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Artur Bartoszewicz, Institute of Mathematics, Łódź Technical University, ul. Wólczańska 215, 93-005 Łódź, Poland. email: `arturbar@p.lodz.pl`
Elżbieta Kotlicka, Center of Mathematics and Physics, Łódź Technical University, al. Politechniki 11, 90-924 Łódź, Poland. email: `ekot@p.lodz.pl`

RELATIONSHIPS BETWEEN CONTINUITY AND ABSTRACT MEASURABILITY OF FUNCTIONS

Abstract

Making use of ideas of Marczewski and Sierpiński we propose a general approach to studies on connections between measurability, continuity and relative continuity of functions. Theorem 2.1 shows that a well-known characterization of (s)-measurable Marczewski functions can be extended to the case of functions measurable with respect to a wide class of algebras involved with a topology. Theorem 2.2 generalizes the Denjoy-Stepanoff theorem and shows that the Denjoy-Stepanoff property stating the continuity of $\mathcal{A}$-measurable functions at all points of a co-negligible set is quite common while an algebra $\mathcal{A}$ and an ideal $\mathcal{J}$ are the results of operations S and S_0 on $\tau \setminus \mathcal{I}$ for a given topology τ. Also from the obtained results we conclude new theorems concerning the algebras associated with product ideals (Theorems 3.12 and 3.13).

1 Introduction.

There are well-known theorems on equivalence between measurability of functions with respect to some σ-algebras of sets and relative continuity or continuity on "large" sets. Let us recall the most important of these facts. Denote by $\mathcal{L}$ and $\mathcal{B}$ the σ-algebras of Lebesgue measurable sets and of sets with the Baire property on $\mathbb{R}$, and by $\mathcal{N}$ and $\mathcal{M}$, the σ-ideals of sets of measure zero and of sets of first category, respectively. Let us quote some known characterizations of Lebesgue measurable functions and functions with the Baire property:

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Theorem 1.1 (Lusin). *A function $f\colon \mathbb{R} \to \mathbb{R}$ is $\mathcal{L}$-measurable if and only if for any positive ε there exists a set $E \subset \mathbb{R}$ such that $\mu(E^c) \le \varepsilon$, where μ denotes Lebesgue measure, and the restriction of f to E, $f \mid E$ is continuous.*

Theorem 1.2 (Nikodym). *A function $f\colon \mathbb{R} \to \mathbb{R}$ is $\mathcal{B}$-measurable if and only if there exists a set $E \subset \mathbb{R}$ such that $E^c \in \mathcal{M}$ and $f \mid E$ is continuous.*

Theorem 1.3 (Denjoy, Stepanoff). *A function $f\colon \mathbb{R} \to \mathbb{R}$ is $\mathcal{L}$-measurable if and only if there exists a set $E \subset \mathbb{R}$ such that $E^c \in \mathcal{N}$ and f is approximately continuous at every point of E.*

Theorem 1.4 (Wilczyński [20]). *A function $f\colon \mathbb{R} \to \mathbb{R}$ is $\mathcal{B}$-measurable if and only if there exists a set $E \subset \mathbb{R}$ such that $E^c \in \mathcal{M}$ and f is $\mathcal{M}$-approximately continuous at every point of E.*

(By $\mathcal{M}$-approximate continuity we mean the category analogue of approximate continuity introduced by Wilczyński.)

Let us also recall some type of measurability connected with perfect sets, due to Sierpiński and Marczewski. Sierpiński introduced in [18] the class of real functions with the following property.

Every perfect set$P \subset \mathbb{R}$ has a perfect subset Q such that $f \mid Q$ is continuous. (Si)

Marczewski [19] invented the class of (s)-sets defined as follows.

$$E \in (s) \Leftrightarrow \forall_{P\in\mathcal{F}}\ \exists_{Q\in\mathcal{F}}\ \ (Q \subset P \cap E \text{ or } Q \subset P \setminus E)$$

where $\mathcal{F}$ denotes the collection of all perfect sets on the real line. He showed that the (s)-sets form a σ-algebra and that the (s_0)-sets defined by

$$E \in (s_0) \Leftrightarrow \forall_{P\in\mathcal{F}}\ \exists_{Q\in\mathcal{F}}\ \ (Q \subset P \setminus E)$$

form a σ-ideal contained in (s). Marczewski proved the following theorem.

Theorem 1.5. *A function $f\colon \mathbb{R} \to \mathbb{R}$ is (s)-measurable if and only if every perfect set $P \subset \mathbb{R}$ has a perfect subset Q such that $f \mid Q$ is continuous; i.e., iff f has property* (Si).

Theorems 1.1–1.5 can be extended to more general cases. For example, in Theorems 1.1, 1.2 and 1.5 we can consider functions defined on a Polish space with values in a separable metric space. In turn Theorems 1.3 and 1.4 are special cases of the following result.

Theorem 1.6 ([12, Thm 6.39 and Exercise 6.E.16])**.** *Let $\mathcal{A}$ be a σ-algebra of subsets of X and let $\mathcal{I} \subset \mathcal{A}$ be a σ-ideal. Let moreover $\tau_d \subset \mathcal{A}$ be a topology determined by a lower density operator from $\mathcal{A}$ into $\mathcal{A}$. Then a function $f\colon X \to \mathbb{R}$ is $\mathcal{A}$-measurable if and only if there exists a set $E \subset X$ such that $E^c \in \mathcal{I}$ and f is τ_d-continuous on E.*

Notice that the property equivalent to measurability in Theorem 1.5 is slightly different from the conditions of Theorems 1.1–1.4. Here we don't have continuity on a "large" set but on some subset of any set from a fixed collection of sets (in this case, of perfect sets). Recently in several papers, the Marczewski construction has been applied to an arbitrary family $\mathcal{F}$ of nonempty subsets of some set X. It is easy to observe ([4],[15]) that the collections of sets

$$S(\mathcal{F}) = \{E \subset X : (\forall P \in \mathcal{F})(\exists Q \in \mathcal{F})(Q \subset P \cap E \text{ or } Q \subset P \setminus E)\}$$

and

$$S_0(\mathcal{F}) = \{E \subset X : (\forall P \in \mathcal{F})(\exists Q \in \mathcal{F})(Q \subset P \setminus E)\}$$

constitute an algebra and an ideal of subsets of X, respectively. An old result of Burstin [9] states that the pair $\langle \mathcal{L}, \mathcal{N} \rangle$ is of the form $\langle S(\mathcal{F}), S_0(\mathcal{F}) \rangle$ where $\mathcal{F}$ consists of perfect sets with positive measure.

We say that the pair $\langle \mathcal{A}, \mathcal{I} \rangle$, where $\mathcal{A}$ is an algebra and $\mathcal{I}$ an ideal of sets, has *Marczewski-Burstin representation* (*MB-representation*) if there exists a family of sets $\mathcal{F}$ such that $\langle \mathcal{A}, \mathcal{I} \rangle = \langle S(\mathcal{F}), S_0(\mathcal{F}) \rangle$. The representation is called *inner* if $\mathcal{F} \subset S(\mathcal{F})$. (For results concerning MB-representations see [2, 3, 4, 8, 10, 16].) Brown and Elalaoui-Talibi [8] observed that the pair $\langle \mathcal{B}, \mathcal{M} \rangle$ has inner MB-representation given by the family $\mathcal{F}$ of all G_δ-sets of the second category. They obtained results, similar to the Marczewski theorem, for Lebesgue measurable sets and for sets with the Baire property:

Theorem 1.7. *A function $f\colon \mathbb{R} \to \mathbb{R}$ is $\mathcal{L}$-measurable if and only if every perfect set $P \in \mathcal{L} \setminus N$ has a perfect subset $Q \in \mathcal{L} \setminus N$ such that $f \mid Q$ is continuous.*

Theorem 1.8. *A function $f\colon \mathbb{R} \to \mathbb{R}$ is $\mathcal{B}$-measurable if and only if every $P \in \mathcal{F}$ has a subset $Q \in \mathcal{F}$ such that $f \mid Q$ is continuous, where $\mathcal{F}$ is the family of G_δ-sets of the second category.*

More precisely, Brown and Elalaoui-Talibi proved Theorems 1.7–1.8 in the more general case of a function $f : X \to Y$ with a Polish space X and a separable metric space Y.

The proofs of implications "measurability" $\Rightarrow$ "relative continuity" make use of Theorems 1.1, 1.2. The proofs of the reverse implications are more complicated and they are only slightly connected with MB-representations of $\langle\mathcal{L},\mathcal{N}\rangle$ and $\langle\mathcal{B},\mathcal{M}\rangle$.

The aim of our paper is to propose a general approach to the problem of connections between MB-representability and such properties as measurability, relative continuity and continuity. From the main results presented in the next section, we derive Theorems 1.3–1.8 as well as new theorems concerning the algebras generated by Borel subsets of the plane and the product ideals $\mathcal{N}\widetilde{\otimes}\mathcal{M}$ and $\mathcal{M}\widetilde{\otimes}\mathcal{N}$ (see Section 3). In particular, Theorem 3.13 gives the solution of the problem posed in [5].

2 General Results.

Let us consider topological spaces $(X,\tau),(Y,\tau_Y)$, an algebra $\mathcal{A}$ of subsets of X, an ideal $\mathcal{I}\subset\mathcal{A}$, and a collection $\mathcal{F}$ of nonempty subsets of X. Let f be a function defined on X with values in Y.

Definition 2.1. We say that a function f is $\mathcal{A}$-measurable if, for any $U\in\tau_Y$, the set $f^{-1}(U)$ belongs to $\mathcal{A}$.

