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A CHARACTERIZATION OF THE GH_k INTEGRAL

Abstract

A concept of a general derivative and a notion of bounded variation have been produced leading to the presentation of the Fundamental Theorem of Calculus for the GH_k integral with a characterization of the integral.

1 Introduction.

Throughout the paper we shall consider a, b to be fixed real numbers such that $a < b$, and k to be a fixed positive integer greater than 1.

Bhattacharaya and Das [1] gave the definition of a Lebesgue type integral, the LS_k integral, on $[a, b]$ with respect to a gk-measure introduced by them induced by a k-convex function. In the development of the theory of the integral, they introduced the definitions of gk-derivative and of gk-absolute continuity so as to obtain a characterization of the LS_k integral as follows. *Let $g : [a, b] \to \mathbb{R}$ be a k-convex function such that $g_+^{k-1}(a)$ and $g_-^{k-1}(b)$ exist. For a function $f : [a, b] \to \mathbb{R}$, $(f, g) \in LS_k[a, b]$ if and only if there exists a function $F : [a, b] \to \mathbb{R}$ such that F is AC_{gk} on $[a, b]$ and $F'_{gk}(x) = f(x)$ for all $x \in [a, b]$ except possibly for a set of gk-measure zero.*

Following the definition of a Riemann type integral, the RS_k^* integral, in Ray and Das [11], Das, Nath and Sahu [6] obtained the definition of a Henstock type integral, the HS_k integral, simply changing Riemann partitions to Henstock partitions. The integral is additive over the division of the defining interval, linear, satisfies the Saks-Henstock lemma and the Cauchy extension formula.

Key Words: k-convex function, LS_k, HS_k, GR_k, GR_k^*-integrals, δ-fine tagged elementary k-system, GH_k integral, k-variationally bounded function, gk-derivative, GH_k primitive, fundamental theorem of calculus
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Subsequently Das and Sahu [8], using the Vitali covering theorem analog and the notion of gk-derivative from [1], proved ([7], Theorem 4.1) this theorem.
Let g be k-convex on $[a, b]$ with one sided $(k-1)$th derivatives existing at end points, and let $f : [a, b] \to \mathbb{R}$ be such that $(f, g) \in HS_k[a, b]$. If F is the HS_k primitive, then $F'_{gk}(x) = f(x)$ gk-almost everywhere on $[a, b]$.

The authors there continued to give the definitions of $[BV_{gk}G^*]$, $[AC_{gk}G^*]$ functions and proved a Denjoy equivalent version that could be stated combining Theorem 3.3, Theorem 4.1, Theorem 4.6 of [8] as follows.
Let g be k-convex on $[a, b]$ with one sided $(k-1)$th derivatives existing at end points, and let $f : [a, b] \to \mathbb{R}$. Then $(f, g) \in HS_k[a, b]$ if and only if there exists a function $F : [a, b] \to \mathbb{R}$ such that
(i) F is $[AC_{gk}G^*]$ on $[a, b]$, and
(ii) $F'_{gk}(x) = f(x)$ gk-almost everywhere on $[a, b]$.

Following the development of the Henstock type integral investigated by Schwabik [14], Das and Sahu [7] obtained a definition of an integral which they called the generalized Schwabik-Henstock integral, which is similar to the HS_k integral in [6], keeping in mind the existence of one sided $(k-1)$th derivatives of a k-convex function or a function of bounded kth variation.

The present authors ([5]) introduced the definition of a generalized Henstock integral, the GH_k-integral, along with its standard properties including the Saks-Henstock lemma and the Cauchy extension formula.

In the present paper the authors deal with the Stieltjes form of the GH_k integral with integrand $f : [a, b] \to \mathbb{R}$ and integrator $g : [a, b]^k \to \mathbb{R}$, not necessarily a k-convex or a BV_k function. The concept of the (gk)derivative has been introduced, that leads to the presentation of the Fundamental Theorem of Calculus and a characterization of the integral in terms of a certain concept of k-variationally bounded function introduced herein. Owing to non-availability of measure and Vitali covering, the authors follow a line of developments analogous to the ones in Cabral and Lee [3, 4].

2 Preliminaries.

The developments of the series of integrals mentioned above in the introduction require the definitions and the properties of *k-convex* functions and functions of *bounded kth variation*, the BV_k functions, for which we refer to Russell [12]. A deep study for k-convex functions is also available in Bullen [2], and a partial presentation of these two notions, as are necessary for the development quoted above, may be found in Das and Kundu [5].

We give the definition of the GH_k integral and the Saks-Henstock lemma analogously as in [5] for ready references.

Let

$$x_{1,0} < x_{1,1} < \cdots < x_{1,k} \leq x_{2,0} < x_{2,1} < \cdots < x_{2,k} \leq \cdots \leq x_{n,0} < x_{n,1} < \cdots < x_{n,k}$$

be any system of points in $[a,b]$. We say that the intervals $[x_{i,0}, x_{i,k}], i = 1, 2, \ldots, n$ form an elementary system

$$\{(x_{i,1}, x_{i,2}, \ldots, x_{i,k-1}) : [x_{i,0}, x_{i,k}],\ i = 1, 2, \ldots, n\}$$

in $[a,b]$. If each interval $[x_{i,0}, x_{i,k}]$ with a set of $(k-1)$ interior points $x_{i,1} < x_{i,2} < \cdots < x_{i,k-1}$ is tagged with $\xi_i \in [x_{i,0}, x_{i,k}]$ we call the system a *tagged elementary k-system* and denote it by

$$\{(\xi_i; x_{i,1}, x_{i,2}, \ldots, x_{i,k-1}) : [x_{i,0}, x_{i,k}],\ i = 1, 2, \ldots, n\}.$$

A tagged elementary k-system $\{(\xi_i; x_{i,1}, x_{i,2}, \ldots, x_{i,k-1}) : [x_{i,0}, x_{i,k}],\ i = 1, 2, \ldots, n\}$ is a *tagged k-partition* of $[a,b]$ if $\bigcup_{i=1}^{n} [x_{i,0}, x_{i,k}] = [a,b]$.

