

# AN INEQUALITY FOR SUBORDINATE ANALYTIC FUNCTIONS

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**1. Introduction.** If  $f(z)$  and  $F(z)$  are nonconstant analytic functions, regular in the unit circle  $|z| < 1$ , then  $f(z)$  is called *subordinate* to  $F(z)$ , written

$$(1) \quad f(z) \prec F(z), \quad |z| < 1,$$

provided there exists a function  $\omega(z)$ , regular in the unit circle, with

$$\omega(0) = 0, \quad |\omega(z)| < 1, \quad \text{and} \quad f(z) = F[\omega(z)].$$

The concept of subordination has proved useful in studies of the range of values of analytic functions [2, pp.163-171]. The following interesting result has recently been established by G.M. Goluzin [1]:

**GOLUZIN'S THEOREM.** *Let*

$a(r)$  = area of the region on the Riemann surface onto which the disk  $|z| < r$  is mapped by  $f(z)$ ,

and

$A(r)$  = area of the region on the Riemann surface onto which the disk  $|z| < r$  is mapped by  $F(z)$ .

Then

$$(2) \quad a(r) \leq A(r) \quad \text{for} \quad r \leq \frac{1}{\sqrt{2}};$$

further, if  $r < 1/\sqrt{2}$ , equality in (2) can be achieved only in the trivial case

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$$\omega(z) = \eta z. {}^1$$

It should be remarked that Goluzin's Theorem is intuitively obvious for the case where  $\omega(z)$  is schlicht, for then the Riemann surface corresponding to  $|z| < r$  under  $f(z)$  is a subset of the corresponding surface for  $F(z)$ , and (2) is therefore obtained for any  $r < 1$ . If  $\omega(z)$  is not restricted to be schlicht, the example

$$f(z) = z^2, \quad F(z) = z$$

shows that (2) does not always hold for all  $r < 1$ ; in fact for this case, as pointed out by Goluzin,

$$a(r) > A(r) \text{ for any } r > \frac{1}{\sqrt{2}},$$

while

$$a\left(\frac{1}{\sqrt{2}}\right) = A\left(\frac{1}{\sqrt{2}}\right),$$

so that the range  $0 \leq r \leq 1/\sqrt{2}$  is the best possible one for the inequality (2).

**2. Theorem.** In this paper we extend Goluzin's Theorem to the complete interval  $0 \leq r < 1$ ; that is, we explicitly find a universal function  $T(r)$  (which turns out to be a continuous function of  $r$ , made up of arcs of polynomials), such that, for any  $f(z), F(z)$  satisfying (1),

$$(3) \quad \frac{a(r)}{A(r)} \leq T(r) \quad (0 \leq r < 1),$$

where we denote  $\lim_{r \rightarrow 0} a(r)/A(r)$  by  $a(0)/A(0)$ , and where the inequality (3) is the best possible one, in the sense that for any  $r$  ( $0 \leq r < 1$ ) it is possible to find a pair of functions  $f(z), F(z)$ , satisfying (1), such that (3) is an equality for that particular  $r$ . (Of course, it follows from Goluzin's Theorem that  $T(r) = 1$  for  $0 \leq r^2 \leq 1/2$ .) We supplement our result by making a complete enumeration of the function pairs  $[f(z), F(z)]$  for which equality in (3) can be achieved. The final result is as follows.