Note that in our definition of measurability we use an algebra of sets which need not be a σ-algebra. So we can not equivalently say that for any Borel set $B\subset Y$ we have $f^{-1}(B)\in\mathcal{A}$.

Definition 2.2. We say that a function f satisfies the *Sierpiński condition with respect to the collection of sets* $\mathcal{F}$ if for any $P\in\mathcal{F}$ there exists $Q\in\mathcal{F}$ such that $Q\subset P$ and $f\mid Q$ is continuous. Denote the class of such functions by $S(X,Y,\mathcal{F})$.

The main purpose of this part of the paper is to discuss the following problems:

Problem 2.1. When does the $\mathcal{A}$-measurability of f imply the Sierpiński condition and vice versa?

Problem 2.2. When is the $\mathcal{A}$-measurability of f equivalent to the continuity of f at each point of some large set; i.e., of a set belonging to $\mathcal{A}\setminus\mathcal{I}$, for a given ideal $\mathcal{I}$ contained in an algebra $\mathcal{A}$?

Let us introduce some definitions.

Definition 2.3. Let $\mathcal{A}$ be an algebra of subsets of X, let $\mathcal{F}$ be a fixed nonempty collection of nonempty subsets of X, and let τ be a topology in X. Let κ be an infinite cardinal number. We say that $\langle\mathcal{A},\mathcal{F},\tau\rangle$ has the *Marczewski property for* κ (in short $\langle\mathcal{A},\mathcal{F},\tau\rangle\in M(\kappa)$) if for any family of sets

$\mathcal{E} \subset \mathcal{A}$ of cardinality $|\mathcal{E}| < \kappa$ and for any set $P \in \mathcal{F}$ there exists a set $Q \in \mathcal{F}$ such that $Q \subset P$ and, for any $E \in \mathcal{E}$, the set $Q \cap E$ is relatively open in Q. Formally it means that

$$\forall_{\alpha<\kappa}\ \forall_{\{E_\gamma:\gamma<\alpha\}\subset\mathcal{A}}\ \forall_{P\in\mathcal{F}}\ \exists_{Q\in\mathcal{F},\ Q\subset P}\ \exists_{\{G_\gamma:\gamma<\alpha\}\subset\tau}\ \forall_{\gamma<\alpha}\ \ (Q\cap E_\gamma = Q\cap G_\gamma). \tag{M(κ)}$$

Definition 2.4. Let $\mathcal{A}$ be an algebra and $\mathcal{I} \subset \mathcal{A}$ be an ideal of subsets of X. Let moreover $\tau \subset \mathcal{A}$ be a topology in X. We say that $\langle \mathcal{A}, \mathcal{I}, \tau\rangle$ satisfies the *Denjoy-Stepanoff condition with respect to a topological space* (Y, τ_Y) (in short, $\langle \mathcal{A}, \mathcal{I}, \tau\rangle \in \mathcal{D}(Y)$) if for any function $f : X \to Y$ the following conditions are equivalent:

- ($*$) f is $\mathcal{A}$-measurable,
- ($**$) there exists a set $E \subset X$ such that $E^c \in \mathcal{I}$ and f is continuous at every point of E.

If $\langle \mathcal{A}, \mathcal{I}, \tau\rangle \in \mathcal{D}(Y)$ for every (Y, τ_Y) with weight smaller than κ, we say that $\langle \mathcal{A}, \mathcal{I}, \tau\rangle$ satisfies the *Denjoy-Stepanoff condition with respect to the cardinal* κ (in short, $\langle \mathcal{A}, \mathcal{I}, \tau\rangle \in \mathcal{D}(\kappa)$).

Now, we are in a position to formulate the main results of this section being the (partial) solutions of Problems 2.1 and 2.2.

Theorem 2.1. *Let $\langle \mathcal{A}, \mathcal{F}, \tau\rangle$ have the Marczewski property for $\kappa \geq \omega_1$, $\mathcal{A} = S(\mathcal{F})$, $\tau \subset \mathcal{A}$ and τ_Y have a weight smaller than κ. Then the following conditions are equivalent:*

- *(i) $f : X \to Y$ is $\mathcal{A}$-measurable,*
- *(ii) $f^{-1}(B) \in \mathcal{A}$ for any Borel set B,*
- *(iii) f satisfies the Sierpiński condition with respect to $\mathcal{F}$.*

Theorem 2.2. *Let $\mathcal{A}$ be an algebra of subsets of X, and let $\mathcal{I} \subset \mathcal{A}$ be an ideal. Moreover let $\tau \subset \mathcal{A}$ be a topology in X, and let κ be a cardinal number. Consider the following conditions:*

- (a) *$\langle \mathcal{A}, \mathcal{I}, \tau\rangle$ satisfies the Denjoy-Stepanoff condition with respect to some non-trivial topological space (Y, τ_Y),*
- (b) $\operatorname{Int} E \neq \emptyset$ *for each* $E \in \mathcal{A} \setminus \mathcal{I}$,
- (c) $\langle \mathcal{A}, \mathcal{I}\rangle = \langle S(\tau \setminus \mathcal{I}), S_0(\tau \setminus \mathcal{I})\rangle$,
- (d) $E \in \mathcal{A}$ *if and only if* $E = U \cup N$ *for some* $U \in \tau$ *and* $N \in \mathcal{I}$,

(e) $\langle \mathcal{A}, \mathcal{I}, \tau \rangle$ *satisfies the Denjoy-Stepanoff condition with respect to the cardinal* κ*,*

(f) $S_0(\tau^*) \subset \mathcal{I}$ *and* $S(\tau^*) \subset \mathcal{A} \subset S(\mathcal{A} \setminus S_0(\tau^*))$*, where* $\tau^* = \tau \setminus \{\emptyset\}$*.*

Then (a)$\Rightarrow$(b)$\Leftrightarrow$(c)$\Leftrightarrow$(d)$\Rightarrow$(f)*. Moreover, if* $\mathcal{I}$ *is* κ*-additive, then* (b)$\Rightarrow$(e)*.*

We divide our considerations into several steps which lead to the above theorems. Some propositions may be of independent interest.

Definition 2.5. We say that $E \in S(\mathcal{F}, \tau)$ if and only if for any set $P \in \mathcal{F}$ there exists $Q \in \mathcal{F}$ such that $Q \subset P$ and the set $Q \cap E$ is relatively open in Q. Formally

$$S(\mathcal{F}, \tau) = \{E : (\forall P \in \mathcal{F})(\exists Q \in \mathcal{F})(\exists G \in \tau)(Q \subset P \text{ and } Q \cap E = Q \cap G)\}.$$

We use the same symbol S for $S(\mathcal{F})$ and $S(\mathcal{F}, \tau)$ which however will not cause confusion since they depend on one and two variables, respectively. The same remark concerns Definition 2.2.

Proposition 2.3. *Assume that* $\tau \subset S(\mathcal{F})$*. Then* $S(\mathcal{F}) = S(\mathcal{F}, \tau)$*.*

PROOF. Let $E \in S(\mathcal{F})$. Pick an arbitrary $P \in \mathcal{F}$. Then there exists a set $Q \in \mathcal{F}$ such that either $Q \subset E \cap P$ or $Q \subset P \setminus E$. In the first case we have $Q \cap E = Q \cap X = Q$, and in the second one, $Q \cap E = Q \cap \emptyset = \emptyset$. Hence $S(\mathcal{F}) \subset S(\mathcal{F}, \tau)$.

On the other hand, consider an $E \in S(\mathcal{F}, \tau)$ and an arbitrary $P \in \mathcal{F}$. Then for some $Q_1 \in \mathcal{F}$ and $Q_1 \subset P$ there exists a $G \in \tau$ such that $Q_1 \cap G = Q_1 \cap E$. By $\tau \subset S(\mathcal{F})$ we can find $Q \in \mathcal{F}$ such that either $Q \subset Q_1 \cap G$ or $Q \subset Q_1 \setminus G$. Then either $Q \subset Q_1 \cap E \subset P \cap E$ or $Q \subset Q_1 \setminus E \subset P \setminus E$. Hence $S(\mathcal{F}, \tau) \subset S(\mathcal{F})$. $\square$

We say that the families of sets $\mathcal{F}_1, \mathcal{F}_2$ are *mutually coinitial* if for any $F_1 \in \mathcal{F}_1$ there exists $F_2 \in \mathcal{F}_2$ such that $F_2 \subset F_1$, and vice versa. Recall [4] that for mutually coinitial families $\mathcal{F}_1, \mathcal{F}_2$ we have $\langle S(\mathcal{F}_1), S_0(\mathcal{F}_1)\rangle = \langle S(\mathcal{F}_2), S_0(\mathcal{F}_2)\rangle$. We obtain some more consequences.

Remark 2.4. Assume that $\mathcal{F}_1, \mathcal{F}_2$ are mutually coinitial families of sets, and a topology τ is contained in $S(\mathcal{F}_1)$ (or, equivalently, in $S(\mathcal{F}_2)$). Then $S(\mathcal{F}_1, \tau) = S(\mathcal{F}_2, \tau)$. The same concerns the Sierpiński condition and the Marczewski property. More precisely $f \in S(X, Y, \mathcal{F}_1) \Leftrightarrow f \in S(X, Y, \mathcal{F}_2)$ and $\langle A, \mathcal{F}_1, \tau\rangle \in M(\kappa) \Leftrightarrow \langle A, \mathcal{F}_2, \tau\rangle \in M(\kappa)$.

