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ON THE RIGHT PREPONDERANT LIMIT

Abstract

In article [4] D. N. Sarkhel investigates the right preponderant limit of a function and he proves that a such finite limit is of Baire 1 class. In this article I generalize this Sarkhel's result.

Let $\mathbb{R}$ be the set of all reals. Denote by μ the Lebesgue measure in $\mathbb{R}$ and by μ_e the outer Lebesgue measure in $\mathbb{R}$. For a set $A \subset \mathbb{R}$ and a point x we define the upper (lower) outer right density $D_u^+(A,x)$ ($D_l^+(A,x)$) of the set A at the point x as

$$\limsup_{h\to 0^+} \frac{\mu_e(A\cap[x,x+h])}{h}$$

$$(\liminf_{h\to 0^+} \frac{\mu_e(A\cap[x,x+h])}{h} \text{ respectively}).$$

In [4] D. N. Sarkhel investigates the following notion:

A function $F : \mathbb{R} \to \mathbb{R}$ is said to have finite right preponderant limit p at a point $c \in \mathbb{R}$, if there is a number $r \in [0, \frac{1}{2})$ so that for each $\eta > 0$ the upper right density

$$D_u^+(\{x \in (c,\infty); |F(x)-p| \geq \eta\}, c) \leq r$$

Moreover in [4] Sarkhel proves that if $F : [a,b] \to \mathbb{R}$ has finite right preponderant limit $f(x)$ at each point $x \in [a,b)$ then f is Baire one on $[a,b)$.

In this article I consider a more general property of f which imply that f is Baire 1.

Remark 1. Firstly we observe that each function $F : [a,b] \to \mathbb{R}$ having finite right preponderant limit at each point $x \in [a,b)$ is measurable (in the Lebesgue sense).

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PROOF. Assume, to a contrary that F is not measurable. Then there are reals c, d and a measurable set $A \subset [a,b)$ such that $c < d$, $\mu(A) > 0$ and

$$\mu(A) = \mu_e(A_1 = \{x \in A; F(x) < c\}) = \mu_e(A_2 = \{x \in A; F(x) > d\}).$$

There is a point $y \in A_1$ with $D_l^+(A_1, y) = D_l^+(A_2, y) = 1$. So for $\eta = \frac{d-c}{3}$ and for each real $p \in \mathbb{R}$ we have

$$D_u^+(\{x; |F(x) - p| \geq \eta\}, y) = 1 > \frac{1}{2},$$

and F does not have any finite right preponderant limit at y. This finishes the proof. $\square$

Let $F : \mathbb{R} \to \mathbb{R}$ be a measurable function and let $c \in \mathbb{R}$ be a point. We will say that a real $p \in L_r(F, c)$ if for each real $\eta > 0$ there is a positive real r_c such that for each real $h \in (0, r_c]$ the inequalities

$$\frac{\mu([c, c+h] \cap F^{-1}((p - \eta, \infty)))}{h} > \frac{1}{2}$$

and

$$\frac{\mu([c, c+h] \cap F^{-1}((-\infty, p + \eta)))}{h} > \frac{1}{2}.$$

are true.

Evidently if a real $p \in \mathbb{R}$ is a right preponderant limit of a function $F : \mathbb{R} \to \mathbb{R}$ at a point c then $p \in L_r(F, c)$. The following example shows that the inverse implication is not true.

Example. For $n \geq 1$ there are closed intervals

$$I_n = [\frac{1}{n+1}, \frac{1}{n}], \quad J_n = [a_n, b_n] \text{ and } K_n = [c_n, d_n]$$

such that

$$K_n \subset int(J_n) \subset J_n \subset int(I_n)$$

and

$$D_l^+(\bigcup_n [\frac{1}{n+1}, a_n], 0) = D_l^+(\bigcup_n [b_n, \frac{1}{n}], 0) = \frac{1}{2},$$

and for each real $h > 0$ the inequalities

$$\frac{\mu([0, h] \cap \bigcup_n ([\frac{1}{n+1}, a_n] \cup K_n))}{h} > \frac{1}{2}$$

and

$$\frac{\mu([0,h]\cap\bigcup_n([b_n,\frac{1}{n}]\cup K_n))}{h}>\frac{1}{2}$$

are true.

Let

$$F(x)=0 \text{ for } x\in(-\infty,0]\cup\bigcup_n K_n,$$

$$F(x)=-1 \text{ for } x\in[\frac{1}{n+1},a_n],\ \ n\geq 1,$$

$$F(x)=1 \text{ for } x\in[1,\infty)\cup\bigcup_n[b_n,\frac{1}{n}],$$

and F is linear on the intervals $[a_n,c_n]$ and $[d_n,b_n]$, where $n\geq 1$. Then the values $F(x)\in L_r(F,x)$ for each point $x\in\mathbb{R}$, but F does not have any right preponderant limit at 0.