**THEOREM.** *The function  $T(r)$  is given by the formula*

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<sup>1</sup>In this paper  $\eta$  always denotes an arbitrary complex constant of unit absolute value.

$$(4) \quad T(r) = mr^{2m-2}$$

in the range

$$\frac{m-1}{m} \leq r^2 \leq \frac{m}{m+1} \quad (m = 1, 2, \dots).$$

Equality in (3) for a given  $r$  is possible under, and only under, the following circumstances:

$$(5) \left\{ \begin{array}{l} \text{(i) } r^2 < 1/2 \text{ and } \omega(z) = \eta z \text{ (as remarked above);} \\ \text{(ii) } r^2 = 1/2, \text{ and either} \\ \qquad \qquad \qquad \qquad \qquad \qquad \qquad \omega(z) = \eta z \\ \text{or} \\ \qquad \qquad \qquad \qquad \qquad \qquad \qquad F(z) = C_0 + C_1 z, \quad f(z) = C_0 + C_1 \eta z^2; \\ \text{(iii) } \frac{m-1}{m} < r^2 < \frac{m}{m+1} \qquad \qquad \qquad (m \geq 2), \\ \text{and} \\ \qquad \qquad \qquad \qquad \qquad \qquad \qquad F(z) = C_0 + C_1 z, \quad f(z) = C_0 + C_1 \eta z^m; \\ \text{(iv) } r^2 = \frac{m}{m+1} \qquad \qquad \qquad (m \geq 2), \\ \text{and either} \\ \qquad \qquad \qquad \qquad \qquad \qquad \qquad F(z) = C_0 + C_1 z, \quad f(z) = C_0 + C_1 \eta z^m \\ \text{or} \\ \qquad \qquad \qquad \qquad \qquad \qquad \qquad F(z) = C_0 + C_1 z, \quad f(z) = C_0 + C_1 \eta z^{m+1}. \end{array} \right.$$

It is very easy to check that for the cases listed above one actually obtains equality in (3), so that if (3) can be shown to be generally true it follows that it is the best possible inequality.

Since  $f(z) = 0$  and  $|f(z)| < 1$  imply that  $f(z) \prec z$ , we can, as an application of the theorem, immediately state the following:

COROLLARY. If  $f(z) = 0$  and  $|f(z)| < 1$  for  $|z| < 1$ , then

$$a(r) \leq \pi m r^{2m}$$

in the range

$$\frac{m-1}{m} \leq r^2 \leq \frac{m}{m+1} \quad (m = 1, 2, \dots).$$

This is the best possible inequality. [The cases of equality can be obtained by putting  $C_0 = 0$ ,  $C_1 = 1$  in (5)].

**3. Lemmas.** Before proceeding with the proof proper of the theorem we shall state some known results, and derive some others.

The following notation will be adopted as standard:

$$f(z) = c_0 + c_1 z + c_2 z^2 + \dots, \quad F(z) = C_0 + C_1 z + C_2 z^2 + \dots,$$

$$\omega(z) = \beta_1 z + \beta_2 z^2 + \dots.$$

Thus

$$c_0 = C_0.$$

Furthermore, as an easy computation shows,

$$a(r) = \pi \sum_{k=1}^{\infty} k |c_k|^2 r^{2k}, \quad A(r) = \pi \sum_{k=1}^{\infty} k |C_k|^2 r^{2k}.$$

LEMMA 1 [2]. If  $\omega(z) \not\equiv \eta z$ , then  $|c_1| < |C_1|$ ,  $|\beta_1| < 1$ .

This is an immediate consequence of the Lemma of Schwarz.

LEMMA 2 [2].  $|\beta_2| \leq 1 - |\beta_1|^2$ .

This is proved by applying the Lemma of Schwarz to the function

$$\frac{\omega(z)/z - \beta_1}{1 - \bar{\beta}_1 \omega(z)/z}.$$

LEMMA 3 [2].  $\max_{|z|=r} |f(z)| \leq \max_{|z|=r} |F(z)| \quad (r < 1)$ .

Also a consequence of Schwarz's Lemma.

LEMMA 4.

$$(6) \quad \sum_{k=1}^{\infty} |c_k|^2 r^{2k} \leq \sum_{k=1}^{\infty} |C_k|^2 r^{2k} \quad (r < 1).$$

This is a special case of a more general inequality for mean values of arbitrary nonnegative order on the circle  $|z| = r$ . The proof by Littlewood [2] uses subharmonic functions. A different proof has been given by Goluzin [1].