Given a positive function $\delta : [a,b] \to (0, \infty)$, a tagged elementary k-system and in particular a tagged k-partition

$$\{(\xi_i; x_{i,1}, x_{i,2}, \ldots ; x_{i,k-1}) : [x_{i,0}, x_{i,k}],\ i = 1, 2, \ldots, n\}$$

of $[a,b]$ is said to be *δ-fine* if $\xi_i \in [x_{i,0}, x_{i,k}] \subset (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i))$ for all $i = 1, 2, \ldots, n$. We shall often call such a positive function δ a *gauge* on $[a,b]$. We note that a δ-fine tagged k-partition

$$\{(\xi_i; x_{i,1}, x_{i,2}, \ldots, x_{i,k-1}) : [x_{i,0}, x_{i,k}],\ i = 1, 2, \ldots, n\}$$

of $[a,b]$ exists because it is simply a usual δ-fine tagged partition

$$\{\xi_i; [x_{i,0}, x_{i,k}],\ i = 1, 2, \ldots, n\}$$

of $[a,b]$ along with a set of $(k-1)$ points $x_{i,1} < x_{i,2} < \cdots < x_{i,k-1}$ in $(x_{i,0}, x_{i,k}), i = 1, 2, \ldots, n$. Clearly for $k = 1$, a δ-fine tagged k-partition is a δ-fine tagged partition.

Definition 2.1 ([5], Definition 2.1). A function $U : [a,b]^{k+1} \to \mathbb{R}$ is called GH_k integrable on $[a,b]$ if there is an element $I \in \mathbb{R}$ such that, given $\epsilon > 0$ there is a gauge δ on $[a,b]$ such that

$$\Big| \sum_{i=1}^{n} [U(\xi_i; x_{i,1}, \ldots, x_{i,k}) - U(\xi_i; x_{i,0}, \ldots, x_{i,k-1})] - I \Big| < \epsilon$$

for every δ-fine tagged k-partition

$$P = \{(\xi_i; x_{i,1}, x_{i,2}, \ldots, x_{i,k-1}) : [x_{i,0}, x_{i,k}], i = 1, 2, \ldots, n\}$$

of $[a, b]$. The real number I is called the GH_k integral of U on $[a, b]$ and we write $(GH_k) \int_a^b U = I$.

If $(GH_k) \int_a^b U$ exists, we often write $U \in GH_k[a, b]$. We use the notation $S(U, P)$ for the Riemann type sum

$$\sum_{i=1}^{n} [U(\xi_i; x_{i,1}, \ldots, x_{i,k}) - U(\xi_i; x_{i,0}, \ldots, x_{i,k-1})]$$

corresponding to the function U and the k-partition

$$P = \{(\xi_i; x_{i,1}, x_{i,2}, \ldots, x_{i,k-1}) : [x_{i,0}, x_{i,k}], i = 1, 2, \ldots, n\}.$$

For $k = 1$, the GH_k integral is the generalized Perron integral of Schwabik [14]. This GH_k integral includes the Henstock, the generalized Perron and the Stieltjes type integrals on $[a, b]$.

For $k > 1$, we set

$$U(\tau; t_1, t_2, \ldots, t_k) = f(\tau)\alpha(t_1, t_2, \ldots, t_k)$$

where $f : [a, b] \to \mathbb{R}$, $\alpha : [a, b]^k \to \mathbb{R}$ so as to obtain a kth Riemann Stieltjes type sum

$$\sum_{i=1}^{n} f(\xi_i)[\alpha(x_{i,1}, \ldots, x_{i,k}) - \alpha(x_{i,0}, \ldots, x_{i,k-1})].$$

If the integral exists, we often write $(f, \alpha) \in GH_k[a, b]$ and the integral will be denoted by $(GH_k) \int_a^b f d\alpha$. In particular, for $f : [a, b] \to \mathbb{R}$, and $h : [a, b] \to \mathbb{R}$, if we put

$$U(\tau; t_1, t_2, \ldots, t_k) = f(\tau)Q_{k-1}(h; t_1, t_2, \ldots, t_k)$$

where $Q_{k-1}(h; t_1, t_2, \ldots, t_k)$ is the $(k-1)$th divided difference of h (see [5], p. 59), we get the kth Riemann-Stieltjes sum

$$s(f, h; P) = \sum_{i=1}^{n} f(\xi_i)[Q_{k-1}(h; x_{i,1}, \ldots, x_{i,k}) - Q_{k-1}(h; x_{i,0}, \ldots, x_{i,k-1})]$$

corresponding to the k-partition

$$P = \{(\xi_i; x_{i,1}, x_{i,2}, \ldots, x_{i,k-1}) : [x_{i,0}, x_{i,k}], i = 1, 2, \ldots, n\}$$

of $[a, b]$ (cf. Russell [13]). We note further that the approximating sums for the resulting integrals do not involve repeated points interior to any partition subinterval $[x_{i,0}, x_{i,k}]$ as in [13], [9] and in some other authors.

