p$.

Throughout this paper X is a Banach space with dual X^*. The closed unit ball of X^* is denoted by $\mathcal{B}(X^*)$.

A function $g : [0,1] \to X$ is said to be: *weakly measurable* if for each $x^* \in X^*$ the real function x^*g is measurable; *absolutely measurable* if the real function $\|g\|$ is measurable; *strongly measurable*, or simply *measurable* if, there is a sequence of simple functions g_n with $\lim_n \|g_n(t) - g(t)\| = 0$, for almost all $t \in [0,1]$.

Let $g : [0,1] \to X$ be a function. We set $\sigma(g,\mathcal{P}) = \sum_{i=1}^{p} g(t_i)|I_i|$ for each partition $\mathcal{P} = \{(I_1,t_1),\ldots,(I_p,t_p)\}$ of $[0,1]$.

3 Kurzweil-Henstock Integrals

Definition 1. A function $g : [0,1] \to X$ is said to be *Kurzweil-Henstock integrable*, or simply *KH-integrable*, on $[0,1]$ if there exists $w \in X$ with the following property: for every $\epsilon > 0$ there exists a gauge δ on $[0,1]$ such that $||\sigma(g,\mathcal{P}) - w|| < \varepsilon$ for each δ-fine Perron partition $\mathcal{P}$ of $[0,1]$. We set $w =: (KH)\int_0^1 g$.

We denote the set of all *KH*-integrable functions $g : [0,1] \to X$ by $\mathcal{KH}([0,1],X)$.

We start by recalling some well known facts concerning the *KH*-integral (see [2], [8], [15]).

Theorem 1. *Let $g : [0,1] \to X$ be KH-integrable on $[0,1]$ and let $G(t) = (KH)\int_0^t g$, for each $t \in [0,1]$.*

(a) For each $x^* \in X^*$ the function x^*g is KH-integrable on each interval $J \subset [0,1]$ and $(KH)\int_J x^*g = x^*(KH)\int_J g$.

(b) The function g is scalarly measurable.

(c) If $g = 0$ almost everywhere on $[0,1]$, then g is KH-integrable with integral equal to zero.

The generalization of the Pettis integral obtained by replacing the Lebesgue integrability of the functions by the Kurzweil-Henstock integrability produces the Kurzweil-Henstock-Pettis integral (for the definition of Pettis integral see [3]).

Definition 2. A function $g : [0,1] \to X$ is said to be *scalarly Kurzweil-Henstock integrable*, or simply *scalarly KH-integrable*, if for each $x^* \in X^*$, the function x^*g is Kurzweil-Henstock integrable on $[0,1]$. If for each subinterval $[a,b]$ of $[0,1]$ there exists a vector $w_{[a,b]} \in X$ such that $x^*w_{[a,b]} = (HK)\int_a^b x^*g$, then g is said to be *Kurzweil-Henstock-Pettis integrable*, or simply *KHP-integrable*, on $[0,1]$ and we set $w_{[a,b]} =: (KHP)\int_a^b g$.

We denote the set of all *KHP*-integrable functions $g : [0,1] \to X$ by $\mathcal{KHP}([0,1], X)$.

Remark 1. By (a) of Theorem 1 it follows that each *KH*-integrable function is also *KHP*-integrable. The reverse implication is not true. In fact let us consider the following example. Let $A_n = [a_n, b_n] \subseteq [0,1]$ be a sequence of intervals such that $a_1 = 0$, $b_n < a_{n+1}$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} b_n = 1$ and define $g : [0,1] \to c_0$ by

$$g(t) = \left(\frac{1}{2|A_{2n-1}|}\chi_{A_{2n-1}}(t) - \frac{1}{2|A_{2n}|}\chi_{A_{2n}}(t)\right)_{n=1}^{\infty}.$$

In [10] it is proved that g is a measurable KHP-integrable function which is not Pettis integrable. In [5] it is showed that the same function is not Kurzweil-Henstock integrable. Then the family of all Kurzweil-Henstock-Pettis integrable functions is larger than the family of all Kurzweil-Henstock integrable ones. □

Definition 3. Let Y be a Banach space. A family $\mathcal{A} \subset \mathcal{KH}([0,1], Y)$ is said to be *Kurzweil-Henstock equiiintegrable*, or simply *KH-equiintegrable*, on $[0,1]$ if for every $\epsilon > 0$ there exists a gauge δ on $[0,1]$ such that

$$\sup_{g \in \mathcal{A}} \left|\left|\sigma(g, \mathcal{P}) - (KH)\int_0^1 g\right|\right| < \varepsilon \,.$$

for each δ-fine Perron partition $\mathcal{P}$ of $[0,1]$.

