A NOTE ON THE DIMENSION THEORY OF RINGS

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1. Introduction. Let O be an integral domain. If in O there is a proper chain

$$(0) \subset P_1 \subset P_2 \subset \cdots \subset P_n \subset (1)$$

of prime ideals, but no such chain

$$(0) \subset P'_1 \subset \cdots \subset P'_{n+1} \subset (1),$$

then O will be said to be *n-dimensional.* Let O be of dimension n: the question is whether the polynomial ring $O[x]$ is necessarily $(n+1)$-dimensional. Here, as throughout, x is an indeterminate.

By an *F-ring* we shall mean a 1-dimensional ring O such that $O[x]$ is not 2-dimensional (i. e., the proposed assertion that $O[x]$ is necessarily 2-dimensional fails). Given an F-ring, we try by definite constructions to pass to a larger F-ring having the same quotient field: this restricts the class of rings in which to look for an F-ring—a priori we do not know they exist. In this way we also come (in Theorem 8 below) to a complete characterization of F-rings: if O is 1-dimensional, then $O[x]$ is 2-dimensional if and only if every quotient ring of $\bar{O}$, the integral closure of O, is a valuation ring. The rings $\bar{O}$ thus coincide (for dimension 1) with Krull's Multiplikationsringe [5; p. 554].

2. Preliminary results. The first five theorems are of a preparatory character, and the proofs offer no difficulties.

THEOREM 1. *Let O be an arbitrary commutative ring with* 1, P_1, P_2, P_3 *distinct ideals in* $O[x]$. *If* $P_1 \subset P_2 \subset P_3$, *and* P_2 *and* P_3 *are prime ideals, then* P_1, P_2, P_3 *cannot have the same contraction to* O.

Proof. Let

$$P_1 \cap O = P_2 \cap O = p,$$

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and consider

$$O[x]/P_2 = \bar{O}[\bar{x}],$$

where $\bar{x}$ is the residue of x and $\bar{O} \simeq O/p$. Since

$$\bar{O}[x] \cdot p \subseteq P_1 \subset P_2,$$

$\bar{x}$ is algebraic over the integral domain $\bar{O}$. Let $\bar{P}_3$ be the image of P_3; then $\bar{P}_3 \neq (0)$; but also $\bar{P}_3 \cap \bar{O} \neq (0)$. In fact, let $y \in \bar{P}_3$, $y \neq 0$. Then

$$c_0 y^n + c_1 y^{n-1} + \cdots + c_n = 0$$

for some $c_i \in \bar{O}$, $c_n \neq 0$; and $c_n \in \bar{P}_3 \cap \bar{O}$. Hence also $P_3 \cap O \neq p$,

COROLLARY. *If O is 1-dimensional, and P_1, P_2, P_3 are distinct prime ideals in $O[x]$ different from (0) with $P_1 \subset P_2 \subset P_3$, then $P_1 \cap O = (0)$, P_2 is the extension of its contraction to O, and P_3 is maximal.*

Proof. If $P_1 \cap O \neq (0)$, then P_1, P_2, P_3 would all have to contract to the same maximal ideal in O. So

$$P_1 \cap O = (0) \text{ and } P_2 \cap O = p \neq (0).$$

Were $O[x] \cdot p \subset P_2$ properly, then, since $O[x] \cdot p$ is prime,

$$O[x] \cdot p \cap O = (0),$$

whereas

$$O[x] \cdot p \cap O = p.$$

So $O[x] \cdot p = P_2$. Were P_3 not maximal, we would have $P_2 \cap O = (0)$.

For the foregoing theorem, see also [4; Th. 10, p. 375].