Recall that a pair $\langle \mathcal{A}, \mathcal{I} \rangle$ of an algebra and an ideal is said to satisfy the *hull property* if for every $A \subset X$ there is a $B \in \mathcal{A}$ such that $A \subset B$, and whenever $C \in \mathcal{A}$ is such that $A \subset C$, we have $B \setminus C \in \mathcal{I}$ (or equivalently, there is a $B \in \mathcal{A}$ such that $A \supset B$, and whenever $C \in \mathcal{A}$ is such that $A \supset C$, we have $C \setminus B \in \mathcal{I}$).

In the sequel we shall need the following theorem concerning conditions sufficient for the existence of the canonical MB-representation of a pair $\langle \mathcal{A}, \mathcal{I} \rangle$. Recall that for a given topology τ, by τ^* we will mean the family $\tau \setminus \{\emptyset\}$.

Proposition 2.5. *Let $\mathcal{I}$ be an ideal of sets contained in an algebra $\mathcal{A}$. Consider the following conditions:*

(I) $\langle \mathcal{A}, \mathcal{I} \rangle$ *has the inner MB-representation,*

(II) $\langle \mathcal{A}, \mathcal{I} \rangle$ *has the hull property,*

(III) *there exists a topology $\tau \subset \mathcal{A}$ such that τ^* is coinitial with $\mathcal{A} \setminus \mathcal{I}$,*

(IV) *there exists a topology $\tau \subset \mathcal{A}$ such that*

$$\forall_{A \in \mathcal{A}} \quad (A \in \mathcal{A} \setminus \mathcal{I} \Rightarrow \operatorname{Int} A \neq \emptyset). \tag{1}$$

If one of the conditions (I)–(IV) *holds, then $\langle \mathcal{A}, \mathcal{I} \rangle = \langle S(\mathcal{A} \setminus \mathcal{I}), S_0(\mathcal{A} \setminus \mathcal{I}) \rangle$.*

PROOF. The first and the second sufficiencies were proved in [3] and [6], respectively. It is easy to check that condition (IV) follows from (III). So, to complete the proof we show that (IV) implies (II).

Let us take any set $A \subset X$. Then $\operatorname{Int} A$ (with respect to topology τ) belongs to $\mathcal{A}$. Consider a set $C \in \mathcal{A}$ such that $C \subset A$. One can see that $C \setminus \operatorname{Int} A$ does not contain any nonempty set open in τ. Hence using assumption (1) we obtain that $C \setminus \operatorname{Int} A \in \mathcal{I}$. □

Proposition 2.6. *Let $\mathcal{F}$ be a collection of nonempty subsets of X. Then we have:*

(i) if τ_Y is a nontrivial topology and any function f satisfying the Sierpiński condition with respect to $\mathcal{F}$ is $\mathcal{A}$-measurable, then $S(\mathcal{F}) \subset \mathcal{A}$;

(ii) if $\tau \subset S(\mathcal{F})$, then any function f satisfying the Sierpiński condition with respect to $\mathcal{F}$ is $S(\mathcal{F})$-measurable.

PROOF. (i) Suppose that $S(\mathcal{F}) \not\subset \mathcal{A}$. Then there exists a set $E \in S(\mathcal{F})$ such that $E \notin \mathcal{A}$. Consider a function f defined by

$$f(x) = \begin{cases} y_1 & \text{for } x \in E, \\ y_2 & \text{for } x \notin E \end{cases}$$

where y_1, y_2 are two different points of Y such that there exists $U \in \tau_Y$ for which $y_1 \in U$ and $y_2 \notin U$. Hence $f^{-1}(U) \notin \mathcal{A}$ and f is not $\mathcal{A}$-measurable. On the other hand $E \in S(\mathcal{F})$; so for any $P \in \mathcal{F}$ there exists $Q \in \mathcal{F}$ contained either in $P \cap E$ or in $P \setminus E$. Hence $f \mid Q = y_1$ or $f \mid Q = y_2$ and $f \mid Q$ is a continuous constant function. This is a contradiction, so we have $S(\mathcal{F}) \subset \mathcal{A}$.

(ii) Take an open set $U \subset Y$. Let $f \in S(X, Y, \mathcal{F})$. Consider $E = f^{-1}(U)$ and an arbitrary $P \in \mathcal{F}$. Then there exists $Q \in \mathcal{F}$ such that $Q \subset P$ and $f \mid Q$ is continuous. Hence $(f \mid Q)^{-1}(U)$ is open in Q. So $(f \mid Q)^{-1}(U) = G \cap Q$ for some $G \in \tau$. Thus we have

$$Q \cap E = Q \cap f^{-1}(U) = (f \mid Q)^{-1}(U) = Q \cap G,$$

which means that $E \in S(\mathcal{F}, \tau)$. By the assumption $\tau \subset S(\mathcal{F})$ and Proposition 2.3, we have $S(\mathcal{F}, \tau) = S(\mathcal{F})$ and consequently, f is $S(\mathcal{F})$-measurable. □

Remark 2.7. Condition (ii) can not be reversed. Indeed, consider the following example. Let $X = X_1 \cup X_2$, $X_1 \cap X_2 = \emptyset$, $\mathcal{A} = \{X_1, X_2, X, \emptyset\}$, $\tau = \{X, \emptyset, G\}$ where $G \cap X_1 \neq \emptyset \neq G \cap X_2$ and $G \neq X$. Let $\mathcal{F} = \{X_1, X_2, X\}$. Then $S(\mathcal{F}) = \mathcal{A}$. A real function f belongs to $S(X, \mathbb{R}, \mathcal{F})$ if and only if f is constant on X_1 and constant on X_2. Hence if $f \in S(X, \mathbb{R}, \mathcal{F})$ then f is $S(\mathcal{F})$-measurable but $\tau \not\subset S(\mathcal{F})$.

Remark 2.8. If we assume that $\mathcal{F}$ is coinitial with $\mathcal{A} \setminus \mathcal{I}$ (for some ideal $\mathcal{I} \subset \mathcal{A}$), $\tau \subset S(\mathcal{F})$ and τ_Y is nontrivial, then $\langle \mathcal{A}, \mathcal{I} \rangle = \langle S(\mathcal{A} \setminus \mathcal{I}), S_0(\mathcal{A} \setminus \mathcal{I}) \rangle$ if and only if every $f \in S(X, Y, \mathcal{F})$ is $S(\mathcal{A})$-measurable. Indeed, we have $S(\mathcal{F}) = S(\mathcal{A} \setminus \mathcal{I})$ and $\mathcal{A}$ is always contained in $S(\mathcal{A} \setminus \mathcal{I})$. So Remark 2.8 follows from Proposition 2.6.

Remark 2.9. The proof of (ii) is a bit simpler than the proofs of implications "relative continuity" ⇒ "measurability" in particular cases of Lebesgue measurable sets and sets with the Baire property, due to Brown and Elalaoui-Talibi [8].

It is interesting to establish whether the $\mathcal{A}$-measurability of f can be equivalent to $f \in S(X, Y, \mathcal{F})$ for some collection $\mathcal{F}$. An example presented below witnesses that conditions considered in Remark 2.8 are not sufficient for the reverse implication "measurability" ⇒ "relative continuity".

Example 2.10. Let $\mathcal{A}$ be an algebra of all sets of nowhere dense boundary with respect to τ_1, where τ_1 stands for the natural topology in $\mathbb{R}$ (we write $\mathcal{A} = \mathcal{NDB}(\tau_1)$). By $H(\mathcal{A})$ we will denote the ideal of hereditary sets in $\mathcal{A}$. Since $H(\mathcal{A})$ equals the collection of nowhere dense sets $\mathcal{ND}(\tau_1)$ and the family τ_1^* is coinitial with $\mathcal{A} \setminus H(\mathcal{A})$, we get

$$\langle \mathcal{A}, H(\mathcal{A})\rangle = \langle S(\tau_1^*), S_0(\tau_1^*)\rangle = \langle S(\mathcal{A} \setminus H(\mathcal{A})), S_0(\mathcal{A} \setminus H(\mathcal{A}))\rangle.$$

On the other hand, there exists an $\mathcal{A}$-measurable real function f which does not satisfy the Sierpiński condition with respect to $\mathcal{A} \setminus H(\mathcal{A})$. (More exactly, f does not satisfy the Sierpiński condition with respect to τ_1^*, but according to Remark 2.4 the both statements are equivalent.) Namely, let $\{q_n : 1 \leq n < \omega\}$ stand for the set of all rational numbers. Put $f(x) = \sum_{n \in N(x)} \frac{1}{2^n}$ where $N(x) = \{k < \omega : q_k < x\}$ for $x \in \mathbb{R}$. It is easy to check that such a function has the desired properties. Indeed, the $\mathcal{A}$-measurability of f follows from monotonicity and failure of the Sierpiński condition due to the discontinuity on a dense set (the set of rationals).