Definition 2.2 ([5], Definition 2.3). A function $U : [a, b]^{k+1} \to \mathbb{R}^n$ is called GH_k integrable on $[a, b]$ if there exists an element $I \in \mathbb{R}^n$ such that, given $\epsilon > 0$, there exists a gauge δ on $[a, b]$ such that

$$\| S(U, P) - I \| = \| \sum_{i=1}^{n} [U(\xi_i; x_{i,1}, \ldots, x_{i,k}) - U(\xi_i; x_{i,0}, \ldots, x_{i,k-1})] - I \| < \epsilon$$

for any δ-fine tagged k-partition

$$P = \{(\xi_i; x_{i,1}, x_{i,2}, \ldots, x_{i,k-1}) : [x_{i,0}, x_{i,k}], i = 1, 2, \ldots, n\}$$

of $[a, b]$. The number $I \in \mathbb{R}^n$ is called the GH_k integral of U on $[a, b]$ and we write $(GH_k) \int_a^b U = I$.

If the integral exists, we often write $U \in GH_k[a, b]$. Here the norm $\|.\|$ is any norm in $\mathbb{R}^n$, for example, the Euclidean one.

Note 2.3. *Following Schwabik* [14] *it is not difficult to show that an $\mathbb{R}^n$-valued function $U : [a, b]^{k+1} \to \mathbb{R}^n$, $U = (U_1, U_2, \ldots, U_n)$, is GH_k integrable if and only if every component $U_m, m = 1, 2, \ldots, n$, is GH_k integrable in the sense of Definition 2.1.*

Theorem 2.4 ([5], Corollary 3.2). *Let $U : [a, b]^{k+1} \to \mathbb{R}^n$ be GH_k integrable on $[a, b]$. Then to each $\epsilon > 0$ there exists a gauge δ on $[a, b]$ such that for every δ-fine tagged k-partition*

$$P = \{(\xi_i; x_{i,1}, \ldots, x_{i,k-1}) : [x_{i,0}, x_{i,k}],\ i = 1, 2, \ldots, n\}$$

of $[a, b]$

$$\sum_{i=1}^{n} \| [U(\xi_i; x_{i,1}, \ldots, x_{i,k}) - U(\xi_i; x_{i,0}, \ldots, x_{i,k-1}) - (GH_k) \int_{x_{i,0}}^{x_{i,k}} U] \| < \epsilon.$$

Pal, Ganguly and Lee [9] gave the following definition of the GR_k integral that requires repeated overlapping division points.

Definition 2.5 ([9], p. 854). Let $f : [a,b] \to \mathbb{R}$ and $g : [a,b]^{k+1} \to \mathbb{R}$. The function f is said to be GR_k integrable with respect to g if there exists a real number I on $[a,b]$ such that for every $\epsilon > 0$ there is a positive function δ on $[a,b]$ such that for any δ^k-fine division $P = \{([x_i, x_{i+k}], \xi_i)\}_{i=0}^{n-k}$ of $[a,b]$ we have

$$\Big|\sum_{i=0}^{n-k} f(\xi_i) g(x_i, x_{i+1}, \ldots, x_{i+k}) - I\Big| < \epsilon.$$

For $k = 1$, if we put $g(x_i, x_{i+1}) = \alpha(x_{i+1}) - \alpha(x_i)$, we get the classical Henstock-Stieltjes integral.

In a subsequent paper Pal, Ganguly and Lee [10] introduced the definition of the GR_k^* integral.

Definition 2.6 ([10], Definition 3.1). Let $f : [a,b] \to \mathbb{R}$ and $g : [a,b]^{k+1} \to \mathbb{R}$ such that $J(g;c)$ exists for all $c \in (a,b)$. We say that f is GR_k^* integrable with respect to g on $[a,b]$ if there exists a real number A such that for all $\epsilon > 0$ there exists $\delta : [a,b] \to \mathbb{R}_+$ such that for any regulated δ^k-fine division $\{x_i, D_i, [a_i, b_i]\}_{i=1}^{p}$ of $[a,b]$ we have

$$\Big|\sum_{i=1}^{p}\sum_{j=0}^{n_i-k} f(\xi_{j,i}) g(x_{j,i}, x_{j+1,i}, \ldots, x_{j+k,i}) + \sum_{i=1}^{p-1} (k-1) f(b_i) J(g; b_i) - A\Big| < \epsilon.$$

It was defined in [9] (also repeated in [10]) that

$$J(g;x) = \lim_{x_0 \to x, x_k \to x} g(x_0, x_1, \ldots, x_k),$$

where $x \in [x_0, x_k]$ and $x_0 < x_1 < \cdots < x_k$. By a *regulated* δ^k-fine division of $[a,b]$ the authors [10] mean a δ^k-fine division of $[a_i, b_i]$, where $\{([a_i, b_i], x_i),\ i = 1, 2, \ldots, p\}$ is a δ-fine division of $[a,b]$ with tags x_i coinciding with a_i or b_i.

Standard properties of the GR_k- and GR_k^*-integrals were studied in [9] and [10].

Note 2.7. *The claim in Pal et al. [10] that the GR_k^* integral includes the GR_k integral is not true. In fact, using the Saks-Henstock lemma for the GR_k^*integral, it is not difficult to prove that a GR_k^* integrable function f is necessarily GR_k integrable. On the other hand, it may happen that $(f,g) \in GR_k[a,b]$ without the existence of $J(g,x)$ everywhere in $[a,b]$. For example, the rational indicator function f is GR_k integrable on $[a,b]$ with respect to g defined by*

$$g(x_0, x_1, \ldots, x_k) = \begin{cases} (x_k - x_0) Q_k(h; x_0, x_1, \ldots, x_k) & \text{if } a \le x_0 < \cdots < x_k \le b \\ 0 & \text{otherwise} \end{cases}$$

where $h : [a,b] \to \mathbb{R}$ is BV_k on $[a,b]$. Since $h^{(k-1)}(x)$ exists n.e. on $[a,b]$, it follows that $J(g,x)$ may not exist for all $x \in (a,b)$ and so $(f,g) \notin GR_k^*[a,b]$. Hence the class of GR_k^* integrable functions is a proper subclass of the class of GR_k integrable functions.