THEOREM 2. *If O is n-dimensional, then $O[x]$ is at least $(n+1)$-dimensional and at most $(2n+1)$-dimensional.*

Proof. Let

$$(0) \subset P_1 \subset P_2 \subset \cdots \subset P_n \subset (1)$$

be a proper chain of prime ideals in O. Then

$$(0) \subset O[x] \cdot P_1 \subset O[x] \cdot P_2 \subset \cdots \subset O[x] \cdot P_n \subset (1)$$

is also a proper chain of prime ideals in $O[x]$; and $O[x] \cdot P_n$ is not maximal, since, for example,

$$O[x] \cdot P_n \subset (O[x] \cdot P_n, x) \subset (1).$$

(Here, as throughout, we use the symbol $\subset$ for proper inclusion.) Hence $O[x]$ is at least $(n+1)$-dimensional. Let now O be n-dimensional, and consider a chain

$$(0) \subset P'_1 \subset \cdots \subset P'_m \subset (1)$$

of prime ideals in $O[x]$. Let there be s distinct ideals among the contractions

$$(0) \cap O, P'_1 \cap O, \cdots, P'_m \cap O.$$

Then

$$m+1 < 2s \leq 2(n+1), \text{ so } m \leq 2n+1.$$

THEOREM 3. *If O is n-dimensional but $O[x]$ is not $(n+1)$-dimensional, then for at least one minimal prime ideal $\mathfrak{p}$ of O either the quotient ring $O_{\mathfrak{p}}$ is an F-ring or $O/\mathfrak{p}$ is m-dimensional and $O/\mathfrak{p}[x]$ is not $(m+1)$-dimensional, and $m < n$.*

Proof. Suppose that for some minimal prime ideal p of O, $O[x] \cdot p$ is not minimal in $O[x]$; that is, there exists a prime ideal P such that

$$(0) \subset P \subset O[x] \cdot p.$$

Then

$$(0) \subset O_p[x] \cdot P \subset O_p[x] \cdot p$$

is also a chain of prime ideals in $O_p[x]$, as one easily verifies. Since $O_p[x] \cdot p$ is not maximal, this shows that O_p is an F-ring. We pass then to the case that $O[x] \cdot p$ is minimal for every minimal prime ideal p of O. Let

$$(0) \subset P'_1 \subset \cdots \subset P'_{n+2} \subset (1)$$

be a chain of prime ideals in $O[x]$. If

$$P'_1 \cap O = \mathfrak{p} \neq (0),$$

then O/p is at most $(n-1)$-dimensional, and $O[x]/O[x] \cdot p$ is a polynomial ring in one variable over O/p and is at least $(n+1)$-dimensional. So we must suppose

$$P_1' \cap O = (0);$$

but then

$$P_2' \cap O = p_2 \neq (0);$$

let p be a minimal prime ideal contained in p_2—such exists since O is finite dimensional; then $O[x] \cdot p \subset P_2'$, properly, since $O[x] \cdot p$ is minimal but P_2' is not. Replacing P_1' by $O[x] \cdot p$, we come back to a previous case, and the proof is complete.

COROLLARY. *If O is an F-ring, then so is some quotient ring of O.*

The foregoing theorem shows that if for some n there exists a ring O which is n-dimensional, while $O[x]$ is not $(n+1)$-dimensional, then there exist F-rings. Thus we may provisionally confine our attention to 1-dimensional rings O.

THEOREM 4. *If O is 1-dimensional, and O is a valuation ring, then $O[x]$ is 2-dimensional.*

Proof. Let p be a proper prime ideal of O, and let

$$(0) \subset P \subseteq O[x] \cdot p,$$

where P is prime. Let

$$f(x) \in P, \quad f(x) \neq 0.$$

Then one can factor out from $f(x)$ a coefficient of least value, that is, write

$$f(x) = c \cdot g(x),$$

where $c \in p$, and $g(x)$ has at least one coefficient equal to 1; in particular, then $g(x) \notin O[x] \cdot p$; hence $c \in P$. So $P \cap O \neq (0)$, whence

$$P \cap O = p \quad \text{and} \quad P = O[x] \cdot p.$$

This proves that $O[x]$ is 2-dimensional (see Corollary to Theorem 1).

Theorem 4 restricts the size of an F-ring, since a maximal ring is a valuation ring. The following theorem reduces the considerations to integrally closed rings.

THEOREM 5. *Let $\overline{O}$ be the integral closure of the integral domain O. Then O is an F-ring if and only if $\overline{O}$ is an F-ring.*

Proof. Let R be an integral domain integrally dependent on O; a basic theoem of Krull (see, for example, [2; Th. 4, p. 254]) says that if $P_1 \subset P_2$ are prime ideals in R, then they contract to distinct prime ideals in O; hence dim $R \leq$ dim O. Another theorem (loc. cit., p. 254) says that if $p_1 \subset p_2$ are prime ideals in O, and p_1 is a prime ideal in R contracting to p_1, then there exists a prime ideal P_2, $P_2 \supset P_1$, contracting to p_2. Hence dim $R \geq$ dim O, and so dim R = dim O. Hence $\overline{O}$ is 1-dimensional if and only if O is 1-dimensional, and $\overline{O}[x]$ is 2-dimensional if and only if $O[x]$ is 2-dimensional.