Theorem 1. *Let $F:\mathbb{R}\to\mathbb{R}$ be a measurable function. If a function $f:\mathbb{R}\to\mathbb{R}$ is such that $f(x)\in L_r(F,x)$ for all $x\in\mathbb{R}$ then f is Baire one class.*

Proof. Assume, to a contradiction that f is not Baire one class. Then there is a nonempty perfect set A such that $osc(f/A)(x)>0$ at each point $x\in A$. So for each point $x\in A$ there is a pair $(u(x),v(x))$ of rationals such that

$$u(x)<v(x) \text{ and } x\in cl(\{t\in A;f(t)\leq u(x)\})\cap cl(\{t\in A;f(t)\geq v(x)\}),$$

($cl(X)$ denotes the closure of X). Let $((u_n,v_n))$ be an enumeration of all pairs of rationals with $u_n<v_n$ for $n\geq 1$ and let $A_n=\{x\in A;(u(x),v(x))=(u_n,v_n)\}$. Observe that each set A_n is closed and

$$A=\bigcup_n A_n.$$

Since A is a complete metric space, it is of the second category in itself and consequently there is a positive integer k such that A_k is of the second category in A. There is an open interval I such that $\emptyset\neq I\cap A\subset A_k$. Let t, z be reals such that $u_k<t<z<v_k$. Since

$$I\cap A=\{x\in I\cap A;f(x)>t\}\cup\{x\in I\cap A;f(x)<z\},$$

at least one of the sets

$$E_1=\{x\in I\cap A;f(x)>t\} \text{ and } E_2=\{x\in I\cap A;f(x)<z\}$$

is of the second category in A. Without loss of the generality we may assume that the set E_1 is of the second category in A.

For each point $x \in E_1$ there is a rational $r(x) > 0$ such that for each real $h \in (0, r(x)]$ the inequality

$$\frac{\mu([x, x+h] \cap \{y; F(y) > t\})}{h} > \frac{1}{2}$$

is true. Enumerate all positive rationals in a sequence (r_n) and put

$$H_n = \{t \in E_1; r(x) = r_n\} \text{ for } n \geq 1.$$

Since

$$E_1 = \bigcup_n H_n,$$

there is a positive integer i such that the set H_i is of the second category in $I \cap A$. There is an open interval $J \subset I$ such that $\emptyset \neq J \cap A \subset cl(J \cap H_i)$. Since the intersection $J \cap A_k = J \cap A$, there is a point $b \in A \cap J$ with $f(b) \leq u_k < t$. But $f(b) \in L_r(F, b)$, so there is an interval $K \subset J$ of the form $[b, b+h_1]$ such that $h_1 < r_i$ and

$$\frac{\mu(K \cap \{y; F(y) < t\})}{h_1} > \frac{1}{2}.$$

Consider two cases:

(1) b is not isolated on the right hand in $A \cap J$;

(2) b is isolated on the right hand in $A \cap J$.

(1) Since the function

$$0 \neq h \to \frac{\mu([h, b+h_1] \cap \{y; F(y) < t\})}{b+h_1-h}$$

is continuous at b, there is a real $c \in (b, b+h_1) \cap H_i$ such that

$$\frac{\mu([c, b+h_1] \cap \{y; F(y) < t\})}{b+h_1-c} > \frac{1}{2}$$

and $b + h_1 - c < r_i$. Since $c \in H_i$ and $b + h_1 - c < r_i$, we have

$$\frac{\mu([c, b+h_1] \cap \{y; F(y) > t\})}{b+h_1-c} > \frac{1}{2}.$$

On the other hand

$$\frac{\mu([c, b+h_1] \cap \{y; F(y) < t\})}{b+h_1-c} > \frac{1}{2}.$$

Consequently, there is a point y_1 with $F(y_1) < t$ and $F(y_1) > t$, a contradiction.

(2) Since the function

$$0 \neq h \to \frac{\mu([h, b+h_1] \cap \{y; F(y) < t\})}{b + h_1 - h}$$

is continuous at b, there is a real $c \in (\infty, b) \cap H_i$ such that

$$\frac{\mu([c, b+h_1] \cap \{y; F(y) < t\})}{b + h_1 - c} > \frac{1}{2}$$

and $b + h_1 - c < r_i$. Since $c \in H_i$ and $b + h_1 - c < r_i$, we have

$$\frac{\mu([c, b+h_1] \cap \{y; F(y) > t\})}{b + h_1 - c} > \frac{1}{2}.$$

On the other hand

$$\frac{\mu([c, b+h_1] \cap \{y; F(y) < t\})}{b + h_1 - c} > \frac{1}{2}.$$

Consequently, there is a point y_1 with $F(y_1) < t$ and $F(y_1) > t$, a contradiction.

In the remaining cases we reason similarly. So the proof is finished. □

Professor B. S. Thomson observed the following remark.

Remark 2. Since the main argument in the proof of Theorem 1 uses an intersection condition, Theorem 1 may be deduced from a general Thomson's Theorem 33.1 in [5], p. 74.

References

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