Using the notion of *KH*-equiintegrability, we may characterize the vector valued *KH*-integrable functions.

Theorem 2. *A function $g : [0,1] \to X$ is KH-integrable on $[0,1]$ if and only if the family $\{x^*g : \ x^* \in \mathcal{B}(X^*)\}$ is KH-equiintegrable on $[0,1]$.*

PROOF. Since

$$||\sigma(g, \mathcal{P}) - w|| = \sup_{x^* \in \mathcal{B}(X^*)} |\sigma(x^*g, \mathcal{P}) - x^*w|$$

the "only if" part follows.

To get the "if" part it suffices to show that there exists $w \in X$ such that $(KH)\int_0^1 x^*g = x^*w$ for all $x^* \in X^*$. We define the linear functional $T_g : X^* \to \mathbb{R}$, by setting $T_g(x^*) = (KH)\int_0^1 x^*g$. We start by proving that T_g is w^*-continuous; i.e., that for each real α both the sets $Q(\alpha) := \{x^* \in X^* : T_g(x^*) \leq \alpha\}$ and $P(\alpha) := \{x^* \in X^* : T_g(x^*) \geq \alpha\}$ are w^*-closed. We consider first $Q(\alpha)$. Since $Q(\alpha)$ is convex, according to the Banach-Dieudonné Theorem it suffices to show that $Q(\alpha) \cap B(X^*)$ is w^*-closed. Let x_0^* be a w^*-cluster point of $Q(\alpha) \cap \mathcal{B}(X^*)$ and let $(x_\gamma^*)_{\gamma \in \mathcal{I}} \subset Q(\alpha) \cap B(X^*)$ be a net converging to x_0^* in the w^*-topology. Now by the assumption of *KH*-equiintegrability, for each given $\varepsilon > 0$ we find a gauge δ and a δ-fine Perron partition $\{(I_1, t_1), \ldots, (I_p, t_p)\}$ of $[0,1]$ such that

$$\sup_{\|x^*\| \leq 1} \left| (KH)\int_0^1 x^*g\,dt - \sum_{i=1}^p x^*g(t_i)|I_i| \right| < \varepsilon \,. \tag{1}$$

Moreover, using the convergence of $(x_\gamma^*)_{\gamma \in \mathcal{I}}$ we choose an index $\gamma_0 \in \mathcal{I}$ such that

$$\sum_{i=1}^p \left|\, x_{\gamma_0}^* g(t_i) - x_0^* g(t_i) \,\right| < \varepsilon \,. \tag{2}$$

Since $x_0^* \in \mathcal{B}(X^*)$, by (1) and (2) we have

$$\begin{aligned} T_g(x_0^*) \leq & |T_g(x_0^*) - \sum_{i=1}^{p} x_0^* g(t_i)|I_i|| + \sum_{i=1}^{p} |x_0^* g(t_i) - x_{\gamma_0}^* g(t_i)||I_i| \\ & + |\sum_{i=1}^{p} x_{\gamma_0}^* g(t_i)|I_i| - (KH)\int_0^1 x_{\gamma_0}^* g| + T_g(x_{\gamma_0}^*) < \alpha + 3\varepsilon. \end{aligned}$$

Since ε is arbitrary, we deduce that $x_0^* \in Q(\alpha) \cap B(X^*)$. Changing x^* into $-x^*$, we have that also $P(\alpha) \cap B(X^*)$ is w^*-closed. Consequently, the functional T_g is w^*-continuous. Then according to fact that X is the w^*-dual of X^*, there exists $w \in X$ such that $T_g(x^*) = x^* w$, and this ends the proof. □

Remark 2. In the Definitions 1 and 3 if we replace the term "Perron partition" by "partition" we obtain the definitions of *McShane integrability* and of *McShane equiintegrability*, respectively. For the McShane integral, a result analogous to that in Theorem 2 holds, using of course the notion of McShane equiintegrability. This result has been proved in [23] Lemma 18, under the additional hypothesis that the function g is Pettis integrable.