Proposition 2.11. *If $\langle \mathcal{A}, \mathcal{F}, \tau\rangle \in M(\kappa)$, $\mathcal{A} = S(\mathcal{F})$ and $\tau \subset \mathcal{A}$, then $\mathcal{A}$ is κ-additive. In particular for $\kappa \geq \omega_1$, $\mathcal{A}$ is a σ-algebra.*

PROOF. Take $\mathcal{E} = \{E_\gamma : \gamma < \alpha\} \subset \mathcal{A}$ with $\alpha < \kappa$ and $P \in \mathcal{F}$. Then there exists $Q \in \mathcal{F}$ such that $Q \subset P$ and for every $E_\gamma \in \mathcal{E}$ we have $Q \cap E_\gamma = Q \cap G_\gamma$ for some open set G_γ. Define $G = \bigcup_{\gamma<\alpha} G_\gamma$. We have

$$Q \cap \bigcup \mathcal{E} = \bigcup_{\gamma<\alpha} (Q \cap E_\gamma) = \bigcup_{\gamma<\alpha} (Q \cap G_\gamma) = Q \cap G.$$

Hence $\bigcup \mathcal{E} \in S(\mathcal{F}, \tau) = S(\mathcal{F}) = \mathcal{A}$. □

Proposition 2.12. *Assume that $\langle \mathcal{A}, \mathcal{F}, \tau\rangle \in M(\kappa)$ and a topology τ_Y has a base $\mathcal{U}$ such that $|\mathcal{U}| < \kappa$. Then every $\mathcal{A}$-measurable function $g : X \to Y$ satisfies the Sierpiński condition with respect to $\mathcal{F}$.*

PROOF. Consider $\mathcal{E} = \{E : (\exists U \in \mathcal{U})(E = f^{-1}(U))\}$. By the measurability of f we have $\mathcal{E} \subset \mathcal{A}$. Take an arbitrary $P \in \mathcal{F}$. Then there exists $Q \in \mathcal{F}$ such that $Q \subset P$ and all the sets $Q \cap E$ for $E \in \mathcal{E}$ are open in Q. So $f \mid Q$ is continuous. □

Some kind of a reverse proposition is the following.

Proposition 2.13. *Assume that for any cardinal $\beta < \kappa$ there exists a family $\mathcal{G}$ of disjoint, nonempty open sets in Y such that $|\mathcal{G}| = \beta$. Moreover, let $\mathcal{A}$ be κ-additive and every $\mathcal{A}$-measurable function $f : X \to Y$ satisfy the Sierpiński condition with respect to a family $\mathcal{F}$. Then for any family $\mathcal{E}$, with cardinality $|\mathcal{E}| < \kappa$, of disjoint sets in $\mathcal{A}$, and for an arbitrary $P \in \mathcal{F}$ there exists a subset Q of P, $Q \in \mathcal{F}$, such that $Q \cap E$ is open in Q for all $E \in \mathcal{E}$.*

PROOF. Let $\alpha < \kappa$ and $\mathcal{E} = \{E_\xi : \xi < \alpha\}$. Let $\mathcal{G} = \{U_\xi : \xi < \alpha\}$ be a family of disjoint open nonempty sets in Y such that $\bigcup \mathcal{G} \neq Y$. Choose $y_\xi \in U_\xi$, $\xi < \alpha$, and define $f(x) = y_\xi$ for $x \in E_\xi$. (To define f for all $x \in X$ we can take $f(x) = y$ for $x \notin \bigcup \mathcal{E}$ where $y \notin \bigcup \mathcal{G}$.) Thus f is $\mathcal{A}$-measurable, so for any $P \in \mathcal{F}$ there exists $Q \in \mathcal{F}, Q \subset P$, such that $f \mid Q$ is continuous. Hence every set of the form $(f \mid Q)^{-1}(U_\xi)$ is open in Q. It means that $E_\xi \cap Q = f^{-1}(U_\xi) \cap Q = (f \mid Q)^{-1}(U_\xi)$ is open in Q. □

Now, let us prove our first main theorem.

PROOF OF THEOREM 2.1.
(i) $\Rightarrow$ (ii) This follows from the fact that $\mathcal{A}$ is a σ-algebra (Proposition 2.11).
(ii) $\Rightarrow$ (i) Obvious.
(i) $\Rightarrow$ (iii) By Proposition 2.12.
(iii) $\Rightarrow$ (i) By Proposition 2.6. □

Applying Theorem 2.1 and Proposition 2.5 with $\mathcal{F} = \mathcal{A} \setminus \mathcal{I}$ we conclude the following

Corollary 2.14. *Assume that a pair $\langle \mathcal{A}, \mathcal{I} \rangle$ of an algebra $\mathcal{A}$ and an ideal $\mathcal{I}$ has the hull property. Let τ be a topology contained in $\mathcal{A}$, such that $\langle \mathcal{A}, \mathcal{A} \setminus \mathcal{I}, \tau \rangle \in M(\kappa)$, and let τ_Y have a weight smaller than κ. Then the following conditions are equivalent:*

(i) $f : X \to Y$ *is $\mathcal{A}$-measurable,*

(ii) $f \in S(X, Y, \mathcal{A} \setminus \mathcal{I})$.

Proposition 2.15. *Let $\mathcal{A}$ be one of the following algebras:*

(1) algebra (s) of Marczewski sets,

(2) algebra $\mathcal{B}$ of the sets with Baire property,

(3) algebra $\mathcal{L}$ of Lebesgue measurable sets.

Then $\langle \mathcal{A}, \mathcal{A} \setminus H(\mathcal{A}), \tau_1 \rangle \in M(\omega_1)$ where τ_1 is the natural topology in $\mathbb{R}$ and $H(\mathcal{A})$ denotes the ideal of hereditary sets in $\mathcal{A}$.

PROOF. (1) Marczewski in [19] proved that $\langle (s), \mathcal{F}, \tau_1 \rangle \in M(\omega_1)$ where $\mathcal{F}$ is the collection of all perfect sets in $\mathbb{R}$. By Remark 2.4, $\langle (s), (s) \setminus (s_0), \tau_1 \rangle \in M(\omega_1)$ because $\mathcal{F}$ and $(s) \setminus (s_0)$ are coinitial. It is known that $H(s) = (s_0)$.

(2) Let $\mathcal{E} = \{E_n : n < \omega\}$ be a countable family of sets with the Baire property. Then $E_n = U_n \triangle B_n$ where U_n is open in τ_1 and B_n is of the first category. Let P be a set of the second category with the Baire property. Define $B = \bigcup_{n<\omega} B_n$ and $Q = P \setminus B$. Then we have

$$Q \cap E_n = Q \cap (U_n \triangle B_n) = Q \cap ((U_n \setminus B_n) \cup (B_n \setminus U_n)) = (Q \cap U_n) \setminus B_n = Q \cap U_n.$$

(3) Let $\mathcal{E} = \{E_n : n < \omega\}$ be a family of Lebesgue measurable sets. Let P be an arbitrary set with positive Lebesgue measure. Take $\varepsilon < \frac{\mu(P)}{2}$. Then there exists a family $\mathcal{G} = \{G_n : n < \omega\}$ of open sets such that $E_n \subset G_n$ and $\mu(G_n \setminus E_n) < \frac{\varepsilon}{2^n}$. Define $Q = P \setminus \bigcup_{n<\omega} (G_n \setminus E_n)$. Then $\mu(Q) > 0$ and:

$$\begin{aligned} Q \cap E_n &= Q \cap (E_n \setminus \bigcup_{k<\omega} (G_k \setminus E_k)) \\ &= Q \cap ((G_n \setminus \bigcup_{k<\omega} (G_k \setminus E_k)) \\ &= (Q \setminus \bigcup_{k<\omega} (G_k \setminus E_k)) \cap G_n = Q \cap G_n. \end{aligned}$$ □

The next corollary is a generalization of Theorem 1.4 of Marczewski and Theorems 1.5–1.6 of Brown and Elalaoui-Talibi.

Corollary 2.16. *Let τ_Y have a countable base. For any algebra described in the previous proposition, the following conditions are equivalent:*

(a) $f : X \to Y$ is $\mathcal{A}$-measurable,

(b) for any $\mathcal{F}$ coinitial with $\mathcal{A} \setminus H(\mathcal{A})$ we have $f \in S(\mathbb{R}, Y, \mathcal{F})$,

(c) for some $\mathcal{F}$ coinitial with $\mathcal{A} \setminus H(\mathcal{A})$ we have $f \in S(\mathbb{R}, Y, \mathcal{F})$.

PROOF. The assertion follows immediately from the previous proposition and Theorem 2.1. □

Let us turn to our second main result, Theorem 2.2. Note that it generalizes Theorem 1.3 (of Denjoy and Stepanoff) as well as Theorems 1.5 and 1.6.