LEMMA 5. *If*

$$s_n = \sum_{k=1}^n |c_k|^2, S_n = \sum_{k=1}^n |C_k|^2,$$

then

$$(7) \quad s_n \leq S_n,$$

with equality for a particular  $n$  implying that

$$(8) \quad c_{n+1} = \beta_1^{n+1} C_{n+1}.$$

The inequality (7) is a known result [1], but we shall repeat the proof to show how (8) follows.

*Proof.* Let

$$s_n(z) = \sum_{k=1}^n c_k z^k, S_n(z) = \sum_{k=1}^n C_k z^k, R_n(z) = \sum_{k=n+1}^{\infty} C_k z^k.$$

Then

$$\begin{aligned} C_0 + s_n(z) + \sum_{n+1}^{\infty} c_k z^k &= f(z) = F(\omega(z)) = S_n(\omega(z)) + R_n(\omega(z)) + C_0 \\ &= C_0 + S_n(\omega(z)) + \sum_{n+1}^{\infty} c_k^{(n)} z^k, \end{aligned}$$

where

$$c_{n+1}^{(n)} = C_{n+1} \beta_1^{n+1},$$

and so on. We have

$$S_n(\omega(z)) = s_n(z) + \sum_{n+1}^{\infty} d_k^{(n)} z^k,$$

where

$$d_k^{(n)} = c_k - c_k^{(n)},$$

so that, for example,

$$d_{n+1}^{(n)} = c_{n+1} - C_{n+1} \beta_1^{n+1}.$$

Now, by the fundamental definition of subordination,

$$S_n[\omega(z)] \prec S_n(z).$$

Therefore, using (6) in the equation above, we have

$$\sum_1^n |c_k|^2 r^{2k} + \sum_{n+1}^{\infty} |d_k^{(n)}|^2 r^{2k} \leq \sum_1^n |C_k|^2 r^{2k}.$$

Thus (7) follows, and equality certainly implies

$$d_{n+1}^{(n)} = c_{n+1} - C_{n+1} \beta_1^{n+1} = 0,$$

as was to be shown.

LEMMA 6 [1]. *If  $\{\lambda_k\}$  are such that*

$$(9) \quad \lambda_k \geq 0, \lambda_k \geq \lambda_{k+1} \quad (k = 1, 2, \dots),$$

*then*

$$(10) \quad \sum_{k=1}^{\infty} \lambda_k |c_k|^2 \leq \sum_{k=1}^{\infty} \lambda_k |C_k|^2.$$

Proof by partial summation, using Lemma 5.

LEMMA 6.1. Under the hypothesis (9), and with  $s_n$  and  $S_n$  defined by (7), equality in (10) implies

$$(11) \quad (\lambda_k - \lambda_{k+1}) \cdot (S_k - s_k) = 0 \quad (k = 1, 2, \dots).$$

Proof. Let

$$r_n = \sum_{k=1}^n \lambda_k |c_k|^2, R_n = \sum_{k=1}^n \lambda_k |C_k|^2, s_0 = S_0 = 0.$$

The hypothesis is

$$\lim_{n \rightarrow \infty} (R_n - r_n) = 0.$$

Thus

$$\begin{aligned} 0 &= \lim_{n \rightarrow \infty} \left[ \sum_{k=1}^n \lambda_k (S_k - S_{k-1}) - \sum_{k=1}^n \lambda_k (s_k - s_{k-1}) \right] \\ &= \lim_{n \rightarrow \infty} \left[ \sum_{k=1}^n \lambda_k S_k - \sum_{k=0}^{n-1} \lambda_{k+1} S_k - \sum_{k=1}^n \lambda_k s_k + \sum_{k=0}^{n-1} \lambda_{k+1} s_k \right] \\ &= \lim_{n \rightarrow \infty} \left[ \sum_{k=1}^{n-1} (\lambda_k - \lambda_{k+1}) S_k + \lambda_n S_n - \sum_{k=1}^{n-1} (\lambda_k - \lambda_{k+1}) s_k - \lambda_n s_n \right] \\ &= \lim_{n \rightarrow \infty} \left[ \sum_{k=1}^{n-1} (\lambda_k - \lambda_{k+1}) (S_k - s_k) + \lambda_n (S_n - s_n) \right]. \end{aligned}$$

Since the terms in the square brackets are all nonnegative, (11) follows.