It can easily be verified that $(f,h) \in GH_k[a,b]$. On the other hand, given $g : [a,b]^{k+1} \to \mathbb{R}$ and $\alpha : [a,b]^k \to \mathbb{R}$ satisfying

$$g(t_0, t_1, \ldots, t_k) = \alpha(t_1, \ldots, t_k) - \alpha(t_0, \ldots, t_{k-1})$$

for all choices of $a \le t_0 < t_1 < \cdots < t_k \le b$, it is clear that $(f,\alpha) \in GH_k[a,b]$ whenever $(f,g) \in GR_k^*[a,b]$. So, (GR_k^*) is a proper subclass of (GH_k). In view of Theorem 3.6 of Das and Kundu [5], we have $(f,g) \in GR_k[a,b]$ whenever $(f,\alpha) \in GH_k[a,b]$, provided that both

$$J^+(f,\alpha;a) = \lim_{t_k \to a} \lim_{t_{k-1} \to a} \cdots \lim_{t_1 \to a} f(a)\alpha(t_1, t_2, \ldots, t_k)$$

where $a \le t_1 < t_2 < \cdots < t_k < b$ and

$$J^-(f,\alpha;b) = \lim_{t_1 \to b} \lim_{t_2 \to b} \cdots \lim_{t_k \to b} f(b)\alpha(t_1, t_2, \ldots, t_k)$$

where $a < t_1 < t_2 < \cdots < t_k \le b$ (see [5], Definition 2.9) exist. Thus it appears that for such representation of the integrator g in terms of α, we have the inclusion chain $(GR_k^*) \subset (GH_k) \subset (GR_k)$.

In order to obtain results analogous to those in Cabral and Lee [3], Pal et al. [10] obtained the following definition of *g-regularity*. The definition needs the terminology " *g-nearly additivity*": A function $F : [a,b] \to \mathbb{R}$ is said to be *g-nearly additive* with respect to f on $[a,b]$ if for every $c \in (a,b)$, we have

$$F(b) - F(a) = F(c) - F(a) + (k-1)f(c)J(g;c) + F(b) - F(c).$$

Definition 2.8 ([10], Definition 4.1). Let $J(g,x)$ exist for all $x \in [a,b]$ and let F be a function g-nearly additive with respect to f on $[a,b]$. F will be said to be g-regular with respect to f at $x \in [a,b]$ if for all $\epsilon > 0$ there exists a function $\delta > 0$, defined on $[a,b]$ such that for all δ^k-fine divisions $P = \{[x_i, x_{i+k}], \xi_i\}_{i=0}^{n-k}$ of $[u,v] \subset [a,b]$ locally tagged at x we have

$$\Big|F(v) - F(u) - (P)\sum_{i=0}^{n-k} f(\xi_i)g(x_i, \ldots, x_{i+k})\Big| < \epsilon \sum_{i=0}^{n-k} \big|g(x_i, \ldots, x_{i+k})\big|.$$

It can clearly be noted that given $g \in LBV^k[a,b]$ (for the definition of LBV^k see [10]), F is g-regular at x if and only if F is GR_k integrable on every

δ-fine interval $[x, v] \subset [x, x + \delta(x))$ or $[u, x] \subset (x - \delta(x), x]$. This idea is a departure from the idea of defining a derivative at a point. In classical cases we say that a function is *regular* at a point x if its derivative exists at *each point* in some neighborhood of x.

In the next section we define a derivative suitable for the development of our integral, the GH_k integral, according to our proposal in its introduction in [5].

3 The GH_k Primitive and Its Characterization.

At the beginning of this section we give the definition of k-variationally bounded function that is essential to provide an existence criterion for the GH_k integral in terms of our proposed derivative (Theorem 3.6 below) and also is useful for subsequent development of the section.

Definition 3.1. Let $g : [a, b]^k \to \mathbb{R}$ and let $\delta : [a, b] \to (0, \infty)$ be any positive function on $[a, b]$. For $X \subset [a, b]$ let

$$V_k(g; X, \delta, P) = \sum_{\xi_i \in X} \big| g(x_{i,1}, \ldots, x_{i,k}) - g(x_{i,0}, \ldots, x_{i,k-1}) \big|$$

for every δ-fine elementary tagged k-system

$$P = \{(\xi_i; x_{i,1}, \ldots, x_{i,k-1}) : [x_{i,0}, x_{i,k}],\ i = 1, 2, \ldots, n\}$$

in $[a, b]$. If

$$V_k(g; X) = \inf_{\delta} \sup_{P} V_k(g; X, \delta, P) < +\infty,$$

we say that g is k-variationally bounded, $(V_k B)$, on X, and in symbols $g \in V_k B(X)$.

If $g \in V_k B(X)$, then there exist $\delta : [a, b] \to (0, \infty)$ and a positive constant M such that for all δ-fine elementary tagged k-system

$$P = \{(\xi_i; x_{i,1}, \ldots, x_{i,k-1}) : [x_{i,0}, x_{i,k}],\ i = 1, 2, \ldots, n\}$$

in $[a, b]$, we have

$$\sum_{\xi_i \in X} |g(x_{i,1}, \ldots, x_{i,k}) - g(x_{i,0}, \ldots, x_{i,k-1})| < M.$$

Definition 3.2. A function $F : [a, b] \to \mathbb{R}$ is said to have a (gk)-derivative f at $x \in [a, b]$ if for each $\epsilon > 0$ there exists $\delta : [a, b] \to (0, \infty)$ such that for every δ-fine singleton elementary k-system $P = \{(x; t_1, \ldots, t_{k-1}) : [t_0, t_k]\}$ in $[a, b]$ tagged at x, we have

$$\begin{aligned}|F(t_k) - F(t_0) - f(x)[g(t_1, \ldots, t_k) - g(t_0, \ldots, t_{k-1})]| \\ < \epsilon|g(t_1, \ldots, t_k) - g(t_0, \ldots, t_{k-1})|.\end{aligned}$$

If the function F possesses a (gk) derivative f at x we write $DF_{gk}(x) = f(x)$. One sided derivatives $D^+F_{gk}(x)$ and $D^-F_{gk}(x)$ are defined in the usual way.