Thus if there exist F-rings, then there exist integrally closed F-rings, and, taking an appropriate quotient ring, we see that there would exist an integrally closed F-ring O having just one proper prime ideal. In view of Theorem 4 (and the close association of integrally closed rings with valuation rings) one may ask whether an integrally closed ring with only one proper prime ideal is necessarily a valuation ring. Were it so, there would be no F-rings, but it is not so: Krull has an example [6; p. 670f]. For convenience, we may mention the example: let K be an algebraically closed field, x and y indeterminates; O consists of the rational functions $r(x, y)$ which, when written in lowest terms, have denominators not divisible by x, and which are such that $r(0, y) \in K$.

3. Principal results. We now establish:

THEOREM 6. *If O is integrally closed with only one maximal ideal p, α an element of the quotient field of O, and $1/\alpha \notin O$, then $O[\alpha] \cdot p$ is prime. If also $\alpha \notin O$, then $O[\alpha] \cdot p$ is not maximal.*

Proof. We first observe that

$$(O[\alpha] \cdot p, \alpha) \neq (1),$$

as an equation

$$1 = c_0 + c_1 \alpha + \cdots + c_s \alpha^s \qquad (c_0 \in p,\ c_i \in O),$$

leads to an equation of integral dependence for $1/\alpha$ over O. Let now $g(x) \in O[x]$ be a monic polynomial of positive degree. We may assume, trivially, that $\alpha \notin O$; then $g(\alpha) = c \in O$ is impossible, as $g(\alpha) - c = 0$ would be an equation of integral dependence for α over O; in particular, $g(\alpha) \neq 0$. Also $1/g(\alpha) \notin O$, for if it were in O, it would be a nonunit in O, and hence would be in p, so that

$$1 \in g(\alpha) \cdot p \subseteq O[\alpha] \cdot p,$$

and this is not so. By the result on α,

$$(O[g(\alpha)] \cdot p,\ g(\alpha)) \neq (1).$$

Since α satisfies $g(x) - g(\alpha) = 0$, $O[\alpha]$ is integral over $O[g(\alpha)]$; over any prime ideal in $O[g(\alpha)]$ containing $(O[g(\alpha)] \cdot p,\ g(\alpha))$, there lies a prime ideal in $O[\alpha]$, hence

$$(O[\alpha] \cdot p,\ g(\alpha)) \neq (1).$$

Since $1 + g(x)$ is monic of positive degree, also

$$(O[\alpha] \cdot p,\ 1 + g(\alpha)) \neq (1).$$

This shows that $g(\alpha) \notin O[\alpha] \cdot p$, a conclusion that also holds if $g(x)$ is of degree zero; that is, $g(x) = 1$.

We now prove that under the homomorphism $g(x) \longrightarrow g(\alpha)$ of $O[x]$ onto $O[\alpha]$, the inverse image of $O[\alpha] \cdot p$ is $O[x] \cdot p$; this will complete the proof, as $O[x] \cdot p$ is prime but not maximal. Let, then,

$$g(x) \in O[x],\ g(x) \notin O[x] \cdot p.$$

We write

$$g(x) = g_1(x) + g_2(x),$$

where $g_2(x) \in O[x] \cdot p$ and no coefficient of $g_1(x)$ is in p; in particular, this is so for the leading coefficient c. Then $g_1(\alpha)/c \notin O[\alpha] \cdot p$, since $g_1(x)/c$ is monic. A fortiori, $g_1(\alpha) \notin O[\alpha] \cdot p$, whence also $g(\alpha) \notin O[\alpha] \cdot p$.