4 Absolute Integrability.

We recall that a function g is called *KH-absolutely integrable* if both g and $\|g\|$ are *KH*-integrable.

Proposition 1. *Let $g : [0,1] \to X$ be an absolutely KH-integrable function on $[0,1]$. Then g is Pettis integrable and for any $E \in \mathcal{M}$ we have*

$$\left\|(KH)\int_E g\right\| \leq \int_E \|g\| \ . \tag{3}$$

PROOF. The Pettis integrability of g is proved in [4] Proposition 3. Moreover for any $E \in \mathcal{M}$ we have

$$\left\|(KH)\int_E g\right\| = \sup_{x^* \in \mathcal{B}(X^*)} \left|\int_E x^* g\right| \leq \sup_{x^* \in \mathcal{B}(X^*)} \int_E |x^* g| \leq \int_E \|g\|$$

and this ends the proof. □

Remark 3. As it has been observed in [4] the Pettis (and then the McShane integrability) of a *KH*-integrable function may not suffice to guarantee the absolute *KH*-integrability of the function. Indeed there are strongly measurable Pettis integrable functions that are not Bochner.

For real valued functions, the absolute integrability of a KH-integrable function g is characterized by the notion of bounded variation of its primitive $G(t) = (KH)\int_0^t g$ (see [24], Theorem 3.4.1).

For functions taking values in an infinite dimensional Banach space, the previous condition is only necessary (taking into account inequality (3), the necessity can be proved as in [24] after trivial changes), but in general it is not sufficient, as the following example, given in [14] Example 14, shows. Let $E \subset [0,1]$ be nonmeasurable and define $g : [0,1] \to L^\infty([0,1])$ by $g(t) = \chi_{\{t\}}$ if $t \in E$, $g(t) = \phi$ if $t \notin E$, where ϕ is the null function in $[0,1]$. Then g is Riemann integrable (the reader should see [14] for the definition of Riemann integral) and consequently KH-integrable. Moreover $G(t) = (KH)\int_0^t g$ is absolutely continuous on $[0,1]$ (see [14], Theorem 8). But $\|g\| = \chi_E$ is not measurable.

We need the following proposition (see also Theorem 4.1 and Remark 4.1 of [20]).

Proposition 2. *Let $g : [0,1] \to X$ be an absolutely measurable, Pettis integrable function. Then for each $E \in \mathcal{M}$ we have $|\nu_g|(E) \le \int_E \|g\|$, where ν_g is the indefinite Pettis integral of g and $|\nu_g|$ is its variation. Moreover, if the Banach space X is such that the unit ball of X^* is weak* separable, then $|\nu_g|(E) = \int_E \|g\|$ for each $E \in \mathcal{M}$.*

PROOF. If $E \in \mathcal{M}$, then for each $x^* \in \mathcal{B}(X^*)$,

$$|x^*\nu_g(E)| \le \int_E |x^*g| \le \int_E \|g\|.$$

Hence $|\nu_g|(E) \le \int_E \|g\|$.

Let assume now the weak* separability of $\mathcal{B}(X^*)$ and let $D = \{x_j^*\}$ be a dense countable subset of $\mathcal{B}(X^*)$ in the weak* topology. The Pettis integrability of g implies that the measure $|\nu_g|$ is absolutely continuous with respect to the Lebesgue measure. Then by the Radon-Nikodym Theorem there exists a non-negative measurable function h on $[0,1]$ such that