LEMMA 7. If

- (i)  $f(z) = c_0 + c_1 z + c_2 z^2 \prec F(z) = c_0 + C_1 z + C_2 z^2,$
- (ii)  $|c_1| < |C_1|,$
- (iii)  $|c_1|^2 + |c_2|^2 = |C_1|^2 + |C_2|^2,$

then

$$(12) \quad f(z) = c_0 + C_1 \eta z^2, \quad F(z) = c_0 + C_1 z.^2$$

*Proof.* As indicated by (ii), we have  $|C_1| > 0$ , so we can normalize as follows:

$$(13) \quad f(z) = c_1 z + c_2 z^2, \quad F(z) = z + C_2 z^2.$$

Let

$$p = |c_1|, \quad q = |c_2|, \quad Q = |C_2|.$$

We have, according to (ii) and (iii),

$$(14) \quad p < 1,$$

$$(15) \quad p^2 + q^2 = 1 + Q^2.$$

Now

$$\max_{|z|=1} |c_1 z + c_2 z^2| = |c_1| + |c_2| = p + q,$$

as is obvious if  $c_1 = 0$ , and follows by taking  $z = e^{-i \arg(c_2/c_1)}$  if  $c_1 \neq 0$ . Similarly,

$$\max_{|z|=1} |z + C_2 z^2| = 1 + Q.$$

Applying Lemma 3, we get

$$(16) \quad p + q \leq 1 + Q.$$

The shaded triangle  $OL_1L_2$  in the facing figure corresponds to the inequality (16). The quarter-circle  $P_1P_2P_3$  corresponds to the equality (15);  $P_1$  lies below  $L_1$ , and  $P_3$  to the left of  $L_2$ , because

$$\sqrt{1+Q^2} \leq 1 + Q \text{ for any } Q \geq 0.$$

The circle and the hypotenuse of the triangle intersect in two points with coordinates  $(Q, 1)$  and  $(1, Q)$ , respectively (merging into a single point if  $Q = 1$ ).

<sup>2</sup> A more general result is stated, but not proved, in a footnote on page 56 of Rogosinski's paper [3].

When  $Q \geq 1$ , the point  $L_3$  in the figure has the coordinates  $(1, Q)$ , and  $L_4$  has coordinates  $(Q, 1)$ . When  $Q \leq 1$ , then  $L_3 = (Q, 1)$ ,  $L_4 = (1, Q)$ . In either case, (14) dictates that eligible points  $(p, q)$  lie to the left of  $L_4$ , and therefore on the arc  $P_1L_3$ . Therefore, since the ordinate of  $L_3$  is never less than 1, it is always true that

$$(17) \quad q \geq 1.$$

We proceed by considering two cases:

- (I)  $Q = 0$ ,
- (II)  $Q > 0$ ,

and show that (II) leads to a contradiction. Thus, assume  $Q > 0$ . We have

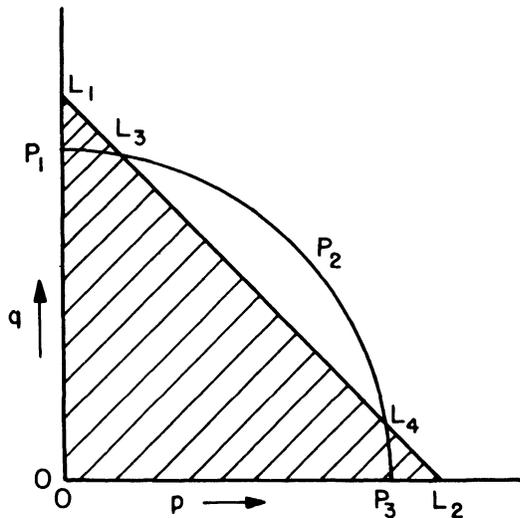
$$c_1z + c_2z^2 = \omega + C_2\omega^2, \quad C_2 \neq 0.$$