COROLLARY. *In the case $\alpha \notin O$, if $g(x) \in O[x]$ and $g(\alpha) \in O[\alpha] \cdot p$, then $g(x) \in O[x] \cdot p$.*

THEOREM 7. *Let O be an integrally closed integral domain, p a proper ideal therein, a an element in the quotient-field of O, but $a \notin O_p$, $1/a \notin O_p$. Then $O[a] \cdot$ p is prime but not maximal; in fact,*

$$O[\alpha] \cdot \mathrm{p} \cap O = \mathrm{p} \quad \textit{and} \quad O[\alpha]/O[\alpha] \cdot \mathrm{p} \simeq O/\mathrm{p}[x].$$

Proof. We know that $O_p[\alpha] \cdot$ p is prime, and

$$O_p[\alpha] \cdot \mathrm{p} \cap O[\alpha] = O[\alpha] = O[\alpha] \cdot p$$

by the last corollary (and the fact that $O_p \cdot$ p $\cap\ O = p$). Hence $O[\alpha] \cdot p$ is prime. Also here, as in the corollary, we have that if $g(x) \in O[x]$ and $g(\alpha) \in O[\alpha] \cdot p$, then $g(x) \in O[x] \cdot p$; the required isomorphism follows at once.

Theorem 7 is known in the case that O is a finite discrete principal order [3, §49, p. 134-136]. The class of rings dealt with in the theorem includes this class properly; for example, the ring O of the example of Krull is not a finite discrete principal order, as $xy^{\rho} \in O$ for all ρ, but $y \notin O$.

THEOREM 8. *If O is 1-dimensional, then $O[x]$ is 2-dimensional if and only if every quotient ring of the integral closure of O is a valuation ring.*

Proof. By Theorem 5, we may assume O to be integrally closed. If O is an F-ring, then so is one of its quotient rings (Theorem 3, Corollary). This quotient ring is not a valuation ring (Theorem 4). Conversely, suppose some quotient ring $O_1 = O_p$ is not a valuation ring. Let α be an element of the quotient field of O_1 such that $\alpha \notin O_1$ and $\alpha^{-1} \notin O_1$. Then $O_1[\alpha]$ is at least 2-dimensional, by Theorem 6, and $O_1[x]$ is at least 3-dimensional, as one sees by considering the homomorphism of $O_1[x]$ onto $O_1[\alpha]$ determined by mapping x into α. So O_1 is an F-ring. Thus $O_p[x] \cdot p$ is not minimal in $O_p[x]$, and it follows at once that $O[x] \cdot p$ is not minimal in $O[x]$, whence O is an F-ring.

Let O be the ring of Krull's example above, and let X be an indeterminate. The single prime ideal p in O is constituted by the rational fractions $r(x, y)$ which, when written in lowest terms, have numerator divisible by x, i.e., are of the form $x\, g(x, y)$, where $g(x, y) \in K[x, y]$. The polynomials in $O[X]$ which vanish for $X = y$ form a prime ideal, different from (0) since $xX - xy$ is in it, properly contained in $O[X] \cdot p$.

The following theorem is well known [4, Th. 13, p. 376].

THEOREM 9. *If O is a Noetherian ring of dimension n, then $O[x]$ is $(n+1)$-dimensional.*

Proof. Taking a quotient ring or residue class does not destroy the Noetherian character of O, so by Theorem 3 we may suppose O is 1-dimensional. Let then p be a proper prime ideal in O. Then $O[x] \cdot p$ is minimal for every principal ideal $O[x] \cdot (a)$, where $a \in p$, $a \neq 0$, so by the Principal Ideal Theorem [3, p. 37], $O[x] \cdot p$ is minimal in $O[x]$, and $O[x]$ is 2-dimensional by Theorem 1, Corollary. — Instead of the Principal Ideal Theorem, one could use instead that the integral closure $\overline{O}$ is also Noetherian (see, for example, [1, Th. 3, p. 29]; see also [3, §39, p. 108]). Neither proof makes use of the full force of the quoted theorems, so it might be of some interest to find a direct proof using less technical means.

NOTE. In a forthcoming paper we will show that if O is a 1-dimensional ring such that $O[x]$ is 2-dimensional, then $O[x_1, \cdots, x_n]$ is $(n+1)$-dimensional. Theorem 2, above, will also be completed by examples showing that for any m, n with $n+1 \leq m \leq 2n+1$, there exist n-dimensional rings such that $O[x]$ is m-dimensional.

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