We observe that, if $k = 1$, then the (gk) derivative is the classical g derivative. Again if further $k = 1$ and $g(x) = x$ for all $x \in [a, b]$, we obtain the definition of classical ordinary derivative.

Definition 3.3. Let $F : [a, b] \to \mathbb{R}$ and $g : [a, b]^k \to \mathbb{R}$. Given $\epsilon > 0$ arbitrarily and a positive function $\delta : [a, b] \to (0, \infty)$, we define

$$\begin{aligned}\Gamma_{\epsilon,\delta} = \{(x, P) : P = \{(x; t_1, \ldots, t_{k-1}) : [t_0, t_k]\} &\text{ is a } \delta\text{-fine singleton} \\ &\text{elementary } k\text{-system in } [a, b]\text{, tagged at } x\text{, such that} \\ &\mid F(t_k) - F(t_0) - f(x)[g(t_1, \ldots, t_k) - g(t_0, \ldots, t_{k-1})] \mid \\ &\geq \epsilon \mid g(t_1, \ldots, t_k) - g(t_0, \ldots, t_{k-1}) \mid\}\end{aligned}$$

and

$$X(\epsilon, \delta) = \{\ x \in [a, b] : (x, P) \in \Gamma_{\epsilon,\delta}\ \}.$$

Definition 3.4. If $(f, g) \in GH_k[a, b]$, then by Theorem 3.8 of [5] $(f, g) \in GH_k[a, x]$ for $a < x \leq b$. Define F on $[a, b]$ by

$$F(x) = F(a) + (GH_k)\int_a^x f\, dg \quad \text{if } a < x \leq b$$

and call F the primitive of the GH_k-integrable function f.

Theorem 3.5 (Fundamental Theorem of Calculus). *If $g \in V_kB[a, b]$ and $DF_{gk}(x) = f(x)$ for all x in $[a, b]$, then $(f, g) \in GH_k[a, b]$ and $F(b) - F(a) = (GH_k)\int_a^b f dg$.*

Proof. Since $g \in V_kB[a, b]$, we can find $\delta_1 : [a, b] \to (0, \infty)$ and $M > 0$ such that for all δ_1-fine elementary tagged k-systems

$$P = \{(\xi_i; x_{i,1}, \ldots, x_{i,k-1}) : [x_{i,0}, x_{i,k}],\ i = 1, 2, \ldots, n\}$$

in $[a,b]$ we have

$$V_k(g;[a,b],\delta_1,P) < M.$$

Since $DF_{gk}(x) = f(x)$ for all $x \in [a,b]$, there exists $\delta : [a,b] \to (0,\infty)$, $\delta(x) < \delta_1(x)$ for all $x \in [a,b]$ such that for every choice of δ-fine singleton elementary k-systems $P = \{(x;t_1,\dots,t_{k-1}) : [t_0,t_k]\}$ tagged at x, we have

$$\begin{aligned}&|F(t_k) - F(t_0) - f(x)[g(t_1,\dots,t_k) - g(t_0,\dots,t_{k-1})]| \\ &\quad < \frac{\epsilon}{M}|g(t_1,\dots,t_k) - g(t_0,\dots,t_{k-1})|.\end{aligned}$$

Now let $P = \{(\zeta_i;z_{i,1},\dots,z_{i,k-1}) : [z_{i,0},z_{i,k}], i = 1,2\dots,n\}$ be any δ-fine tagged k-partition of $[a,b]$. Then

$$\begin{aligned}&|F(b) - F(a) - \sum_{i=1}^{n} f(\zeta_i)[g(z_{i,1},\dots,z_{i,k}) - g(z_{i,0},\dots,z_{i,k-1})]| \\ &\le \sum_{i=1}^{n} |F(z_{i,k}) - F(z_{i,0}) - f(\zeta_i)[g(z_{i,1},\dots,z_{i,k}) - g(z_{i,0},\dots,z_{i,k-1})]| \\ &< \frac{\epsilon}{M}\sum_{i=1}^{n} |g(z_{i,1},\dots,z_{i,k}) - g(z_{i,0},\dots,z_{i,k-1})| \frac{\epsilon}{M} M = \epsilon.\end{aligned}$$

Hence $(f,g) \in GH_k[a,b]$ and $F(b) - F(a) = (GH_k)\int_a^b f\,dg$. □

Theorem 3.6. *Let $f : [a,b] \to \mathbb{R}$ and $g \in V_kB[a,b]$. Then $(f,g) \in GH_k[a,b]$ with primitive $F : [a,b] \to \mathbb{R}$ if and only if for all $\epsilon > 0$ there exists $\delta : [a,b] \to (0,\infty)$ such that for all δ-fine elementary tagged k-system*

$$P = \{(\xi_i;x_{i,1},\dots,x_{i,k-1}) : [x_{i,0},x_{i,k}],\ i = 1,2,\dots,p\}$$

in $[a,b]$ with each $\xi_i \in X(\epsilon,\delta)$

we have

$$\sum_{i=1}^{p} |f(\xi_i)[g(x_{i,1},\dots,x_{i,k}) - g(x_{i,0},\dots,x_{i,k-1})]| < \epsilon, \tag{1}$$

$$\sum_{i=1}^{p} |F(x_{i,k}) - F(x_{i,0})| < \epsilon. \tag{2}$$

PROOF. Let the given conditions hold. By the definition of $X(\epsilon,\delta)$ and for every choice of δ-fine singleton elementary k-system $P = \{(x;t_1,\dots,t_{k-1}) : [t_0,t_k]\}$ in $[a,b]$ tagged at

