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REGULAR SEQUENCES AND LIFTING PROPERTY

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Let A be a commutative noetherian ring, E a finite A-module and let M be an arbitrary A-module. Let $\varphi: E \to M$ be a homomorphisn of A-modules.

In this note we prove in an elementary way that an M-sequence $\underset{\sim}{x} = (x_1, \cdots, x_n)$ being taken to lie in the (Jacobson-) radical $\mathrm{rad}(A)$ of A, is also an E-sequence if $\underset{\sim}{x}E$ is the contraction $\varphi^{-1}(\underset{\sim}{x}M)$ of $\underset{\sim}{x}M$ in E.

As a corollary of this lifting property we obtain very easily the so-called delocalization-lemma for regular sequences (also **[2]**, Cor. 1 for local rings A and **[4]** Chap. I, §4). Then we exemplify that the condition $\varphi^{-1}(\underset{\sim}{x}M) = \underset{\sim}{x}E$ is not necessary for the statement of our theorem (see Example 3); otherwise it is easily seen that generally the theorem (especially Corollary 2) becomes false without any additional condition (see Examples 1 and 2).

Recall that a sequence $x_1, \cdots, x_n$ of elements of A is said to be (M-regular or) an M-sequence if, for each $0 \leqq i \leqq n-1$, a_{i+1} is a nonzerodivisor on $M/(x_1, \cdots, x_i)M$ and $M \neq (x_1, \cdots, x_n)M$.

2. First we consider the case $n = 1$.

LEMMA. *The notations being as above. Let x be a M-regular element in the radical* $\mathrm{rad}(A)$ *of A and suppose that*

(1) $$\ker \varphi \subseteq xE^{1}.$$

Then x is an E-regular element too and φ is injective.

Proof. We put $F = \ker \varphi$. Clearly x is E/F-regular, hence $xE \cap F = xF$, hence $F = xF$ by (1). Therefore we get $F = 0$ by Nakayama's lemma, hence φ is injective and x is E-regular.

THEOREM. *Let E be a finite A-module, M an arbitrary A-module and $\varphi: E \to M$ a module-homomorphism. Let $\underset{\sim}{x} = (x_1, \cdots, x_n)$ be an M-sequence in* $\mathrm{rad}(A)$ *and suppose that*

[1] We denote by xE or $\underset{\sim}{x}E$ the product $(x)E$ or $(\underset{\sim}{x})E$ respectively, where (x) or $(\underset{\sim}{x})$ is the ideal generated by x or $x_1, \cdots, x_n$ respectively.

(2) $$\varphi^{-1}(\underline{x}M) = \underline{x}E^1.$$

Then $\underline{x}$ is an E-sequence and φ is injective.

Proof. (By induction on n): Note that $\ker\varphi \subseteq \varphi^{-1}(\underline{x}M)$, hence the case $n = 1$ results from the lemma.

For $n > 1$ we put $\underline{x}' = (x_1, \cdots, x_{n-1}), E' = E/\underline{x}'E, M' = M/\underline{x}'M$ and $\varphi' = \varphi \otimes 1_{A/\underline{x}'A} : E' \to M'$. Then we have

$$\ker\varphi' = \varphi^{-1}(\underline{x}'M)/\underline{x}'E \cap \varphi^{-1}(\underline{x}'M) = \varphi^{-1}(\underline{x}'M)/\underline{x}'E \subseteq \varphi^{-1}(\underline{x}M)/\underline{x}'E,$$

hence we get by (2):

$$\ker\varphi' \subseteq \underline{x}E/\underline{x}'E = x_n \cdot E'.$$

Since x_n is M'-regular we are in the situation of the lemma with x_n and φ': $E' \to M'$ instead of x and φ. Therefore φ' is injective, i.e.

(3) $$\varphi^{-1}(\underline{x}'M) = \underline{x}'E,$$

and x_n is an E'-regular element. The sequence $\underline{x}'$ is M-regular by assumption, hence by (3) and by induction on n, $\underline{x}'$ is E-regular and φ is injective. This concludes the proof.

COROLLARY 1.1 (*Delocalization Lemma*, 1. *form*): *Let A be a noetherian ring, E a finite A-module and $x_1, \cdots, x_n \in \operatorname{rad}(A)$. Let $U \subset A$ be the set of nonzerodivisors modulo $\underline{x}$ for E. Suppose that $\underline{x}$ is an E_U-sequence. Then $\underline{x}$ is an E-sequence.*

Proof. Let $\varphi : E \to E_U$ be the natural homomorphism. No element of U is zerodivisor for $E/\underline{x}E$, hence $\varphi^{-1}(\underline{x}E_U) = \underline{x}E$, proving the corollary.