PROOF OF THEOREM 2.2.
(a)$\Rightarrow$(b) Assume that for some space Y with a nontrivial topology τ_Y we have $\langle \mathcal{A}, \mathcal{I}, \tau \rangle \in \mathcal{D}(Y)$. Suppose that for some $E_0 \in \mathcal{A} \setminus \mathcal{I}$ the set $\operatorname{Int} E_0$ is empty. Let y_1, y_2 be points in Y for which there exists $U \in \tau_Y$ such that $y_1 \in U$ and $y_2 \in U^c$. Define $f : X \to Y$ by

$$f(x) = \begin{cases} y_1 & \text{for } x \in E_0, \\ y_2 & \text{for } x \in E_0^c. \end{cases} \tag{2}$$

Evidently, f is $\mathcal{A}$-measurable, so there exists a set $E \subset X$ such that $E^c \in \mathcal{I}$ and f is continuous on E. On the other hand, f is not continuous at any point of E_0. Thus $E_0 \subset E^c$ which means that $E_0 \in \mathcal{I}$ – a contradiction.
(b)$\Rightarrow$(c). By (b) and Theorem 2.3 (IV) we have $\langle \mathcal{A}, \mathcal{I} \rangle = \langle S(\mathcal{A} \setminus \mathcal{I}), S_0(\mathcal{A} \setminus \mathcal{I}) \rangle$. Observe that the families $\mathcal{A} \setminus \mathcal{I}$ and $\tau \setminus \mathcal{I}$ are mutually coinitial. Indeed, every set from $\tau \setminus \mathcal{I}$ belongs to $\mathcal{A} \setminus \mathcal{I}$. On the other hand, for $E \in \mathcal{A} \setminus \mathcal{I}$, we have $\operatorname{Int} E \in \tau \setminus \mathcal{I}$ because $E \setminus \operatorname{Int} E \in \mathcal{I}$. Hence $\langle S(\mathcal{A} \setminus \mathcal{I}), S_0(\mathcal{A} \setminus \mathcal{I}) \rangle = \langle S(\tau \setminus \mathcal{I}), S_0(\tau \setminus \mathcal{I}) \rangle$.
(c)$\Rightarrow$(b) Let $E \in \mathcal{A} \setminus \mathcal{I}$. Then $E \in S(\tau \setminus \mathcal{I}) \setminus S_0(\tau \setminus \mathcal{I})$ and consequently, there exists a set $U \in \tau \setminus \mathcal{I}$ such that $U \subset E$. Hence $\operatorname{Int} E \neq \emptyset$.
(b)$\Rightarrow$(d) It is enough to show that any $E \in \mathcal{A}$ is of the form $E = U \cup N$ for some $E \in \tau$ and $N \in \mathcal{I}$. We have $E = \operatorname{Int} E \cup (E \setminus \operatorname{Int} E)$ and $E \setminus \operatorname{Int} E \in \mathcal{I}$.
(d)$\Rightarrow$(b) Let $E \in \mathcal{A} \setminus \mathcal{I}$. Then U in the representation $E = U \cup N$ is nonempty.
(b)$\Rightarrow$(f) Note that $S_0(\tau^*) = NWD(\tau)$. Observe that $S_0(\tau^*) \cap \mathcal{A} \subset \mathcal{I}$ because every set $E \in S_0(\tau^*)$ has empty interior. Hence if $E \in S_0(\tau^*)$ and E is closed, then $E \in \mathcal{I}$. Let now $E \in S_0(\tau^*)$. Then E is nowhere dense and $\overline{E}$ is nowhere dense too. So $\overline{E} \in \mathcal{I}$ and consequently, $E \in \mathcal{I}$. Thus $S_0(\tau^*) \subset \mathcal{I}$.

Let $E \in S(\tau^*)$. Then E is a set with nowhere dense boundary and E is of the form $E = U \cup N$ where $U \in \tau$ and $N \in S_0(\tau^*) = NWD(\tau)$. By $S_0(\tau^*) \subset \mathcal{I}$ and (d) we have $E \in \mathcal{A}$. Thus $S(\tau^*) \subset \mathcal{A} = S(\mathcal{A} \setminus \mathcal{I}) \subset S(\mathcal{A} \setminus S_0(\tau^*))$. The last inclusion holds by [2, Lemma 6].
(b)$\Rightarrow$(e)

Proof of $(*)\Rightarrow(**)$. Assume now that an ideal $\mathcal{I}$ is κ-additive.
First we show that $\langle \mathcal{A}, \mathcal{A} \setminus \mathcal{I}, \tau \rangle \in \mathcal{M}(\kappa)$. Let $\alpha < \kappa$ and let $\mathcal{E} = \{E_\gamma : \gamma < \alpha\}$ be a family of sets contained in $\mathcal{A}$. Then any E_γ is of the form $E_\gamma = U_\gamma \cup N_\gamma$ where $U_\gamma \in \tau$, $N_\gamma \in \mathcal{I}$. Let P be an arbitrary set from $\mathcal{A} \setminus \mathcal{I}$. Define $N = \bigcup_{\gamma<\alpha} N_\gamma$ and $Q = P \setminus N$. By κ-additivity of $\mathcal{I}$ we have $N \in \mathcal{I}$ and hence $Q \in \mathcal{A} \setminus \mathcal{I}$. Since

$$Q \cap E_\gamma = Q \cap (U_\gamma \cup N_\gamma) = (Q \cap U_\gamma) \cup (Q \cap N_\gamma) = Q \cap U_\gamma,$$

$\langle \mathcal{A}, \mathcal{A} \setminus \mathcal{I}, \tau \rangle \in \mathcal{M}(\kappa)$ and consequently, $\langle \mathcal{A}, \tau \setminus \mathcal{I}, \tau \rangle \in \mathcal{M}(\kappa)$. Let the weight of τ_Y be smaller than κ. Hence, by Proposition 2.12, every $\mathcal{A}$-measurable function $f : X \to Y$ satisfies the Sierpiński condition with respect to $\tau \setminus \mathcal{I}$. Thus for each nonempty set $U \in \tau \setminus \mathcal{I}$ there exists $\mathcal{V} \in \tau \setminus \mathcal{I}$ such that $\mathcal{V} \subset U$ and $f \mid \mathcal{V}$ is continuous in τ. Equivalently f is continuous at every point of $\mathcal{V}$. Let $\mathcal{G}$ be a family of all sets $\mathcal{V} \in \tau$ such that f is continuous on $\mathcal{V}$. Put $E = \bigcup \mathcal{G}$. One can see that E^c does not belong to $\mathcal{A} \setminus \mathcal{I}$ because E^c has no nonempty open subset in $\mathcal{A} \setminus \mathcal{I}$. Hence $E^c \in \mathcal{I}$. The function f is obviously continuous at every point of E.

Proof of $(**) \Rightarrow (*)$.

This implication always holds. Indeed, let $U \in \tau_Y$. Then $f^{-1}(U) = (f^{-1}(U) \cap E) \cup (f^{-1}(U) \cap E^c)$. It is clear that $f^{-1}(U) \cap E^c \in \mathcal{I}$. Since f is continuous at every point of E, we have $f^{-1}(U) \cap E = G \cap E$ for some $G \in \tau \subset \mathcal{A}$. Consequently $f^{-1}(U) \in \mathcal{A}$. □

Corollary 2.17. *Assume that, for a topology τ in X, the ideal $S_0(\tau^*)$ has additivity κ. Then $\langle S(\tau^*), S_0(\tau^*), \tau \rangle \in \mathcal{D}(\kappa)$.*

Corollary 2.18. *Let $\mathcal{I}$ be a σ-ideal contained in an algebra $\mathcal{A}$ and let τ_Y have a countable base. Assume that $\tau \subset \mathcal{A}$ is a topology such that τ^* is coinitial with $\mathcal{A} \setminus \mathcal{I}$ (which means that any set $E \in \mathcal{A}$ has nonempty interior if and only if $E \in \mathcal{A} \setminus \mathcal{I}$). Then $\mathcal{A}$ is a σ-algebra and the following conditions are equivalent:*

(i) *$f : X \to Y$ is $\mathcal{A}$-measurable,*

(ii) *there exists a set $E \subset X$ such that $E^c \in \mathcal{I}$ and f is continuous at every point of E.*

Remark 2.19. Implication (b)⇒(e) in Theorem 2.2 (under the assumption that an ideal $\mathcal{I}$ is ω_1-additive) is stronger than Theorem 1.6. Indeed, the assumptions of Theorem 1.6 are equivalent to the fact that $\mathcal{I}$ is the ideal of nowhere dense sets in τ and simultaneously of sets of the first category in τ, and the sets from $\mathcal{I}$ are closed. Even Corollary 2.18 is a bit stronger; the sets belonging to $\mathcal{I}$ need not be closed. (See for example [12, Exercise 62(b)].) Observe that $\langle \mathcal{L}, \mathcal{N}, \tau_d \rangle$ and $\langle \mathcal{B}, \mathcal{M}, \tau_{\mathcal{M}-d} \rangle$, where τ_d and $\tau_{\mathcal{M}-d}$ stand respectively for density and $\mathcal{M}$-density topologies [20], satisfy Theorem 1.6 and our Corollary 2.18. However, the triple $\langle \mathcal{L}, \mathcal{N}, \tau_0 \rangle$, where τ_0 is a topology described in [12, Exercise 62(b)], satisfies Corollary 2.18 but not Theorem 1.6. Note also that there exists a topology τ for which $\langle (s), (s_0), \tau \rangle$ fulfills assumptions of Corollary 2.18 [7] whereas it is not clear whether it satisfies assumptions of Theorem 1.6. Finally, notice that in the case of all triples mentioned above, we have applied Theorem 2.2 with $\kappa = \omega_1$. One can show that for any regular $\kappa > \omega_1$ there exists $\langle \mathcal{A}, \mathcal{I}, \tau \rangle \in D(\kappa)$, where $\mathcal{I}$ is κ-additive and any set $E \in \mathcal{A} \setminus \mathcal{I}$ has nonempty interior [7].

The following example shows that Theorem 2.2 is essentially stronger than Corollary 2.18.

Example 2.20. Let $X_0 = \mathbb{R} \times \{0\}$, $X_1 = \mathbb{R} \times \{1\}$ and $X = X_0 \cup X_1$. Denote by τ_d, τ_1 the density topology and the natural topology in $\mathbb{R}$, respectively. Define $\mathcal{A}$, $\mathcal{I}$ and τ by

$$\begin{aligned}\mathcal{A} &= \{(A \times \{0\}) \cup (B \times \{1\}) \colon A \in \mathcal{L},\ B \in \mathcal{P}(\mathbb{R})\},\\ \mathcal{I} &= \{(A \times \{0\}) \cup (B \times \{1\}) \colon A \in \mathcal{N},\ B \in \mathcal{P}(\mathbb{R})\},\\ \tau &= \{(A \times \{0\}) \cup (B \times \{1\}) \colon A \in \tau_d,\ B \in \tau_1\}.\end{aligned}$$

One can check that the triple of $\langle \mathcal{A}, \mathcal{I}, \tau \rangle$ ($\mathcal{I}$ is a σ-ideal) satisfies condition (b) in Theorem 2.2 but does not satisfy assumptions of Corollary 2.18.