$x \in [a,b] \backslash X(\epsilon,\delta)$, we have

$$|F(t_k) - F(t_0) - f(x)[g(t_1,\dots,t_k) - g(t_0,\dots,t_{k-1})]| \\ < \epsilon |g(t_1,\dots,t_k) - g(t_0,\dots,t_{k-1})|.$$

Since $g \in V_k B[a,b]$, there exist a positive function $\delta_1 (< \delta)$ on $[a,b]$ and $M > 0$ such that for each δ_1-fine elementary tagged k-system

$$P = \{(\xi_i; x_{i,1},\dots,x_{i,k-1}) : [x_{i,0}, x_{i,k}],\ i = 1,2,\dots,n\}$$

in $[a,b]$, we have $V_k(g;[a,b],\delta_1,P) < M$. Let $P = \{(\zeta_i; z_{i,1},\dots,z_{i,k-1}) : [z_{i,0}, z_{i,k}], i = 1,2,\dots,n\}$ be any δ_1-fine tagged k-partition of $[a,b]$. Then

$$\begin{aligned} &|F(b) - F(a) - \sum_{i=1}^{n} f(\zeta_i)[g(z_{i,1},\dots,z_{i,k}) - g(z_{i,0},\dots,z_{i,k-1})]| \\ \le &\sum_{\zeta_i \in X(\epsilon,\delta)} |F(z_{i,k}) - F(z_{i,0})| \\ &+ \sum_{\zeta_i \in X(\epsilon,\delta)} |f(\zeta_i)[g(z_{i,1},\dots,z_{i,k}) - g(z_{i,0},\dots,z_{i,k-1})]| \\ &+ \sum_{\zeta_i \notin X(\epsilon,\delta)} |F(z_{i,k}) - F(z_{i,0}) - f(\zeta_i)[g(z_{i,1},\dots,z_{i,k}) - g(z_{i,0},\dots,z_{i,k-1})]| \\ < &2\epsilon + \epsilon\ V_k(g;[a,b],\delta_1,P) < 2\epsilon + \epsilon\ M = \epsilon(2+M). \end{aligned}$$

Hence, $(f,g) \in GH_k[a,b]$ and F is the GH_k-primitive.

Conversely, let $(f,g) \in GH_k[a,b]$ with primitive F. Let

$$Y_l = \{x : a \le x \le b,\quad l-1 \le |f(x)| < l\}.$$

Then

$$[a,b] = \bigcup_{l=1}^{\infty} Y_l \text{ and } Y_l \cap Y_{l'} = \phi \ for\ l \ne l'.$$

Without any loss of generality we take $0 < \epsilon < 1$. By Theorem 3.1 of [5] (Saks-Henstock lemma), there exists $\delta_l(x) > 0$ on $[a,b]$ such that for every δ_l-fine tagged elementary k-system $\{(\xi_i; x_{i,1},\dots,x_{i,k-1})$: $[x_{i,0}, x_{i,k}],\ i = 1,2,\dots,p\}$ in $[a,b]$ with $\xi_i \in Y_l$ we have

$$\sum_{i=1}^{p} |F(x_{i,k}) - F(x_{i,0}) - f(\xi_i)[g(x_{i,1},\dots,x_{i,k}) - g(x_{i,0},\dots,x_{i,k-1})]| < \frac{\epsilon^2}{2^{l+1}.l}.$$

Let $\delta(x) = \delta_l(x) for\ x \in Y_l, l = 1, 2, \ldots$ and let $\{(\eta_i; y_{i,1}, \ldots, y_{i,k-1})\colon [y_{i,0}, y_{i,k}],$ $i = 1, \ldots, q\}$ be a δ-fine tagged elementary k-system in $[a, b]$ with $\eta_i \in X(\epsilon, \delta)$. Then

$$\begin{aligned}
&\sum_{i=1}^{q} \mid f(\eta_i)[g(y_{i,1}, \ldots, y_{i,k}) - g(y_{i,0}, \ldots, y_{i,k-1})] \mid \\
&= \sum_{l=1}^{\infty} \sum_{\eta_i \in Y_l} \mid f(\eta_i)[g(y_{i,1}, \ldots, y_{i,k}) - g(y_{i,0}, \ldots, y_{i,k-1})] \mid \\
&\le \sum_{l=1}^{\infty} l \sum_{\eta_i \in Y_l} \mid g(y_{i,1}, \ldots, y_{i,k}) - g(y_{i,0}, \ldots, y_{i,k-1}) \mid \\
&\le \sum_{l=1}^{\infty} \frac{l}{\epsilon} \sum_{\eta_i \in Y_l} \mid F(y_{i,k}) - F(y_{i,0}) - f(\eta_i)[g(y_{i,1}, \ldots, y_{i,k}) - g(y_{i,0}, \ldots, y_{i,k-1})] \mid \\
&< \sum_{l=1}^{\infty} \frac{l}{\epsilon} \cdot \frac{\epsilon^2}{2^l.\, l} = \sum_{l=1}^{\infty} \frac{\epsilon}{2^l} = \epsilon.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
&\sum_{i=1}^{q} \mid F(y_{i,k}) - F(y_{i,0}) \mid \\
&\le \sum_{i=1}^{q} \mid [F(y_{i,k}) - F(y_{i,0})] - f(\eta_i)[g(y_{i,1}, \ldots, y_{i,k}) - g(y_{i,0}, \ldots, y_{i,k-1})] \mid \\
&\quad + \sum_{i=1}^{q} \mid f(\eta_i)[g(y_{i,1}, \ldots, y_{i,k}) - g(y_{i,0}, \ldots, y_{i,k-1})] \mid \\
&< \frac{\epsilon}{2} + \sum_{l=1}^{\infty} \frac{\epsilon^2}{2^{l+1}.\, l} \le \frac{\epsilon}{2} + \sum_{l=1}^{\infty} \frac{\epsilon^2}{2^{l+1}} = \frac{\epsilon}{2} + \frac{\epsilon^2}{2} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \qquad \square
\end{aligned}$$