$$|\nu_g|(E) = \int_E h \ , \tag{4}$$

for every $E \in \mathcal{M}$. Then we obtain

$$h \le \|g\| \tag{5}$$

almost everywhere on $[0,1]$. Moreover, since for each $x^* \in \mathcal{B}(X^*)$ the function x^*g is Lebesgue integrable, we have $|x^*\nu_g|(E) = \int_E |x^*g| \le |\nu_g|(E) = \int_E h$, for each $E \in \mathcal{M}$ and for each $x^* \in \mathcal{B}(X^*)$. Therefore there exists a set $N \subset [0,1]$ of zero measure such that $|x^*_j g(t)| \leq h(t)$, for each index j and for all $t \in [0,1] \setminus N$. As D is weak* dense in $\mathcal{B}(X^*)$, we get $|x^* g(t)| \leq h(t)$, for each $x^* \in \mathcal{B}(X^*)$ and for all $t \in [0,1] \setminus N$. Hence

$$\|g\| \leq h \tag{6}$$

almost everywhere on $[0,1]$. By (4), (5) and (6) we infer $|\nu_g|(E) = \int_E \|g\|$, for each $E \in \mathcal{M}$ and this ends the proof. □

Theorem 3. *Let X be a Banach space such that the unit ball of X^* is weak* separable and let $g : [0,1] \to X$ be KH-integrable on $[0,1]$. Then g is absolutely KH-integrable on $[0,1]$ if and only if g is absolutely measurable, Pettis integrable and its indefinite Pettis integral is of bounded variation.*

PROOF. The "only if" part follows from Propositions 1 and 2, by observing that if a function g is absolutely integrable, then it is absolutely measurable. The "if" part follows at once from Proposition 2. □

Remark 4. We observe that the previous theorem is not true for arbitrary Banach spaces. Indeed it is enough to consider the function $g : [0,1] \to l^2([0,1])$ defined by $g(t) = \frac{1}{t}\chi_{\{t\}}$ if $t \neq 0$, $g(t) = \phi$ if $t = 0$, where ϕ is the null function in $[0,1]$. Such a function is "scalarly negligible" and therefore Pettis integrable with Pettis integral equal to zero over any measurable set. (By the term "scalarly negligible" we mean that for each $x^* \in X^*$, $x^* g(t) = 0$ almost everywhere). Since $l^2([0,1])$ is a super-reflexive Banach space, g is also McShane integrable (see Theorem 1 of [6]), and then *KH*-integrable (see [8]). Moreover $\|g(t)\| = \frac{1}{t}$ for all $t \in (0,1]$. Then g is absolutely measurable, but not absolutely integrable. □

5 Convergence Theorems

The notion of *KH*-equiintegrability was first introduced in [17] for real valued functions and permitting the proof of convergence theorem for pointwise convergent sequence of *KH*-integrable functions (see also [18], [19], [15], [12] and [1]).

Recently (see [22] Theorem 1) it has been showed that a pointwise convergent sequence of *KH*-equiintegrable real valued functions converges in the Alexiewicz norm to the pointwise limit, and the result may be easily extended to Banach valued functions.

Our aim in this section is to get for the Kurzweil-Henstock-Pettis integral a convergence result analogous to the one we have for the Pettis integral (see

[11] and [20]), using the notion of *KH*-equiintegrability instead of the notion of uniform integrability (for the Lebesgue integral). To this purpose we cannot directly apply the Banach valued version of Theorem 1 in [22], but the assumption of "pointwise convergence" needs to be weakened to the assumption of "convergence almost everywhere".

As Gordon observed (see [15] p. 209 and [12]), the concept of *KH*-equiintegrability, unlike the concept of uniform integrability, does not allow one to ignore sets of measure zero. The following Theorem 4 is a variant of Theorem 1 in [22] in which the assumption of "pointwise convergence" is relaxed considering sequences of pointwise bounded functions.

We recall that if $g \in \mathcal{KH}([0,1],X)$, the Alexiewicz norm of g is defined by $\|g\|_{KH} = \sup\{\|(KH)\int_0^t g\| : 0 \le t \le 1\}$.

The following lemma may be proved in a standard way (when $X = \mathbb{R}$ and the family $\mathcal{A}$ contains only a function see [15] p. 323).