It results from the following Corollary 1.2 that the conditions for $\underline{x}$ in the two Corrollaries 1.1 and 1.2 are equivalent.

COROLLARY 1.2 (*Delocalization Lemma*, 2. *form*): *Let E be a finitely generated module over a noetherian ring A and $x_1, \cdots, x_r \in$* $\operatorname{rad}(A)$. *If $\underline{x}$ is an $E_{\mathfrak{y}}$-sequence for all $\mathfrak{y} \in Ass(E/\underline{x}E)$, then $\underline{x}$ is an E-sequence.*

Proof. Let $M = \bigoplus E_{\mathfrak{y}}$ for $\mathfrak{y} \in Ass(E/\underline{x}E)$, and φ the homomorphism $E \to M$ defined by $u \to \Sigma_{\mathfrak{y}}\varphi_{\mathfrak{y}}(u)$, where $\varphi_{\mathfrak{y}}$ denotes the natural map $E \to E_{\mathfrak{y}}$. [Note that $Ass\,(E/\underline{x}E)$ is a finite set.]

Since $\underline{x}$ is an $E_{\mathfrak{y}}$-sequence for all $\mathfrak{y} \in Ass(E/\underline{x}E)$, it must be an M-sequence too. We want to apply our theorem to finish the proof. For that we show that $\varphi^{-1}(\underline{x}M) = \underline{x}E$:

Since E is finitely generated, the submodule $\underline{x}E$ has an irredundant primary decomposition $\underline{x}E = Q_1 \cap \cdots \cap Q_r$ corresponding to the ideals $\mathfrak{y}_i \in Ass(E/\underline{x}E)$. Localizing $\underline{x}E$ by any ideal $\mathfrak{y} \in Ass\ (E/\underline{x}E)$ we obtain **[5]**:

$$\varphi_{\mathfrak{y}}^{-1}((Q_i)_{\mathfrak{y}}) = \begin{cases} Q_i & \text{if } \mathfrak{y}_i \subseteq \mathfrak{y} \\ E & \text{if } \mathfrak{y}_i \not\subseteq \mathfrak{y}, \end{cases}$$

hence $\cap_{\mathfrak{y}} \varphi_{\mathfrak{y}}^{-1}(\underline{x}E_{\mathfrak{y}}) = \underline{x}E$.

On the other hand we have $\underline{x}M = \bigoplus_{\mathfrak{y}} \underline{x}E_{\mathfrak{y}}$, hence $\varphi^{-1}(\underline{x}M) = \cap_{\mathfrak{y}} \varphi_{\mathfrak{y}}^{-1}(\underline{x}E_{\mathfrak{y}})$. This concludes the proof of the corollary.

3. Now let $f: A \to B$ be a ring-homomorphism. If $\mathfrak{a}$ and $\mathfrak{b}$ are ideals in A and B respectively we define as usual $\mathfrak{a}^e$ to be the extension $f(\mathfrak{a})B$ of $\mathfrak{a}$ and $\mathfrak{b}^c$ to be the contraction $f^{-1}(\mathfrak{b})$ of $\mathfrak{b}$.

COROLLARY 2. *Let $f: A \to B$ be a homomorphism of noetherian rings. Let $\mathfrak{a}$ be an ideal generated by elements $x_1, \cdots, x_n \in$ $\operatorname{rad}(A)$. Suppose that $f(x_1), \cdots, f(x_n)$ form a B-sequence and suppose that $\mathfrak{a}^{ec} = \mathfrak{a}$. Then $x_1, \cdots, x_n$ is an A-sequence.*

Proof. Regard B as an A-module relatively to f. We consider the module-homomorphism $\varphi: A \to B$ given by $\varphi(a) = f(a)$ for all $a \in A$. Then, by assumption all conditions of the theorem are fulfilled, proving the corollary.

REMARKS. (i) The proof of Corollary 2 shows that the delocalization Lemma 1.1 or 1.2 respectively can be formulated for rings too.

(ii) If $f: A \to B$ is faithfully flat then any sequence $x_1, \cdots, x_n \in \operatorname{rad}(A)$ is A-regular $\Leftrightarrow$ $f(x_1), \cdots, f(x_n)$ is B-regular. This well-known statement (s. **[4]** or **[3]**) is an easy consequence of Corollary 2: f is faithfully flat says that f is flat and the induced map ${}^a f$: Spec $B \to$ Spec A is surjective. But if f is flat then the last condition is equivalent to $\mathfrak{a}^{ec} = \mathfrak{a}$ (s. **[1]**, p. 45), where $\mathfrak{a}$ is generated by $x_1, \cdots, x_n$. Hence Corollary 2 works for $\Leftarrow$; the other direction is trivial.