Theorem 3.7. *Let $f : [a, b] \to \mathbb{R}$ and $g \in V_k B[a, b]$. Then $(f, g) \in GH_k[a, b]$ with primitive $F : [a, b] \to \mathbb{R}$ if and only if for all $\epsilon > 0$ there exists $\delta : [a, b] \to (0, \infty)$ such that for all δ-fine tagged elementary k-systems*

$$P = \{(\xi_i; x_{i,1}, \ldots, x_{i,k-1}) : [x_{i,0}, x_{i,k}],\ i = 1, 2, \ldots, p\}$$

in $[a, b]$ with each $\xi_i \in X(\epsilon, \delta)$ we have

$$\sum_{i=1}^{p} \left| g(x_{i,1}, \ldots, x_{i,k}) - g(x_{i,0}, \ldots, x_{i,k-1}) \right| < \epsilon \text{ and } \sum_{i=1}^{p} \left| F(x_{i,k}) - F(x_{i,0}) \right| < \epsilon.$$

Proof. If $(f,g) \in GH_k[a,b]$, then, in view of the Saks-Henstock lemma (Theorem 2.4 above) with $U(\xi; t_0, t_1, \ldots, t_{k-1}) = f(\xi)g(t_0, t_1, \ldots, t_{k-1})$ and $n = 1$, there exists $\delta : [a,b] \to (0,\infty)$ such that for every δ-fine tagged elementary k-system

$$P = \{(\xi_i; x_{i,1}, \ldots, x_{i,k-1}) : [x_{i,0}, x_{i,k}],\ i = 1, 2, \ldots, p\}$$

in $[a,b]$ we have

$$\sum_{i=1}^{p} \left| F(x_{i,k}) - F(x_{i,0}) - f(\xi_i)[g(x_{i,1}, \ldots, x_{i,k}) - g(x_{i,0}, \ldots, x_{i,k-1})] \right| < \epsilon^2.$$

If now the system P is such that each $\xi_i \in X(\epsilon, \delta)$, then

$$\begin{aligned} &\sum_{i=1}^{p} \left| g(x_{i,1}, \ldots, x_{i,k}) - g(x_{i,0}, \ldots, x_{i,k-1}) \right| \\ \le &\sum_{i=1}^{p} \left| F(x_{i,k}) - F(x_{i,0}) - f(\xi_i)[g(x_{i,1}, \ldots, x_{i,k}) - g(x_{i,0}, \ldots, x_{i,k-1})] \right| / \epsilon \\ < &\epsilon. \end{aligned}$$

That $\sum_{i=1}^{p} |F(x_{i,k}) - F(x_{i,0})| < \epsilon$ follows from the proof of the only if part of Theorem 3.6.

Conversely let the above conditions hold. Let

$$Y_l = \{x : a \le x \le b,\quad l-1 \le |f(x)| < l\}.$$

so that $[a,b] = \bigcup_{l=1}^{\infty} Y_l$ and $Y_l \cap Y_{l'} = \phi$ for $l \neq l'$. Let $\epsilon > 0$ be arbitrary. Then for $\epsilon_l = \frac{\epsilon}{l 2^{l+1}}$, there exists $\delta_l(x) > 0$ such that for every δ_l-fine tagged elementary k-system

$$P = \{(\xi_i; x_{i,1}, \ldots, x_{i,k-1}) : [x_{i,0}, x_{i,k}],\ i = 1, 2, \ldots, p\}$$

in $[a,b]$ with $\xi_i \in X(\epsilon_l, \delta_l)$, we have

$$\sum_{i=1}^{p} \left| g(x_{i,1}, \ldots, x_{i,k}) - g(x_{i,0}, \ldots, x_{i,k-1}) \right| < \epsilon_l$$

and

$$\sum_{i=1}^{p} \left| F(x_{i,k}) - F(x_{i,0}) \right| < \epsilon_l.$$

Let $\delta : [a,b] \to (0,\infty)$ such that $\delta(x) \leq \delta_l(x),\ x \in Y_l$. Then for every δ_l-fine tagged elementary k-system

$$P = \{(\xi_i; x_{i,1}, \ldots, x_{i,k-1}) : [x_{i,0}, x_{i,k}],\ i = 1, 2, \ldots, p\}$$

in $[a,b]$ with $\xi_i \in X(\epsilon, \delta) \subset \bigcap\limits_{l=1}^{\infty} X(\epsilon_l, \delta_l)$, we have

$$\begin{aligned}
&\sum_{i=1}^{p} \left| f(\xi_i)[g(x_{i,1}, \ldots, x_{i,k}) - g(x_{i,0}, \ldots, x_{i,k-1})] \right| \\
&\leq \sum_{l=1}^{\infty} \sum_{\xi_i \in Y_l} l \left| g(x_{i,1}, \ldots, x_{i,k}) - g(x_{i,0}, \ldots, x_{i,k-1}) \right| < \sum_{l=1}^{\infty} \frac{\epsilon}{2^{l+1}} = \frac{\epsilon}{2} < \epsilon.
\end{aligned}$$

In view of the sufficient part of Theorem 3.6, it follows that $(f,g) \in GH_k[a,b]$, and this completes the proof. □

For a positive integer n, let $X_{\frac{1}{n}} = \bigcap\limits_{\delta} X(\frac{1}{n}, \delta)$, and for $F : [a,b] \to \mathbb{R}$, let

$$V(F, X_{\frac{1}{n}}) = \inf_{\delta} \sup_{P} \sum_{i=1}^{p} \left| F(x_{i,k}) - F(x_{i,0}) \right|$$

where the supremum is taken over all δ-fine tagged elementary k-systems

$$P = \{(\xi_i; x_{i,1}, \ldots, x_{i,k-1}) : [x_{i,0}, x_{i,k}],\ i = 1, 2, \ldots, p\}$$

in $[a,b]$ with $\xi_i \in X(\frac{1}{n}, \delta)$.