Lemma 1. *Let Y be a Banach space, let $\mathcal{A}$ be a pointwise bounded family of functions $g : [0,1] \to Y$ and let $M \subset [0,1]$. If $|M| = 0$, then for each $\varepsilon > 0$, there exists a gauge δ on M such that $\sup_{g \in > \mathcal{A}} \sigma(\|g\|, \mathcal{P}) < \varepsilon$, for each partition $\mathcal{P}$ anchored in M.*

Theorem 4. *Let $(g_n \in \mathcal{KH}([0,1],X))_n$ be a sequence of functions and let $M \subset [0,1]$ be a set with $|M| = 0$ such that*

(i) $g_n(t) \to g(t)$ *for* $t \in E$ *where* $E = [0,1] \setminus M$,

(ii) $(g_n)_n$ *is pointwise bounded in* M,

(iii) (g_n) *is KH-equiintegrable on* $[0,1]$.

Then $g \in \mathcal{KH}([0,1],X)$ and $\|g - g_n\|_{KH} \to 0$. Moreover, if δ is the gauge corresponding to ε in the definition of KH-equiintegrability and δ' is the gauge corresponding to ε in Lemma 1 applied to the sequence $(g_n)_n$ in M, then

$$\left\|\sigma(g\chi_E, \mathcal{P}) - (KH)\int_0^1 g\right\| < 3\varepsilon\,, \tag{7}$$

for each δ_0-fine Perron partition $\mathcal{P}$ of $[0,1]$, where $\delta_0(t) = \delta(t)$ if $t \in E$ and $\delta_0(t) = \min(\delta(t), \delta'(t))$ if $t \in M$.

Proof. By conditions (i) and (ii) it follows that $(g_n)_n$ is pointwise bounded in $[0,1]$. Then the sequence $(g_n\chi_E)_n$ is pointwise convergent to $g\chi_E$ on $[0,1]$ and by condition (iii), *KH*-equiintegrable on $[0,1]$. (See [14] p. 361 for the case $X = \mathbb{R}$. The general case is straightforward.) By [22] Theorem 1 $g\chi_E$ is

KH-integrable on $[0,1]$ and $\|g\chi_E - g_n\chi_E\|_{KH} \to 0$. Thus by ($c$) of Theorem 1 also g is *KH*-integrable on $[0,1]$. Since $\|g - g_n\|_{KH} = \|g\chi_E - g_n\chi_E\|_{KH}$, also $\|g - g_n\|_{KH} \to 0$. At last it is easy to check that, if δ is the gauge corresponding to ε in the definition of *KH*-equiintegrability and δ' is the gauge corresponding to ε in Lemma 1 applied to the sequence $(g_n)_n$ in M, then inequality (7) holds. (For the case $X = \mathbb{R}$ see [14] Theorem 13.16 and p. 361. The general case is straightforward.) □

Next theorem is a Kurzweil-Henstock version of the Vitali Theorem for Pettis integral (see [11] and [20]) and its proof is based on the same idea.

Theorem 5. *Let $g : [0,1] \to X$ be a function and assume that there exists a sequence $(g_n \in \mathcal{KHP}([0,1], X))_n$ such that:*

(i) *the family $\{x^* g_n(t) : \ x^* \in \mathcal{B}(X^*), \ n \in \mathbb{N}\}$ is KH-equiintegrable on* $[0,1]$,

(ii) *the sequence $(g_n)_n$ is pointwise bounded in* $[0,1]$,

(iii) *for each $x^* \in X^*$, $\lim_{n\to\infty} x^* g_n = x^* g$ almost everywhere in* $[0,1]$.

Then $g \in \mathcal{KHP}([0,1], X)$ and for each $x^* \in X^*$ we have $\|x^*g - x^*g_n\|_{KH} \to 0$. Moreover, if the Banach space X is separable, then $g \in \mathcal{KH}([0,1], X)$.

