

PLANE GEOMETRIES FROM CONVEX PLATES

MARLOW SHOLANDER

1. Introduction. It is shown below that to each member of a general class of two-dimensional convex bodies there corresponds an affine geometry in the sense of Artin [1] and an S. L. space in the sense of Busemann [4].

A two-dimensional convex body is called a *convex plate*. For the few elementary properties of such plates assumed here, see [3].

Let K be a convex plate, and let K^0 denote its boundary curve. All constructions are to be made in the plane E of K . Consider an arbitrary direction ϕ in E and the two lines of support to K in this direction. Let t_0 be the line of support whose associated half-plane in the direction $\phi + \pi/2$ contains K . Let t_1 be the other line of support. For $0 < i < 1$, let t_i be the line parallel to t_0 which divides line segments extending from t_0 to t_1 in the ratio of i to $1 - i$. Let t_i cut K^0 at points R_i and T_i so that the directed segment $R_i T_i$ has direction ϕ .

For $0 < i < 1$ and $0 < j < 1$, let S_{ij} be the point which divides $R_i T_i$ in the ratio of j to $1 - j$. The set $s_j = \cup_i S_{ij}$ is an open Jordan arc whose endpoints are points of contact of t_0 and t_1 with K . A set s_j is called a *strut*. Other struts may be obtained by varying ϕ . When the direction needs emphasis, the above notations are modified by affixing the angle in parentheses, for example, $R_i(\phi)$ or $s_j(\phi)$. Two struts with no common points or all points in common are called *parallel*. Clearly $s_j(\phi)$ and $s_k(\phi)$ are parallel.

Under the name *Durchlinien*, Zindler [6] studied struts of the form $s_{1/2}(\phi)$. It is easy to see that $s_{1/2}(\phi)$ halves the area of K , and that the centroid of K is contained in the convex hull of this strut.

2. A preliminary theorem. This section is devoted to a proof of the following theorem. An edge of K is defined as a (*maximal*) *line segment in K^0* .

THEOREM. *If for distinct directions ϕ and ψ , struts $s_i(\phi)$ and $s_j(\psi)$ meet at distinct points P and Q , they meet at all points of the segment PQ . Such segments of intersection occur if and only if K has at least two edges.*

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Proof. Let $i = 1/(1 + a)$ and $j = 1/(1 + b)$. From the affine invariant nature of the problem, we may assume ϕ and ψ are respectively the positive x - and positive y -directions in E , where P has been chosen as the origin. We may assume the chords passing through P along the axes are P_3P_1 and P_4P_2 , where P_1, P_2, P_3 , and P_4 have respectively the coordinates $(a, 0)$, $(0, b)$, $(-1, 0)$, and $(0, -1)$. If P_4P_1 is parallel to P_3P_2 , let n be the line parallel to these lines which passes through P . Otherwise let n be the line on P and the point of intersection of these lines. Finally, we may assume that Q lies in the first quadrant on or above the line n . Let Q have coordinates (r, s) .

Let the chords through Q parallel to the axes be Q_3Q_1 and Q_4Q_2 . Coordinates of Q_1, Q_2, Q_3 , and Q_4 have respectively the form $(r + ap, s)$, $(r, s + bq)$, $(r - p, s)$, and $(r, s - q)$. We note that P_4P_1, n , and P_3P_2 have respectively the equations

$$ay = x - a, \quad a(b + 1)y = b(a + 1)x, \quad \text{and} \quad y = bx + b.$$

Since Q is on or above n ,

$$(1) \quad b(a + 1)r \leq a(b + 1)s.$$

Because K is convex, Q_2 cannot be above P_2P_3 ; that is,

$$(2) \quad s + bq \leq b(r + 1).$$

Multiply (2) by a and add to (1). This gives

$$(3) \quad r - a \leq a(s - q);$$

that is, Q_4 is on or above P_1P_4 . Moreover, equality in (3) implies equality in (1) and (2).

Consider first the case $r < a$. Here, since Q_4 cannot be above P_1P_4 , it lies on P_1P_4 . Thus equality holds in (2), and Q_2 lies on P_2P_3 . Since P_4, Q_4 , and P_1 are distinct and collinear, they are on an edge of K . Similarly, Q_2, P_2 , and P_3 lie on an edge.

In the case $r \geq a$,

$$\text{slope } P_4P_1 \leq \text{slope } Q_4Q_1, \quad 1/a \leq q/ap, \quad \text{and} \quad p \leq q.$$

If $s < b$, Q_3 cannot be below P_2P_3 ; that is,

$$(4) \quad b(r + 1) \leq s + bp.$$

Together with (2) this yields $q \leq p$. Hence $p = q$, and equality holds in both (2)

and (4). This shows that Q_2, P_2, Q_3, P_3 are collinear, and hence on an edge of K . Furthermore, slope $P_4 P_1 = \text{slope } Q_4 Q_1$, and P_4, P_1, Q_4, Q_1 are on an edge. If $s \geq b$, Q_3 cannot be above $P_2 P_3$, slope $Q_3 Q_2 \leq \text{slope } P_3 P_2$, $bq/p \leq b$, and $q \leq p$. Again $p = q$, slope $P_4 P_1 = \text{slope } Q_4 Q_1$, and slope $P_3 P_2 = \text{slope } Q_3 Q_2$. An edge of K contains P_4, P_1, Q_4 , and Q_1 , and another edge contains Q_2, Q_3, P_2 , and P_3 .