Therefore

$$(18) \quad \omega(z) = \frac{-1 + \sqrt{1 + 4C_2(c_1z + c_2z^2)}}{2C_2},$$

where, since  $\omega(z)$  is regular in  $|z| < 1$ , we must have

$$(19) \quad 1 + 4C_2(c_1z + c_2z^2) \neq 0 \quad (|z| < 1),$$



and we must take that branch of  $\omega(z)$  which equals zero for  $z = 0$ . Let  $z_1, z_2$  be the roots of (19). Since (19) is to have no roots for  $|z| < 1$ , we have

$$|z_1| \geq 1, \quad |z_2| \geq 1,$$

and therefore, by (19),

$$1 \leq |z_1 z_2| = \left| \frac{1}{4c_2 C_2} \right| = \frac{1}{4qQ},$$

or, in view of (17),

$$(20) \quad Q \leq 1/4.$$

The next step consists in applying Lemma 2. By (18),

$$\beta_1 = \omega'(0) = c_1, \quad \beta_2 = \frac{1}{2} \omega''(0) = c_2 - c_1^2 C_2.$$

Thus, by Lemma 2,

$$|c_2 - c_1^2 C_2| \leq 1 - |c_1|^2,$$

and so

$$|c_2| - |c_1^2 C_2| \leq 1 - |c_1|^2;$$

that is,

$$q + p^2(1 - Q) \leq 1.$$

But  $q \geq 1$  by (17), and  $1 - Q > 0$  by (20). Therefore

$$q = 1 \quad \text{and} \quad p = 0.$$

Equation (15) now implies  $Q = 0$ , showing the impossibility of (II).

Since (I) is the only alternative to (II), we may conclude that

$$(22) \quad Q = |C_2| = 0.$$

Then (14), (15), (17), (22) jointly imply

$$q = 1 \quad \text{and} \quad p = 0.$$

Therefore, by (13),

$$\omega(z) = \eta z^2,$$

and (12) now follows.

LEMMA 8. *Let  $m$  be a fixed positive integer,  $t$  a fixed positive real number, and  $k$  a positive integer. The following relations hold:*

- (i) *for any  $k$ , if  $\frac{m-1}{m} \leq t \leq \frac{m}{m+1}$ , then  $mt^m \geq kt^k$ ;*
- (ii) *if  $k \leq m$ ,  $t \geq \frac{m-1}{m}$ , then  $\{kt^k\}_k$  is an increasing sequence;*
- (iii) *if  $k \leq m$ ,  $t > \frac{m-1}{m}$ , then  $\{kt^k\}_k$  is a strictly increasing sequence;*
- (iv) *if  $k \leq m-1$ ,  $t \geq \frac{m-1}{m}$ , then  $\{kt^k\}_k$  is a strictly increasing sequence;*
- (v) *if  $k \geq m$ ,  $t \leq \frac{m}{m+1}$ , then  $\{kt^k\}_k$  is a decreasing sequence;*
- (vi) *if  $k \geq m$ ,  $t < \frac{m}{m+1}$ , then  $\{kt^k\}_k$  is a strictly decreasing sequence;*
- (vii) *if  $k \geq m+1$ ,  $t \leq \frac{m}{m+1}$ , then  $\{kt^k\}_k$  is a strictly decreasing sequence.*