PROOF. We fix $x^* \in X^*$ and apply Theorem 4 to the sequence $(x^* g_n(t))_n$. Then x^*g is *KH*-integrable, $\|x^*g - x^*g_n\|_{KH} \to 0$ and for each $t \in [0,1]$

$$\lim_{n\to\infty} (KH)\int_0^t x^* g_n = (KH)\int_0^t x^* g\,. \tag{8}$$

Now fix $t_0 \in [0,1]$ and denote by C the weak closure of the set $((KHP)\int_0^{t_0} g_n)_n$. Since $((KHP)\int_0^{t_0} g_n)_n$ is a weakly Cauchy sequence, it is bounded. Moreover $C \setminus \{(KHP)\int_0^{t_0} g_n\}_n$ consists at most of one point. We want to prove that C is weakly compact. We assume by contradiction that C is not weakly compact. Then applying Theorem 1 of [16] ((1) $\longleftrightarrow$ (9)) with $T = X$ and $E = C$, there are $\theta > 0$, $(x_m)_m \subset C$ and a sequence $(y_k^*)_k \subset \mathcal{B}(X^*)$ such that $\langle y_k^*, x_m\rangle = 0$ if $k > m$ and $\langle y_k^*, x_m\rangle > \theta$ if $k \le m$. Thus we can find a subsequence $(h_m)_m$ of $(g_n)_n$ such that:

(j) $(KH)\int_0^{t_0} y_k^* h_m = 0$ if $k > m$,

(jj) $(KH)\int_0^{t_0} y_k^* h_m > \theta$ if $k \le m$,

(jjj) $\lim_{m\to\infty} (KH)\int_0^{t_0} x^* h_m = (KH)\int_0^{t_0} x^* g$, for each $x^* \in X^*$.

By condition (iii) the function g is weakly measurable. Then we may apply Theorem 2F of [7] to the sequence $(y_k^*)_k$ to find a subsequence $(y_{k_j}^*)_j$ of $(y_k^*)_k$ such that $(y_{k_j}^* g)_j$ is almost everywhere convergent in $[0, t_0]$. Now if y_0^* is a weak* cluster point of $(y_{k_j}^*)_j$ (The Banach-Alaoglu-Bourbaki Theorem guarantees the existence of such a point.), then $(y_{k_j}^* g)_j$ converges to $y_0^* g$ almost everywhere in $[0, t_0]$.

By condition (i) it follows that the family $\{x^* h_m(t) :\ x^* \in \mathcal{B}(X^*),\ m \in \mathbb{N}\}$ is *KH*-equiintegrable on $[0,1]$. Now for each j let $N_j \subset [0,1]$ be a set such that $|N_j| = 0$ and $y_{k_j}^* h_m \to y_{k_j}^* g$, everywhere on $[0,1] \setminus N_j$ and set $M = \bigcup_j N_j$. Then we apply in M Lemma 1 to the family $\{y_{k_j}^* h_m :\ j,\ m \in \mathbb{N}\}$, and Theorem 4 to the sequences $(y_{k_j}^* h_m)_m$, $j = 1, 2, \dots$. So taking into account inequality (7), we infer that the sequence $(y_{k_j}^* g \chi_{[0,1]\setminus M})_j$ is *KH*-equiintegrable in $[0,1]$. Therefore, applying once again Theorem 4 and (c) of Theorem 1 we obtain

$$\lim_{j\to\infty} (KH)\!\int_0^{t_0} y_{k_j}^* g = (KH)\!\int_0^{t_0} y_0^* g \, .$$

Moreover by (jj) and (8), for each index j we find

$$\lim_{m\to\infty} (KH)\!\int_0^{t_0} y_{k_j}^* h_m = (KH)\!\int_0^{t_0} y_{k_j}^* g \geq \theta \, .$$

Then

$$(KH)\!\int_0^{t_0} y_0^* g \geq \theta \, . \tag{9}$$

On the other hand, since for each m, h_m is *KHP*-integrable, the functional $x^* \to (KH)\int_0^{t_0} x^* h_m$ is weak*-continuous. Therefore if $(y_\alpha^*)_\alpha$ is a subnet of $(y_{k_j}^*)_j$ weak* converging to y_0^*, by (j) for each m we infer

$$\begin{aligned} \lim_\alpha (KH)\int_0^{t_0} y_\alpha^* h_m &= \lim_\alpha y_\alpha^* (KHP)\int_0^{t_0} h_m \\ &= y_0^* (KHP)\int_0^{t_0} h_m = (KH)\int_0^{t_0} y_0^* h_m = 0. \end{aligned}$$