(iii) Let B be a surjectively-free A-algebra [i.e. $A = \Sigma_{\psi} \psi(B)$, where ψ runs over $\operatorname{Hom}_A(B, A)$]. Then for any ideal $\mathfrak{a}$ of A one has

$$\mathfrak{a}^{ec} = \mathfrak{a}B \cap A = \mathfrak{a},$$

and the induced map Spec $B \to$ Spec A is surjective; see **[5]**, (5, E), p. 37.

4. We are indebted to L. Badescu for pointing to the following

EXAMPLE 1. Consider the ring

$$R = \{f \in k[x,y] \mid f(1,0) = f(-1,0)\} \subset k[x,y],$$

where k denotes say the field of complex numbers. Then R is the finitely generated subring $k[x - x^3, x^2, xy, y]$ of $k[x,y]$; clearly $k[x,y]$ is integrally dependent on R and with the same quotient field. Therefore $Y := \operatorname{Spec} R$ is not normal. Write X for the normal affine variety $\operatorname{Spec}(k[x,y]) \cong k^2$. Let the inclusion of R in $k[x,y]$ define the proper morphism

$$\pi : X \to Y.$$

Then if x_1, x_2 are the points $(1,0)$ and $(-1,0) \in k^2$, we have $\pi(x_1) = \pi(x_2) =: y_0$, and

$$\operatorname{res} \pi : X - \pi^{-1}\{y_0\} \to Y - \{y_0\}$$

is an isomorphism. In particular, Y is normal at all points except y_0 [this is also clear by the connectedness theorem, because $\pi^{-1}\{y_0\}$ is not connected]. To be more in detail, take

$$v_1 = 1 - x^2, v_2 = xy, v_3 = y, v_4 = x - x^3.$$

Then

$$R =$$

$$k[v_1, v_2, v_3, v_4]/(v_4 v_3 - v_2 v_1; v_2^2 - v_3^2 + v_1 v_3^2; v_4^2 + v_1^3 - v_1^2; v_1 v_2 - v_2 v_4 - v_1^2 v_3).$$

So Y can be regarded as an affine surface in k^4, which is nonsingular in codimension 1, but not normal in the origin (corresponds to the point $y_0 \in \operatorname{Spec} R = Y$). Therefore $0_{Y,y_0}$ is not a Cohen Macaulay-ring by the criterion of normality, [3], 5.8.6.

We fix the notations:

$$A = 0_{Y,y_0} = R_{\mathfrak{y}} \qquad \text{with} \quad \mathfrak{y} = (x-1, y) \cap R = (x+1, y) \cap R;$$

$$B = 0_{X,x_1} = k[x,y]_{\mathfrak{y}} \qquad \text{with} \quad \mathfrak{y} = (x-1; y);$$

$f: A \to B$ the corresponding local homomorphism; $a_1 = (x-1)(x+1)$, $a_2 = y$ in A and $f(a_1) = (x-1)(x+1)$, $f(a_2) = y$ regarded as being in B. Then B is a regular local ring, and

$f(a_1)$, $f(a_2)$ generate its maximal ideal $\mathfrak{m}_B$. Since depth $A \lneqq \dim A = 2$ the sequence a_1, a_2 will not be A-regular. And indeed we have $\mathfrak{a}^e = \mathfrak{m}_B$ and $\mathfrak{a}^{ec}\mathfrak{m}_A \neq \mathfrak{a}$.

EXAMPLE 2. Let $(A, \mathfrak{m})$ be a one-dimensional local noetherian ring which is not a Cohen Macaulay-ring. Let $x \in \mathfrak{m}$ be a parameter of A and $\mathfrak{y}$ a minimal prime over-ideal of zero in A. Take $B = A/\mathfrak{y}$. Then by assumptions $f(x)$ is B-regular, but x is not A-regular. And we have $\mathfrak{a}^{ec} \neq \mathfrak{a}$ (otherwise $\mathfrak{y}$ would be zero).

EXAMPLE 3. Let $(A, \mathfrak{m})$ be a local Cohen Macaulay-ring of dimension 1 which is not regular. Then the maximal ideal $\mathfrak{m}$ can be generated in this way:

$$\mathfrak{m} = xA + m_1A + \cdots + m_rA,$$

where x denotes an A-regular element.

Let $\mathfrak{a}$ be the ideal generated by x and $B = A[m_i/x] \subset A_x$. Since x is A-regular the natural homomorphism $f: A \to B$ is injective, and clearly x is B-regular.

But now we have $\mathfrak{a}^e = xB = \mathfrak{m}B$, hence $\mathfrak{a}^{ec} = \mathfrak{m}B \cap A = \mathfrak{m}$, hence $\mathfrak{a}^{ec} \neq \mathfrak{a}$ because A is not regular.

This example shows that condition (2) is not necessary for the statement of the theorem.

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