Theorem 3.8. *Let $F : [a,b] \to \mathbb{R}$ and $g : [a,b]^k \to \mathbb{R}$. Then for every $\epsilon > 0$ there exists a gauge δ on $[a,b]$ such that for every δ-fine tagged elementary k-system*

$$P = \{(\xi_i; x_{i,1}, \ldots, x_{i,k-1}) : [x_{i,0}, x_{i,k}],\ i = 1, 2, \ldots, p\}$$

in $[a,b]$ with $\xi_i \in X(\frac{1}{n}, \delta)$, we have

$$(P) \sum_{i=1}^{p} \left| g(x_{i,1}, \ldots, x_{i,k}) - g(x_{i,0}, \ldots, x_{i,k-1}) \right| < \epsilon$$

and

$$(P) \sum_{i=1}^{p} \left| F(x_{i,k}) - F(x_{i,0}) \right| < \epsilon$$

if and only if for every positive integer n

$$V_k(g, X_{\frac{1}{n}}) = 0 \text{ and } V(F, X_{\frac{1}{n}}) = 0.$$

PROOF. To prove the if part, we note that for every positive integer m there exists a positive function δ_m on $[a, b]$ such that

$$(P)\sum_{i=1}^{p}\left|g(x_{i,1},\ldots,x_{i,k}) - g(x_{i,0},\ldots,x_{i,k-1})\right| < \frac{1}{m}$$

and

$$(P)\sum_{i=1}^{p}\left|F(x_{i,k}) - F(x_{i,0})\right| < \frac{1}{m}$$

whenever

$$P = \{(\xi_i; x_{i,1},\ldots,x_{i,k-1}) : [x_{i,0}, x_{i,k}],\ i = 1, 2, \ldots, p\}$$

is a δ_m-fine tagged elementary k-system in $[a, b]$ with $\xi_i \in X(\frac{1}{n}, \delta_m)$. For every $\epsilon > 0$, there exists a positive integer $m(\epsilon)$ such that $\frac{1}{m(\epsilon)} < \epsilon$. Choose $\delta : [a, b] \to (0, \infty)$ such that $\delta(x) = \delta_{m(\epsilon)}(x)$ for $x \in X(\frac{1}{n}, \delta_{m(\epsilon)})$ and arbitrary otherwise. Then for every δ-fine tagged elementary k-system

$$P = \{(\xi_i; x_{i,1},\ldots,x_{i,k-1}) : [x_{i,0}, x_{i,k}],\ i = 1, 2, \ldots, p\}$$

in $[a, b]$ with $\xi_i \in X(\frac{1}{n}, \delta)$, we have

$$(P)\sum_{i=1}^{p}\left|\, g(x_{i,1},\ldots,x_{i,k}) - g(x_{i,0},\ldots,x_{i,k-1})\,\right| < \frac{1}{m(\epsilon)} < \epsilon$$

with

$$(P)\sum_{i=1}^{p}\left|\, F(x_{i,k}) - F(x_{i,0})\,\right| < \frac{1}{m(\epsilon)} < \epsilon.$$

Conversely, fix $n \in \mathbb{N}$. Then there exists a positive function δ on $[a, b]$ such that for every δ-fine tagged elementary k-system

$$P = \{(\xi_i; x_{i,1},\ldots,x_{i,k-1}) : [x_{i,0}, x_{i,k}],\ i = 1, 2, \ldots, p\}$$

in $[a, b]$ with $\xi_i \in X(\frac{1}{n}, \delta)$, we have

$$(P)\sum_{i=1}^{p}\left|g(x_{i,1},\ldots,x_{i,k}) - g(x_{i,0},\ldots,x_{i,k-1})\right| < \frac{1}{n}$$

and
$$(P)\sum_{i=1}^{p}\big|F(x_{i,k})-F(x_{i,0})\big|<\frac{1}{n}.$$
Since $X(\frac{1}{m},\delta)\subset X(\frac{1}{m+1},\delta)$ for all $m\in\mathbb{N}$, for $m\geq n$ we obtain
$$\begin{aligned}\sup_P V_k\big(g;X(\frac{1}{n},\delta),\delta,P\big)&\leq\sup_P V_k\big(g;X(\frac{1}{m},\delta),\delta,P\big)\\&\leq\sup_P V_k\big(g;X(\frac{1}{m+1},\delta),\delta,P\big)\leq\frac{1}{m+1}.\end{aligned}$$
Hence taking the limit as $m\to\infty$, it follows that
$$V_k(g,X_{\frac{1}{n}})=0 \text{ and } V(F,X_{\frac{1}{n}})=0. \qquad \square$$

We immediately obtain the following.

Corollary 3.9. *Let $g\in V_kB[a,b]$. A function $F:[a,b]\to\mathbb{R}$ is a GH_k primitive of the function $f:[a,b]\to\mathbb{R}$ with respect to g if and only if for every $n\in\mathbb{N}$,*
$$V_k(g,X_{\frac{1}{n}})=0 \text{ and } V(F,X_{\frac{1}{n}})=0.$$

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