3 New Applications.

Let (X, τ) be a topological space and let $\mathcal{B}or(X)$ stand for the family of all Borel subsets of X. For a fixed ideal $\mathcal{I}$ of subsets of X by $\Sigma_{\mathcal{I}}$ we denote the algebra generated by $\mathcal{B}or(X) \cup \mathcal{I}$. In this section we are going to apply general results obtained in the previous section for $\mathcal{A} = \Sigma_{\mathcal{I}}$ and $\mathcal{F} = \Sigma_{\mathcal{I}} \setminus \mathcal{I}$ when $\mathcal{I}$ is equal to the intersection of two given ideals or $\mathcal{I}$ is the product of ideals.

At first however, let us define a condition which, in the case of the considered algebras, seems more convenient to be checked than condition $M(\kappa)$ defined in Section 2.

Definition 3.1. Let $\mathcal{A}$ be an algebra of subsets of X and let $\mathcal{F}$ be a fixed nonempty collection of nonempty subsets of X. We say that $\langle \mathcal{A}, \mathcal{F}, \tau \rangle \in M_*(\kappa)$, where κ is any cardinal number, if

$$\forall_{\alpha<\kappa}\ \forall_{\{E_\gamma : \gamma<\alpha\} \subset \mathcal{A}}\ \forall_{P \in \mathcal{F}}\ \exists_{\{G_\gamma : \gamma<\alpha\} \subset \tau}\ \ \Big(P \setminus \bigcup_{\gamma<\alpha} (E_\gamma \triangle G_\gamma) \in \mathcal{F}\Big). \qquad M_*(\kappa)$$

As the following lemmas show, condition $M_*(\kappa)$ (in general stronger than $M(\kappa)$) turns out to be equivalent to $M(\kappa)$ for $\mathcal{F} = \mathcal{A} \setminus \mathcal{I}$ where $\mathcal{I} \subset \mathcal{A}$ is any ideal.

Lemma 3.1. *If $\langle \mathcal{A}, \mathcal{F}, \tau \rangle \in M_*(\kappa)$, then $\langle \mathcal{A}, \mathcal{F}, \tau \rangle \in M(\kappa)$.*

PROOF. Let $\alpha < \kappa$. By assumption, for any family $\{E_\gamma : \gamma < \alpha\} \subset \mathcal{A}$ and $P \in \mathcal{F}$, there exists a family $\{G_\gamma : \gamma < \alpha\} \subset \tau$ such that $P \setminus \bigcup_{\gamma<\alpha} (E_\gamma \triangle G_\gamma) \in \mathcal{F}$. Put $Q = P \setminus \bigcup_{\gamma<\alpha} (E_\gamma \triangle G_\gamma)$. It is obvious that $Q \in \mathcal{F}$ and $Q \subset P$. Moreover, for any $\gamma < \alpha$, we have $Q \cap (E_\gamma \triangle G_\gamma) = \emptyset$ and hence $Q \cap E_\gamma = Q \cap G_\gamma$. □

Lemma 3.2. *If a κ-additive algebra $\mathcal{A}$ contains a topology τ and $\mathcal{I} \subset \mathcal{A}$ is a fixed ideal of subsets of X, then the following conditions are equivalent:*

(i) $\langle \mathcal{A}, \mathcal{A} \setminus \mathcal{I}, \tau \rangle \in M(\kappa)$,

(ii) $\langle \mathcal{A}, \mathcal{A} \setminus \mathcal{I}, \tau \rangle \in M_*(\kappa)$.

PROOF. By virtue of the previous lemma, it is enough to prove implication (i) ⇒ (ii).

Let $\alpha < \kappa$. Consider a family $\{E_\gamma : \gamma < \alpha\} \subset \mathcal{A}$ and $P \in \mathcal{A} \setminus \mathcal{I}$. By assumption (i), we can find a family $\{G_\gamma : \gamma < \alpha\} \subset \tau$ and a set $Q \in \mathcal{A} \setminus \mathcal{I}$ such that $Q \subset P$ and

$$\forall_{\gamma<\alpha} \ \ (Q \cap E_\gamma = Q \cap G_\gamma). \tag{3}$$

Because $\tau \subset \mathcal{A}$, it is clear that $P \setminus \bigcup_{\gamma<\alpha} (E_\gamma \triangle G_\gamma) \in \mathcal{A}$.

Suppose now that $P \setminus \bigcup_{\gamma<\alpha} (E_\gamma \triangle G_\gamma) \in \mathcal{I}$. From (3) we derive that $Q \cap (E_\gamma \triangle G_\gamma) = \emptyset$ for any $\gamma < \alpha$. Hence

$$\begin{aligned} P \setminus \bigcup_{\gamma<\alpha} (E_\gamma \triangle G_\gamma) &\supset Q \setminus \bigcup_{\gamma<\alpha} (E_\gamma \triangle G_\gamma) \\ &= \bigcap_{\gamma<\alpha} (Q \setminus (E_\gamma \triangle G_\gamma)) = Q \end{aligned}$$

and consequently, $Q \in \mathcal{I}$ which is a contradiction. □

Lemma 3.3. *Assume that a σ-algebra $\mathcal{A}$ contains $\mathcal{B}or(X)$. Let $\mathcal{I} \subset \mathcal{A}$ be a fixed σ-ideal of subsets of X such that*

$$\forall_{A\in\mathcal{A}} \ \exists_{B\in\mathcal{B}or(X)} \ \ (B \subset A \textit{ and } A \setminus B \in \mathcal{I}). \tag{4}$$

Then the following conditions are equivalent:

(i) $\langle \mathcal{A}, \mathcal{A} \setminus \mathcal{I}, \tau \rangle \in M_*(\omega_1)$,

(ii) $\langle \mathcal{B}or(X), \mathcal{B}or(X) \setminus \mathcal{I}, \tau \rangle \in M_*(\omega_1)$.

PROOF. (i) $\Rightarrow$ (ii) Obvious, because $\tau \subset \mathcal{B}or(X)$.
(ii) $\Rightarrow$ (i) Let us start with the observation which will be useful in the sequel.

$$\text{For any sets } E, G \subset X \text{ if } \widetilde{E} \subset E, \text{ then } E \triangle G \subset (\widetilde{E} \triangle G) \cup (E \setminus \widetilde{E}). \tag{5}$$

Take a family $\{E_\gamma : \gamma < \alpha\} \subset \mathcal{A}$ with $\alpha < \omega_1$, and $P \in \mathcal{A} \setminus \mathcal{I}$. By (4), for a fixed $\gamma < \alpha$, there exists a Borel set $\widetilde{E}_\gamma \subset E_\gamma$ such that $E_\gamma \setminus \widetilde{E}_\gamma \in \mathcal{I}$. Similarly we can pick a set $\widetilde{P} \subset P$ satisfying $\widetilde{P} \in \mathcal{B}or(X) \setminus \mathcal{I}$. Then assumption (ii) implies the existence of a family $\{G_\gamma : \gamma < \alpha\} \subset \tau$ such that $\widetilde{P} \setminus \bigcup_{\gamma<\alpha} (\widetilde{E}_\gamma \triangle G_\gamma) \in \mathcal{B}or(X) \setminus \mathcal{I}$. Making use of observation (5) we obtain

$$\begin{aligned} P \setminus \bigcup_{\gamma<\alpha} (E_\gamma \triangle G_\gamma) &\supset \widetilde{P} \setminus \bigcup_{\gamma<\alpha} (E_\gamma \triangle G_\gamma) \\ &\supset \widetilde{P} \setminus \bigcup_{\gamma<\alpha} ((\widetilde{E}_\gamma \triangle G_\gamma) \cup (E_\gamma \setminus \widetilde{E}_\gamma)) \\ &= (\widetilde{P} \setminus \bigcup_{\gamma<\alpha} (\widetilde{E}_\gamma \triangle G_\gamma)) \setminus \bigcup_{\gamma<\alpha} (E_\gamma \setminus \widetilde{E}_\gamma). \end{aligned}$$

Hence $P \setminus \bigcup_{\gamma<\alpha} (E_\gamma \triangle G_\gamma) \in \mathcal{A} \setminus \mathcal{I}$ as desired. □

Notice that if a σ-ideal $\mathcal{I}$ has Borel base (That is, for every set $A \in \mathcal{I}$ there exists $B \in \mathcal{B}or(X) \cap \mathcal{I}$ such that $A \subset B$.), then the σ-algebra $\Sigma_{\mathcal{I}}$ consists of sets of the form $B \cup F$ where $B \in \mathcal{B}or(X)$ and $F \in \mathcal{I}$. Hence one can easily conclude that the assumption (4) in Lemma 3.3 is satisfied in the case of $\mathcal{A} = \Sigma_{\mathcal{I}}$. Thus from Lemmas 3.2 and 3.3 the next assertion follows immediately.

Proposition 3.4. *If a σ-ideal $\mathcal{I}$ of subsets of X possesses Borel base then the following conditions are equivalent:*

(i) $\langle \Sigma_{\mathcal{I}}, \Sigma_{\mathcal{I}} \setminus \mathcal{I}, \tau \rangle \in M(\omega_1)$,

(ii) $\langle \mathcal{B}or(X), \mathcal{B}or(X) \setminus \mathcal{I}, \tau \rangle \in M_*(\omega_1)$.