3. Affine geometries. Consider a convex plate K with the properties:

- (i) K has at most one edge;
- (ii) K has no corners.

Let I be the set of inner points of K . Consider distinct points P and Q in I . Assume, for a given ϕ , that P is on $s_j(\phi)$ and Q is on $s_k(\phi)$, $j < k$. Clearly P is on $s_{1-j}(\phi + \pi)$, and Q is on $s_{1-k}(\phi + \pi)$. From considerations of continuity, there exists a direction ψ such that P and Q are on a strut $s_i(\psi)$. From this and from the previous section we have the following result.

PROPERTY I. Two distinct points in I lie on one and only one strut.

Consider now a strut $s_i(\phi)$ and a point P in I . The strut $s_j(\phi)$ which passes through P is parallel to $s_i(\phi)$. On the other hand, let $s_k(\psi)$, $\psi \neq \phi$, pass through P . Since $s_i(\phi)$ and $s_k(\psi)$ have endpoints which separate one another on K^0 , these struts have some point of I in common, and the following holds.

PROPERTY II. Given a strut s and a point P in I , there is one and only one strut through P and parallel to s .

PROPERTY III. There are three points of I not on a strut.

These three properties are Axioms I, II, and III of Artin [1]. Listed in *Lattice theory* [2, p. 110] as APG1, APG2, and APG3, they classify I as a plane affine geometry.

It would be of interest to know what sets I satisfy Artin's Axiom IV (see Appendix), or even what sets have nontrivial dilatations. An ellipse K yields an I with all the desired properties. To show this it is sufficient to consider the case where K is the circle

$$x^2 + y^2 \leq a^2.$$

Consider the sphere

$$S: x^2 + y^2 + (z - a)^2 = a^2,$$

resting on the origin of the xy -plane E . The line

$$x \cos \phi + y \sin \phi = R$$

in E projects from the center of S into a great half-circle on S . This half-circle projects perpendicularly on E into a half-ellipse, the strut $s_i(\phi)$, where

$$2i = 1 + R/\sqrt{R^2 + a^2}.$$

Thus the mapping which takes (r, θ) in I into the point (R, θ) of E , where

$$R\sqrt{a^2 - r^2} = ar,$$

places the struts in one-to-one correspondence with straight lines. In this example, we have a finite model for Euclidean geometry.

4. Other geometries. In general we may obtain a plane projective geometry from a plane affine geometry by adjoining an ideal line (see [2, p. 110]). In this case K^0 serves as the ideal line. The affine and projective geometries associated with K are examples of matroid lattices.

In § 3 we mapped an elliptical I onto the Euclidean plane E . A similar mapping may be defined for any I so that struts map on curves in E which satisfy the hypotheses of [4, p. 89, Th. 1]. It follows that a metric may be introduced (in E and hence) in I which makes of I an S. L. Space of Busemann: I will be finitely compact, convex in the sense of Menger, externally convex in the sense of Busemann, and the struts will be geodesics under this metric. This S. L. space also satisfies the Euclidean Parallel Axiom. In fact, all Hilbert's (plane) Axioms [5] are satisfied except the congruence axioms. The determination of the conditions under which the latter hold is an open problem.

5. Appendix. Artin's Axiom IV, not readily available to all readers, is given below after necessary introductory material. Using Axioms I-IV, we may assign coordinates (α, β) to points so that the equation of a "strut" is linear.

The set of points considered is called a *plane*. A mapping σ associates with every point P a point $P' = \sigma(P)$. A mapping is called a *dilatation* if to each pair of points P, Q correspond parallel struts s and s' such that P and Q lie on s , and P' and Q' lie on s' . The identity mapping of the plane is denoted by 1. A *translation* is a dilatation which is either 1 or else has no fixed points. A *trace* of a dilatation σ is a strut which contains a point P and its image P' . (If $P \neq P'$, there is a unique trace on P .) A *homomorphism* is a correspondence from translation τ to translation τ^α such that each trace of τ is a trace of τ^α and such that

$$(\tau_1 \tau_2)^\alpha = \tau_1^\alpha \tau_2^\alpha.$$

AXIOM IVa. Given P and Q , there exists a translation carrying P into Q .

AXIOM IVb. Given translations τ_1 and τ_2 (neither equal to 1) with the same traces, there exists a homomorphism τ^α such that $\tau_1^\alpha = \tau_2$.

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WASHINGTON UNIVERSITY

