SOME THEOREMS CONCERNING ABSOLUTE NEIGHBORHOOD RETRACTS

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1. Introduction. If X is a subset of a topological space X^*, and there exists a mapping (continuous function) $\rho : X^* \longrightarrow X$ such that $\rho(x) = x$ for $x \in X$, then X is called a *retract* of X^*, and ρ is called a *retraction.* If U is a neighborhood of X in X^* (that is, a set open in X^* and containing X), and there exists a retraction $\rho' : U \longrightarrow X$, then X is called a *neighborhood retract* of X^*. If X is a separable metric space such that every homeomorphic image of X as a closed subset of a separable metric space M is a neighborhood retract of M, then X is called an *absolute neighborhood retract* or an *ANR*. It is with such spaces that we shall be principally concerned. They are of particular interest because of their usefulness in homotopy theory, and also because every locally-finite polyhedron is an *ANR* [5].

Section 2 is concerned with two theorems of point-set topological nature. For many other theorems along this line, see [3, §10].

In §3, we are concerned with spaces of mappings into *ANR*'s, and with homotopy problems involving *ANR*'s; in particular, we obtain certain restrictions on the cardinality of collections of homotopy classes of mappings into *ANR*'s. For closely related results, see [4], especially the corollary to Theorem 5.

2. Theorems in point-set-topology. The following theorem is a slight generalization of a standard result.

THEOREM 1. *If a subset X_0 of an ANR X is a neighborhood retract of X, then X_0 is an ANR.*

This generalization consists in not requiring that X_0 be closed. For the more restricted theorem and its proof (which in actuality establishes Theorem 1), see [3, §10.1].

An interesting corollary of Theorem 1 is that *an open subset of an ANR is an ANR.*

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We now prove a theorem which shows that the study of ANR's may—for many purposes—be reduced to the study of connected ANR's.

THEOREM 2. *A topological space X is an ANR if and only if X has at most countably many components, each of which is open in X, and each of which is itself an ANR.*

Proof of Theorem 2. Suppose X is an ANR. Then it is well known that X is locally connected, and hence has open components. Since X must be perfectly separable, this implies that there are at most countably many components; and that they are themselves ANR's follows from the corollary to Theorem 1. This proves the necessity.

Now suppose $X = \bigcup C_i$, where each C_i is an open component of X, and also an ANR, and suppose that X is a closed subset of a separable metric space M. For definiteness we assume that the set of C_i's is denumerably infinite; the proof must be simplified slightly if the union is finite.

Since C_1 and $(C_2 \cup C_3 \cup \cdots)$ are disjoint closed subsets of M, we can pick open subsets U_1 and U_1^* of M with $C_1 \subset U_1$ and $(C_2 \cup C_3 \cup \cdots) \subset U_1^*$, and with $U_1 \cap U_1^* = \phi$ (the null set). Then obviously $U_1 \cap (C_2 \cup C_3 \cup \cdots) = \phi$ (where $\overline{U}_1$ is the closure of U_1).

Having picked $U_1, U_2, \cdots, U_{n-1}$ such that each U_i is open, contains C_i, is disjoint from the other U_i, and has a closure disjoint from C_j for all $j \neq i$; we see that the sets

$$\bigcup_1^{n-1} \overline{U}_i \cup \bigcup_{n+1}^{\infty} C_i \equiv K_n$$

and C_n are disjoint closed sets. Hence we may choose disjoint open sets $U_n \supset C_n$ and $U_n^* \supset K_n$.

Then we easily see that by induction we may pick open sets U_i $(i = 1, 2, \cdots)$ so that for each i we have $C_i \subset U_i$, and so that for $i \neq j$ we have $U_i \cap U_j = \phi$.

Since each C_i is a closed ANR subset of M, we may pick open sets V_i $(i = 1, 2, \cdots)$ such that each V_i contains C_i, and retractions $\rho_i\colon V_i \longrightarrow C_i$, $i = 1, 2, \cdots$.

Define

$$\rho : \bigcup_1^{\infty} (U_i \cap V_i) \longrightarrow X$$

by setting $\rho(x) = \rho_i(x)$ for $x \in U_i \cap V_i$. It is clear that ρ retracts a neighborhood of X onto X, as required.

3. Results in homotopy theory. Our next theorem has a corollary concerning homotopy groups and the homotopy classification problem.

THEOREM 3 : *Let X be a compact topological space, and let Y be an ANR. Then the space Y^X of continuous functions on X into Y has open, arcwise-connected components.*

If $X_0 \subset X$, and $y_0 \in Y$, then the space $Y^X\{X_0, y_0\}$ of functions in Y^X which carry X_0 to y_0 also has open, arcwise-connected components.