3.1 Applications to Algebras $\Sigma_{\mathcal{J} \cap \mathcal{K}}$

Let $\mathcal{J}, \mathcal{K}$ be fixed ideals of subsets of X. Note that if the assumptions of Theorem 2.1 are fulfilled for $\mathcal{A} = \Sigma_{\mathcal{I}}$ and $\mathcal{F} = \Sigma_{\mathcal{I}} \setminus \mathcal{I}$ where $\mathcal{I} \in \{\mathcal{J}, \mathcal{K}\}$, then the natural question arises, whether the mentioned theorem works also with $\mathcal{I} = \mathcal{J} \cap \mathcal{K}$. As Theorem 3.5 will show, the answer is affirmative assuming that the both considered ideals have some additional properties.

Theorem 3.5. *Assume that σ-ideals $\mathcal{J}, \mathcal{K}$ of subsets of X possess Borel bases, a topology τ is contained in $\Sigma_{\mathcal{J}} \cap \Sigma_{\mathcal{K}}$ and τ_Y has a countable base. If pairs $\langle \Sigma_{\mathcal{J}}, \mathcal{J} \rangle, \langle \Sigma_{\mathcal{K}}, \mathcal{K} \rangle$ have the hull property and*

$$\langle \mathcal{B}or(X), \mathcal{B}or(X) \setminus \mathcal{J}, \tau \rangle, \ \langle \mathcal{B}or(X), \mathcal{B}or(X) \setminus \mathcal{K}, \tau \rangle \in M_*(\omega_1),$$

then the following conditions are equivalent:

(i) $f : X \to Y$ *is $\Sigma_{\mathcal{J} \cap \mathcal{K}}$-measurable,*

(ii) $f \in S(X, Y, \Sigma_{\mathcal{J} \cap \mathcal{K}} \setminus (\mathcal{J} \cap \mathcal{K}))$.

PROOF. It is fairly easy to show that, under the above assumptions, the ideal $\mathcal{J} \cap \mathcal{K}$ possesses Borel base and $\langle \mathcal{B}or(X), \mathcal{B}or(X) \setminus (\mathcal{J} \cap \mathcal{K}), \tau \rangle \in M_*(\omega_1)$. If the pair $\langle \Sigma_{\mathcal{J} \cap \mathcal{K}}, \mathcal{J} \cap \mathcal{K} \rangle$ had the hull property, we would derive the assertion from Corollary 2.14 and Proposition 3.4. The remaining part of the proof follows from Lemma 3.6. $\square$

Lemma 3.6. *If ideals $\mathcal{J}, \mathcal{K}$ possess Borel bases and pairs $\langle \Sigma_{\mathcal{J}}, \mathcal{J} \rangle, \langle \Sigma_{\mathcal{K}}, \mathcal{K} \rangle$ have the hull property, then $\langle \Sigma_{\mathcal{J} \cap \mathcal{K}}, \mathcal{J} \cap \mathcal{K} \rangle$ has the hull property too.*

PROOF. Take any set $A \subset X$. By the respective assumptions, there are Borel sets B_1, B_2 containing A such that

$$(\forall_{F_1 \in \Sigma_{\mathcal{J}}} \ (A \subset F_1 \Rightarrow B_1 \setminus F_1 \in \mathcal{J})) \text{ and } (\forall_{F_2 \in \Sigma_{\mathcal{K}}} \ (A \subset F_2 \Rightarrow B_2 \setminus F_2 \in \mathcal{K})). \tag{6}$$

Put $B = B_1 \cap B_2$. Then evidently $A \subset B$ and $B \in \Sigma_{\mathcal{J} \cap \mathcal{K}}$. Let $F \in \Sigma_{\mathcal{J} \cap \mathcal{K}} = \Sigma_{\mathcal{J}} \cap \Sigma_{\mathcal{K}}$ be any superset of A. Then by (6) we get $B \setminus F = (B_1 \setminus F) \cap (B_2 \setminus F) \in \mathcal{J} \cap \mathcal{K}$. $\square$

It is known that the pairs $\langle \mathcal{L}, \mathcal{N} \rangle$, $\langle \mathcal{B}, \mathcal{M} \rangle$ have the hull property, so from Theorem 3.5 we conclude

Corollary 3.7. *A function $f : \mathbb{R} \to \mathbb{R}$ is $\mathcal{L}$-measurable and $\mathcal{B}$-measurable simultaneously if and only if f satisfies the Sierpiński condition with respect to $(\mathcal{B} \cap \mathcal{L}) \setminus (\mathcal{N} \cap \mathcal{M})$.*

3.2 Applications to Algebras $\Sigma_{\mathcal{J}\widetilde{\otimes}\mathcal{K}}$

Let $\mathcal{J},\mathcal{K}$ be fixed ideals in X.

Definition 3.2. (Cf. [13]) Define the ideal

$$\mathcal{J}\otimes\mathcal{K} = \left\{A \subset X^2 : \{x : A^x \notin \mathcal{K}\} \in \mathcal{J}\right\},$$

where $A^x = \{y : (x,y) \in A\}$ for $x \in X$. By the *product ideal* we mean the ideal

$$\mathcal{J}\widetilde{\otimes}\mathcal{K} = \left\{A \subset X^2 : (\exists B \in \mathcal{B}or(X^2) \cap (\mathcal{J}\otimes\mathcal{K}))\ A \subset B\right\}.$$

Remark 3.8. It is easy to observe that $\mathcal{J}\widetilde{\otimes}\mathcal{K} \subset \mathcal{J}\otimes\mathcal{K}$ and $\mathcal{B}or(X^2)\setminus \mathcal{J}\widetilde{\otimes}\mathcal{K} = \mathcal{B}or(X^2)\setminus\mathcal{J}\otimes\mathcal{K}$. Moreover, directly from the definition it follows that product ideals possess Borel bases.

Recall that if $\mathcal{J}=\mathcal{K}=\mathcal{N}$ or $\mathcal{J}=\mathcal{K}=\mathcal{M}$, then $\mathcal{J}\widetilde{\otimes}\mathcal{K}$ is (by the Fubini theorem and by the Kuratowski-Ulam theorem) equal to the σ-ideal of plane Lebesgue null sets or plane meager sets, respectively. The mixed product ideals $\mathcal{N}\widetilde{\otimes}\mathcal{M}$ and $\mathcal{M}\widetilde{\otimes}\mathcal{N}$ form new interesting ideals. In [17] it was observed that pairs $\langle\Sigma_{\mathcal{I}},\mathcal{I}\rangle$, for $\mathcal{I}\in\{\mathcal{N}\widetilde{\otimes}\mathcal{M},\mathcal{M}\widetilde{\otimes}\mathcal{N}\}$, have inner MB-representation with $\mathcal{F}=\Sigma_{\mathcal{I}}\setminus\mathcal{I}$. (More precisely this observation follows from the cited result of Baldwin [6] and the fact that the considered pairs have the hull property.) So, by Proposition 3.4, to apply Theorem 2.1, it suffices to verify that $\langle\mathcal{B}or(\mathbb{R}^2),\mathcal{B}or(\mathbb{R}^2)\setminus\mathcal{I},\tau_2\rangle \in M_*(\omega_1)$ where τ_2 stands for the natural topology in $\mathbb{R}^2$. To this aim we will need the following structural theorems due to M. Balcerzak (for some related results, see [11]).

Theorem 3.9. *[1, Prop. 2.1, page 63] For each Borel set $B\subset\mathbb{R}^2$ and for a fixed base $\{U_n : n<\omega\}$ of nonempty open subsets of $\mathbb{R}^2$, there is a sequence $\{F_n : n<\omega\}$ of F_σ sets in $\mathbb{R}$, such that $B\triangle\bigcup_{n<\omega}(F_n\times U_n)\in\mathcal{N}\widetilde{\otimes}\mathcal{M}$.*

Theorem 3.10. *[1, Lemma 2.3, page 64] For each Borel set $B\subset\mathbb{R}^2$ and every $\varepsilon>0$, there exist an open set $G\subset\mathbb{R}^2$ and a meager set $C\subset\mathbb{R}$ such that $B\subset(C\times\mathbb{R})\cup G$ and $\mu((G\setminus B)^x)<\varepsilon$ for each $x\in\mathbb{R}\setminus C$.*

Theorem 3.11. *If $\mathcal{I}\in\{\mathcal{N}\widetilde{\otimes}\mathcal{M},\mathcal{M}\widetilde{\otimes}\mathcal{N}\}$, then $\langle\mathcal{B}or(\mathbb{R}^2),\mathcal{B}or(\mathbb{R}^2)\setminus\mathcal{I},\tau_2\rangle\in M_*(\omega_1)$.*

Proof. • The case of $\mathcal{I} = \mathcal{N} \widetilde{\otimes} \mathcal{M}$

Consider a family $\{E_n : n < \omega\} \subset \mathcal{B}or(\mathbb{R}^2)$ and $P \in \mathcal{B}or(\mathbb{R}^2) \setminus \mathcal{N} \widetilde{\otimes} \mathcal{M}$. Put $\widetilde{P} = \{x \in \mathbb{R}\colon P^x \in \mathcal{B} \setminus \mathcal{M}\}$. Being a Borel set (see [14]), $\widetilde{P}$ is obviously Lebesgue measurable. Thus $\varepsilon = \mu(\widetilde{P})$ is a positive number. Let $\{U_n : n < \omega\}$ stand for the family of all bounded open intervals with rational endpoints. By Theorem 3.9, for each $n < \omega$ there exists a family $\{F_{n,m} : m < \omega\}$ of F_σ subsets of $\mathbb{R}$ such that

$$\forall_{n<\omega} \quad E_n \triangle \bigcup_{m<\omega} (F_{n,m} \times U_m) \in \mathcal{N} \widetilde{\otimes} \mathcal{M}. \tag{7}$$

For fixed $n, m < \omega$ pick open sets $G_{n,m}$ such that

$$F_{n,m} \subset G_{n,m} \text{ and } \mu(G_{n,m} \setminus F_{n,m}) < \frac{\varepsilon}{2^{n+m+1}}.$$

Then for any $n < \omega$ let $F_n = \bigcup_{m<\omega} (F_{n,m} \times U_m)$ and $G_n = \bigcup_{m<\omega} (G_{n,m} \times U_m)$. Clearly, the sets G_n are open in $\mathbb{R}^2$ which implies that $P \setminus \bigcup_{n<\omega} (E_n \triangle G_n)$ is a Borel set.