**4. Proof of the theorem.** For any positive integer  $m$ ,

$$(23) \quad \frac{\alpha(r)}{\pi} = \sum_{k=1}^{\infty} k |c_k|^2 r^{2k}$$

$$= \left[ mr^{2m} \sum_{k=1}^{m-1} |c_k|^2 + \sum_{k=m}^{\infty} k |c_k|^2 r^{2k} \right] - \left[ mr^{2m} \sum_{k=1}^{m-1} |c_k|^2 - \sum_{k=1}^{m-1} k |c_k|^2 r^{2k} \right],$$

where  $\sum_{k=1}^0$  is to be understood to have the value zero; (23) can be rewritten as

$$(24) \quad \frac{\alpha(r)}{\pi} = \sum_{k=1}^{\infty} \lambda_k^{(m)} |c_k|^2 - \sum_{k=1}^{m-1} [(mr^{2m} - kr^{2k}) |c_k|^2],$$

where

$$(25) \quad \lambda_k^{(m)} = \begin{cases} mr^{2m} & \text{if } 1 \leq k \leq m-1, \\ kr^{2k} & \text{if } k \geq m. \end{cases}$$

Similarly,

$$(26) \quad \frac{A(r)}{\pi} = \sum_{k=1}^{\infty} \lambda_k^{(m)} |C_k|^2 - \sum_{k=1}^{m-1} [(mr^{2m} - kr^{2k}) |C_k|^2].$$

Henceforth let  $r$  be positive, and restricted to the interval

$$\frac{m-1}{m} \leq r^2 \leq \frac{m}{m+1}.$$

We see by (25) and Lemma 8(v) that

$$\lambda_1^{(m)} = \lambda_2^{(m)} = \dots = \lambda_m^{(m)} > \lambda_{m+1}^{(m)} > \lambda_{m+2}^{(m)} > \dots > 0$$

if

$$\frac{m-1}{m} \leq r^2 < \frac{m}{m+1},$$

while, by Lemma 8(vii),

$$\lambda_1^{(m)} = \lambda_2^{(m)} = \dots = \lambda_{m+1}^{(m)} > \lambda_{m+2}^{(m)} > \lambda_{m+3}^{(m)} > \dots > 0$$

if

$$r^2 = \frac{m}{m+1}.$$

By Lemmas 6 and 6.1, therefore,

$$(27) \quad \sum_{k=1}^{\infty} \lambda_k^{(m)} |c_k|^2 \leq \sum_{k=1}^{\infty} \lambda_k^{(m)} |C_k|^2 \quad (m = 1, 2, \dots),$$

with equality possible only if

$$(28) \quad S_k = s_k \text{ for all } k \geq m, \text{ providing } \frac{m-1}{m} \leq r^2 < \frac{m}{m+1},$$

or

$$S_k = s_k \text{ for all } k \geq m + 1, \text{ providing } r^2 = \frac{m}{m + 1}.$$

Subtracting (26) from (24) gives, by (27),

$$(30) \quad \frac{a(r) - A(r)}{\pi} \leq \sum_{k=1}^{m-1} [(mr^{2m} - kr^{2k}) |C_k|^2] - \sum_{k=1}^{m-1} [(mr^{2m} - kr^{2k}) |c_k|^2],$$

with equality possible only if conditions (28) and (29) are met. By Lemma (8) (ii), (iii), and (iv), the last sum of (30) is nonnegative, and can vanish only if

$$(31) \quad c_k = 0, \ k = 1, 2, \dots, m - 1, \text{ providing } \frac{m - 1}{m} < r^2 \leq \frac{m}{m + 1},$$

or

$$(32) \quad c_k = 0, \ k = 1, 2, \dots, m - 2, \text{ if } r^2 = \frac{m - 1}{m}.$$

We conclude that

$$(33) \quad \begin{aligned} \frac{a(r) - A(r)}{\pi} &\leq \sum_{k=1}^{m-1} [(mr^{2m} - kr^{2k}) |C_k|^2] \\ &= (mr^{2m-2} - 1) \sum_{k=1}^{m-1} k |C_k|^2 r^{2k} - mr^{2m-2} \sum_{k=1}^{m-1} [(kr^{2k} - r^2) |C_k|^2]. \end{aligned}$$