(We give the function spaces the topology of uniform convergence, which for compact X's coincides with the more general compact-open topology.)

In the case that X is metric, a known theorem tells us that Y^X is itself an ANR, whence the first result follows easily. However, the second result is of perhaps greater interest, because of its bearing on homotopy groups (see below). We shall prove the second result; the proof of the first proceeds similarly, but is slightly easier.

Proof of Theorem 3. Let d be a metric on Y. The topology on our function space is obtained by taking

$$d^*(f, g) = \sup_{x \in X} d(f(x), g(x))$$

as a metric for functions f and g. We first show that *if $f \in Y^X(X_0, y_0)$ then there exists an $\epsilon > 0$ such that if $d^*(f, g) < \epsilon$, then g can be joined to f by an arc in $Y^X\{X_0, y_0\}$.*

By Wojdyslawski's imbedding theorem, we may take Y to be a closed subset of a convex subset Z of a Banach space S. Since Y is an ANR, there exists a retraction $\rho : U \longrightarrow Y$ of a neighborhood U of Y in Z onto Y. Since $f(X)$ is compact and disjoint from the closed subset $Z - U$ of Z, we may choose $\epsilon > 0$ such that the ϵ-neighborhood of $f(X)$ in Z is contained in U.

Now suppose $d^*(f, g) < \epsilon$. Define

$$H(x, t) = (1-t)f(x) + tg(x) \qquad ((t, x) \in X \times I).$$

It is clear that each $H(x, t) \in U$, so we may define

$$h(x, t) = \rho(H(x, t)) \qquad ((t, x) \in X \times I).$$

Plainly $h: X \times I \longrightarrow Y$ is continuous. Hence by a theorem of R.H. Fox [2], the function $\rho^* : I \longrightarrow Y^X$ defined by

$$\rho^*(t)(x) = h(x, t) \qquad (t \in I, x \in I).$$

is also continuous. One sees easily that $\rho^*(0) = f$, $\rho^*(1) = g$, and that the image of ρ^* is actually in $Y^X\{X_0, y_0\}$.

This proves the statement made at the beginning of the proof. This statement makes it obvious that the arc-components are open. One shows from this statement that the arc-components coincide with the components, in exactly the same way that one shows that a connected open subset of Euclidean space is arcwise connected.

A result of Fox [2] implies that if X is either locally compact and regular, or locally separable (that is, satisfies the first axiom of countability), and the function spaces are given the compact-open topology, then the arcs in Y^X and in $Y^X\{X_0, y_0\}$ are essentially identical with the homotopies of functions in these spaces. It follows from Theorem 3 and the equivalence of the topologies for a compact X that *if X is compact and either regular or locally separable, and Y is an ANR, then Y^X and $Y^X\{X_0, y_0\}$ have open components, and these components coincide with the homotopy classes of the function spaces.*

Arens [1] has given a theorem which - specialized to the cases with which we are dealing - states that if X has a basis of cardinal number a, and Y has a basis of cardinal number b, then Y^X (and hence $Y^X\{X_0, y_0\}$ as well) has a basis whose cardinal number does not exceed both a and b. Eliminating trivial cases by assuming that X does not have finite basis, we see that each of our function spaces has a basis of cardinal number not greater than a. This easily implies that the collections of open components of our function spaces cannot have cardinalities greater than a; with the conclusion obtained by use of Fox's theorem, this establishes the following result.

THEOREM 4: *Let the topological space X be compact, and either regular or locally separable. Suppose that X has a basis of (infinite) cardinal number a. Let Y be an ANR. Then the collection of homotopy classes of functions in Y^X, and also the collection of homotopy classes of functions in $Y^X\{X_0, y_0\}$, each has cardinality not greater than a.*

The following theorem is an immediate corollary to the second conclusion of of the above theorem.

THEOREM 5: *A homotopy group of an ANR is at most countable.*

The necessity for strong hypotheses in Theorems 3, 4, and 5 is indicated by examples we have found which show that (*i*) an arcwise-connected, locally-connected, and compact subset of the plane may have an uncountable fundamental group; (*ii*) the homotopy classes of the space of mappings of the 1-sphere into an arcwise-connected and locally-connected subset of 3-space need not coincide with the components of this function space; (*iii*) the homotopy classes of the space of mappings of a separable metric, arcwise-connected, locally-contractible, compact space into itself need not be open in this space of mappings.

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