Hence by (8)

$$\lim_m (KH)\!\int_0^{t_0} y_0^* h_m = (KH)\!\int_0^{t_0} y_0^* g = 0,$$

in contradiction with (9). Thus the set C is weakly compact. By (8) it follows that the sequence $((KHP)\!\int_0^{t_0} g_n)_n$ is weak-Cauchy. Then $\lim_n (KHP)\int_0^{t_0} g_n \in$

X. Since t_0 is arbitrary, the function g is *KHP*-integrable and $\lim_n (KHP)\int_0^{t_0} g_n = (KHP)\int_0^{t_0} g$.

If X is separable, then $\mathcal{B}(X^*)$ is metrizable in the weak* topology. Since $\mathcal{B}(X^*)$ is also weak*-compact, it is weak* separable. Let $D = \{x_j^*\}$ be a countable dense subset of $\mathcal{B}(X^*)$ and, for each j let $E_j \subset [0,1]$ be a set such that $|E_j| = 0$ and $x_j^* g_n \to x_j^* g$, everywhere on $[0,1] \setminus E_j$. Put $S = [0,1] \setminus (\bigcup_j E_j)$. Let now $\varepsilon > 0$ be given. Fix $x_0^* \in \mathcal{B}(X^*)$ and denote by $(x_{j_s}^*)_s$ a subsequence of D weak*-convergent to x_0^*. By conditions (i) and (ii), taking into account Lemma 1, it follows that also the family $\{x_j^* g_n \chi_S : \ n \in \mathbb{N}, \ j \in \mathbb{N}\}$ is *KH*-equiintegrable in $[0,1]$. Let δ be a gauge corresponding to ε in the definition of *KH*-equiintegrability. Then if $\mathcal{P}$ is a δ-fine Perron partition of $[0,1]$, by (8) we have

$$\left|\sigma(x_0^* g\chi_S, \mathcal{P}) - x_0^*(KHP)\int_0^1 g\right| = \lim_s \left|\sigma(x_{j_s}^* g\chi_S, \mathcal{P}) - x_{j_s}^*(KHP)\int_0^1 g\right|$$

$$= \lim_s \lim_n \left|\sigma(x_{j_s}^* g_n\chi_S, \mathcal{P}) - x_{j_s}^*(KHP)\int_0^1 g_n\right| \le \varepsilon \,.$$

Therefore the family $\{x^* g\chi_S : \ x^* \in \mathcal{B}(X^*)\}$ is *KH*-equiintegrable in $[0,1]$ and applying Theorem 2 we get the *KH*-integrability of $g\chi_S$. Then by (c) of Theorem 1 also g is *KH*-integrable and this ends the proof. □

Remark 5. We notice that also a McShane version of Theorem 4 and Theorem 5 holds. In such a case the McShane integral, the Pettis integral and the L_1-norm need to be considered instead of the Kurzweil-Henstock integral, the Kurzweil-Henstock-Pettis integral and the Alexiewicz norm, respectively, and to use Theorem 4 of [21] instead of Theorem 1 of [22]. But in the McShane version of Theorem 5, the second part follows at once, since in a separable Banach space each Pettis integrable function is also McShane integrable (see [13] and [9]). An analogous result is not true for the Kurzweil-Henstock integral. Indeed the example in Remark 1 is a c_0-valued function that is *KHP*-integrable, but not *KH*-integrable.

Remark 6. We observe that in the proof of the second part of Theorem 5 we only use the fact that in a separable Banach space X, the unit ball $\mathcal{B}(X^*)$ is weak* separable. Therefore in the claim we can replace the hypothesis of separability of X by that of weak* separability of $\mathcal{B}(X^*)$.

We note yet that really in the hypotheses of Theorem 5, according to Theorem 2, we require also the *KH*-integrability of each function g_n. But, in general, without the hypothesis of separability of the space X (or of weak* separability of the unit ball $\mathcal{B}(X^*)$), we may not get the *KH*-integrability of the function g. Indeed, at least under the Continuum Hypothesis, there is an example (see [6] Example (CH)) of a scalarly negligible function g which is not McShane integrable. Therefore such a function g is Pettis integrable, and then *KHP*-integrable, but is not *KH*-integrable (see [8]). If we set $g_n \equiv 0$ for each n, the sequence $(g_n)_n$ and the function g satisfy all the hypotheses of Theorem 5.