Suppose now that $P \setminus \bigcup_{n<\omega} (E_n \triangle G_n) \in \mathcal{N} \otimes \mathcal{M}$. By (5) we have

$$\begin{aligned} P \setminus \bigcup_{n<\omega} (E_n \triangle G_n) &\supset P \setminus \bigcup_{n<\omega} ((E_n \triangle F_n) \cup (G_n \setminus F_n)) \\ &= P \setminus \bigcup_{n<\omega} (G_n \setminus F_n)) \setminus \bigcup_{n<\omega} (E_n \triangle F_n). \end{aligned}$$

From (7) we conclude that $P \setminus \bigcup_{n<\omega} (G_n \setminus F_n) \in \mathcal{N} \otimes \mathcal{M}$, which means that $\{x : (P \setminus \bigcup_{n<\omega} (G_n \setminus F_n))^x \notin \mathcal{M}\} \in \mathcal{N}$. To get a contradiction observe that for all $x \notin \bigcup_{n<\omega} \bigcup_{m<\omega} (G_{n,m} \setminus F_{n,m})$ we have

$$\left(\bigcup_{n<\omega} \bigcup_{m<\omega} (G_{n,m} \setminus F_{n,m}) \times U_m \right)^x = \emptyset,$$

and

$$\mu(\bigcup_{n<\omega} \bigcup_{m<\omega} (G_{n,m} \setminus F_{n,m})) < \sum_{n<\omega} \sum_{m<\omega} \frac{\varepsilon}{2^{n+m+1}} = \frac{\varepsilon}{2}.$$

Thus for each $x \notin \bigcup_{n<\omega} \bigcup_{m<\omega} (G_{n,m} \setminus F_{n,m})$ we obtain

$$(P \setminus \bigcup_{n<\omega} (G_n \setminus F_n))^x = P^x \setminus (\bigcup_{n<\omega} \bigcup_{m<\omega} (G_{n,m} \setminus F_{n,m}) \times U_m)^x = P^x,$$

Consequently, $\mu(\{x : (P \setminus \bigcup_{n<\omega} (G_n \setminus F_n))^x \notin \mathcal{M}\}) > \frac{\varepsilon}{2}$.

By Remark 3.8 we get $P \setminus \bigcup_{n<\omega} (E_n \triangle G_n) \in \mathcal{B}or(\mathbb{R}^2) \setminus (\mathcal{N} \widetilde{\otimes} \mathcal{M})$ as desired.

- The case of $\mathcal{I} = \mathcal{M} \widetilde{\otimes} \mathcal{N}$

Take a family $\{E_n : n < \omega\} \subset \mathcal{B}or(\mathbb{R}^2)$ and $P \in \mathcal{B}or(\mathbb{R}^2) \setminus (\mathcal{M} \widetilde{\otimes} \mathcal{N})$. For $k < \omega$ we define the sets $P_k = \{x : \mu(P^x) > \frac{2}{k}\}$. From [14] it follows that all P_k, $k < \omega$, are Borel subsets of $\mathbb{R}$. Moreover, we can choose $k_0 < \omega$ such that $P_{k_0} \notin \mathcal{M}$. Put $\varepsilon_n = \frac{1}{2^n k_0}$ for any $n < \omega$. According to Theorem 3.10, there exist families $\{G_n : n < \omega\} \subset \tau_2$ and $\{C_n : n < \omega\} \subset \mathcal{M}$ such that $E_n \subset G_n \cup (C_n \times \mathbb{R})$ and $\mu([G_n \setminus E_n]^x) < \varepsilon_n$ for all $x \in \mathbb{R} \setminus C_n$ and $n < \omega$. Let $C = \bigcup_{n<\omega} C_n$. Then $C \in \mathcal{M}$ and $C \times \mathbb{R} \in \mathcal{M} \otimes \mathcal{N}$.

As in the previous case, suppose that $P \setminus \bigcup_{n<\omega} (E_n \triangle G_n) \in \mathcal{M} \otimes \mathcal{N}$. From

$$P \setminus \bigcup_{n<\omega} (E_n \triangle G_n) = (P \setminus \bigcup_{n<\omega} (G_n \setminus E_n)) \setminus \bigcup_{n<\omega} (E_n \setminus G_n)$$

it follows that $P \setminus \bigcup_{n<\omega} (G_n \setminus E_n) \in \mathcal{M} \otimes \mathcal{N}$. On the other hand

$$\begin{aligned} \mu((P \setminus \bigcup_{n<\omega} (G_n \setminus E_n))^x) &= \mu(P^x \setminus \bigcup_{n<\omega} (G_n \setminus E_n)^x) \\ &> \mu(P^x) - \sum_{n<\omega} \mu((G_n \setminus E_n)^x) > \frac{1}{k_0} > 0 \end{aligned}$$

for each $x \in (\mathbb{R} \setminus C) \cap P_{k_0} \notin \mathcal{M}$, which is a contradiction. □

Now from Theorems 2.1, 3.5 and 3.11 we derive

Theorem 3.12. *Let $\mathcal{I} \in \{\mathcal{N} \widetilde{\otimes} \mathcal{M},\ \mathcal{M} \widetilde{\otimes} \mathcal{N},\ (\mathcal{N} \widetilde{\otimes} \mathcal{M}) \cap (\mathcal{M} \widetilde{\otimes} \mathcal{N})\}$. Then every function $f : \mathbb{R}^2 \to \mathbb{R}$ is $\Sigma_{\mathcal{I}}$-measurable if and only if f satisfies the Sierpiński condition with respect to the family $\Sigma_{\mathcal{I}} \setminus \mathcal{I}$.*

In the case of product ideals, as for the ideals of all Lebesgue null sets and meager sets in $\mathbb{R}$, one can introduce the notion of a density point of a measurable set (which lead to the notion of a density-type topology) as follows.

Definition 3.3. [5, Def. 2.2] Assume that $\mathcal{J}$ and $\mathcal{K}$ are σ-ideals of subsets of $\mathbb{R}$, invariant with respect to translation. A point $(x_0, y_0) \in \mathbb{R}^2$ is an $\mathcal{J} \widetilde{\otimes} \mathcal{K}$-density point of $E \in \Sigma_{\mathcal{J} \widetilde{\otimes} \mathcal{K}}$ if for each increasing sequence (n_i) of positive integers there are a subsequence (n_{i_k}) and a set $A \in \mathcal{J}$ such that

$$\forall_{x \in (-1,1) \setminus A} \quad \limsup_{k\to\infty} [(-1, 1) \setminus (((n_{i_k}, n_{i_k}) \cdot E)^{x+x_0} - y_0)] \in \mathcal{K}$$

where $(t,t)\cdot E=\{(tx,ty):(x,y)\in E)\}$ for $t\in\mathbb{R}$.

Let $\mathcal{I}\in\{\mathcal{N}\widetilde{\otimes}\mathcal{M},\mathcal{M}\widetilde{\otimes}\mathcal{N}\}$. Denote by $\varphi_{\mathcal{I}}(E)$ the set of all $\mathcal{I}$-density points of $E\in\Sigma_{\mathcal{I}}$. It was proved in [5] that the family $\tau_{\mathcal{I}}=\{E\in\Sigma_{\mathcal{I}}:E\subset\varphi_{\mathcal{I}}(E)\}$ forms a topology in $\mathbb{R}^2$. From [5, Prop. 3.3 and 4.3] it follows that the respective version of the Lebesgue density theorem holds for the operator $\varphi_{\mathcal{I}}$ in both cases. Thus we are in a position to apply Corollary 2.18 (or Theorem 1.6 from [12]) and formulate the following Denjoy-Stepanoff type result.

Theorem 3.13. *For $\mathcal{I}\in\{\mathcal{N}\widetilde{\otimes}\mathcal{M},\mathcal{M}\widetilde{\otimes}\mathcal{N}\}$ the following conditions are equivalent:*

(i) $f:\mathbb{R}^2\to\mathbb{R}$ is $\Sigma_{\mathcal{I}}$-measurable,

(ii) there exists a set $E\subset\mathbb{R}^2$ such that $E^c\in\mathcal{I}$ and f is continuous on E with respect to the topology $\tau_{\mathcal{I}}$.

This solves a problem posed in [5].

Added in Proof. Some similar problems were considered in the paper, *Restrictions to continuous function and Boolean algebras*, Proc. Amer. Math. Soc., **118, no. 3** (1993), 791–796, by I. Recław, who obtained several theorems which however go in a different direction in comparison to ours.

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