Acknowledgment. The author is grateful to the referee for useful comments to the original version of Proposition 2 and Theorem 3.

References

[1] B. Bongiorno, and L. Di Piazza, *Convergence theorem for generalized Riemann-Stieltjes integrals*, Real Anal. Exchange, **17** (1991-92), 339–361.

[2] S. Cao, *The Henstock integral for Banach-valued functions*, SEA Bull. Math., **16 n. 1** (1992), 35–40.

[3] J. Diestel, and J. J. Uhl, *Vector measures*, Math. Surveys, **15** (1977).

[4] L. Di Piazza and V. Marraffa, *The McShane, PU and Henstock integrals of Banach valued functions*, Czechoslovak Math. J., **52 (127) n. 3** (2002), 609–633.

[5] L. Di Piazza and K. Musial, *Set valued Kurzweil-Henstock-Pettis integral*, Set-valued Anal., to appear.

[6] L. Di Piazza and D. Preiss, *When do McShane and Pettis integrals coincide?*, Ill. Math. Jour., **47 n. 4** (2003), 1177–1187.

[7] D. H. Fremlin, *Pointwise compact sets of measurable functions*, Manuscripta Math., **15** (1975), 219–242.

[8] D. H. Fremlin, *The Henstock and McShane integrals of vector-valued functions*, Ill. Math. Jour., **38 n. 3** (1994), 471–479.

[9] D. H. Fremlin and J. Mendoza, *On the integration of vector-valued functions*, Ill. Math. Jour., **38 n.1** (1994), 127–147.

[10] J. L. Gamez and J. Mendoza, *On Denjoy-Dunford and Denjoy-Pettis integrals*, Studia Math., **130** (1998), 115–133.

[11] G. F. Geitz, *Pettis integration*, Proc. Amer. Math. Soc., **82 n. 1** (1981), 81–86.

[12] R. Gordon, *Another look at a convergence theorem for the Henstock integral*, Real Anal. Exchange, **15** (1989/90), 724–728.

[13] R. Gordon, *The McShane integral of Banach-valued functions*, Ill. Math. Jour., **34 n. 3** (1990), 557–567.

[14] R. Gordon, *Riemann integration in Banach spaces*, Rocky Moun. J. Math., **XXI 3**, Summer 1991.

[15] R. Gordon, *The integrals of Lebesgue, Denjoy, Perron, and Henstock*, Graduate Studies in Math., **vol. 4**, Am. Math. Soc. (1991).

[16] R. C. James, *Weak compactness and reflexivity*, Israel J. Math., **2** (1964), 101–119.

[17] J. Kurzweil, *Nichtalsolut Konvergente Integral*, Teubner, Leipzig (1980).

[18] J. Kurzweil and J. Jarnik, *Perron type integrable functions*, Real Anal. Exchange, **17** (1991/92), 110–139.

[19] J. Kurzweil and J. Jarnik, *Equiintegrability and controlled convergence of Perron-type integrable functions*, Real Anal. Exchange, **17** (1991/92), 110–139.

[20] K. Musial, *Topics in the theory of Pettis integration*, Rendiconti Istit. Matem. Univ. Trieste, **XXIII** (1991), 177–262.

[21] S. Swartz, *Uniform integrability and mean convergence for the vector-valued McShane integral*, Real Anal. Exchange, **23** (1) (1997/8), 303 – 312.

[22] S. Swartz, *Norm convergence and uniform integrability for the Kurzweil-Henstock integral*, Real Anal. Exchange, **24 (1)** (1998/9), 423–426.

[23] G. Ye, and S. Schwabik, *The McShane and the Pettis integral of Banach space-valued functions defined on* $\mathbb{R}^n$, Ill. Jour. Math., **46 n. 2** (2002), 1125–1144.

[24] L. P. Yee, and R. Vyborny, *The integral: An easy approach after Kurzweil and Henstock*, Cambridge University Press (2000).