By Lemma (8)(iv), the last sum of (33) is nonnegative, and can vanish only if

$$(34) \quad C_2 = C_3 = \dots = C_{m-1} = 0.$$

Therefore

$$(35) \quad \begin{aligned} \frac{a(r) - A(r)}{\pi} &\leq (mr^{2m-2} - 1) \sum_{k=1}^{m-1} k |C_k|^2 r^{2k} \\ &\leq (mr^{2m-2} - 1) \sum_{k=1}^{\infty} k |C_k|^2 r^{2k} = (mr^{2m-2} - 1) \frac{A(r)}{\pi}, \end{aligned}$$

where the first inequality sign may reduce to an equality sign only if (28), (29), (31), (32), and (34) hold, while the second inequality may become an equality only if either  $m = 1$ , or  $r^2 = 1/2$ , or

$$C_m = C_{m+1} = \dots = 0, r^2 > \frac{1}{2}.$$

From (35) we immediately obtain the desired relation (4). Only the need for examining the possibility of equality in (35) remains.

Collecting the available information for this case, we are led to (5)(iii) when

$$\frac{m-1}{m} < r^2 < \frac{m}{m+1}, m \geq 2.$$

When

$$r^2 = \frac{m}{m+1}, m \geq 2,$$

on the other hand, we obtain

$$f(z) = C_0 + c_m z^m + c_{m+1} z^{m+1}, F(z) = C_0 + C_1 z,$$

with

$$(36) \quad |C_1|^2 = |c_m|^2 + |c_{m+1}|^2.$$

Since  $f(z) \prec F(z)$ , we also have

$$c_m z^m + c_{m+1} z^{m+1} \prec C_1 z.$$

Therefore, using Lemma 3, we get

$$(37) \quad |c_m| + |c_{m+1}| \leq |C_1|.$$

Squaring (37), and subtracting (36), yields the conclusion that either  $c_m = 0$  or  $c_{m+1} = 0$ , and therefore (5)(iv) follows.

For  $r^2 \leq 1/2$  we know that

$$(38) \quad \sum_{k=1}^n |c_k|^2 = \sum_{k=1}^n |C_k|^2 \quad \text{for } n \geq 2.$$

For  $r^2 < 1/2$ , from Lemma 6.1 it is found to be necessary that  $\omega(z) = \eta z$ , as was already shown by Goluzin.

The only remaining case is  $r^2 = 1/2$ . Again  $\omega(z) = \eta z$  gives equality in (3). We leave this trivial possibility aside by restricting our attention (see Lemma 1) to

$$(39) \quad |c_1| < |C_1|, \quad |\beta_1| < 1.$$

Applying Lemma 5 to (38) yields

$$(40) \quad c_k = \beta_1^k C_k \quad (k = 3, 4, \dots),$$

or, by (38) and (40),

$$|c_k| = |\beta_1|^k |C_k| = |C_k| \quad (k \geq 3).$$

Therefore, in view of (39),

$$c_k = C_k = 0 \quad (k \geq 3).$$

This shows that  $f(z)$  and  $F(z)$  are respectively of the forms

$$(41) \quad f(z) = c_0 + c_1 z + c_2 z^2 \quad \text{and} \quad F(z) = c_0 + C_1 z + C_2 z^2.$$

Equations (38) and (41) contain the hypotheses of Lemma 7, and the result (5)(ii) of the theorem therefore follows.

In conclusion, it may be remarked that due to Lemma (8)(i), the function  $T(r)$  may be written in the compact form

$$T(r) = \max_{k=1,2,\dots} (kr^{2k-2}) \quad (0 < r \leq 1).$$

As an immediate consequence, we have the much weaker result that

$$T(r) < \sum_{k=1}^{\infty} kr^{2k-2} = \frac{1}{(1-r^2)^2} \quad (r > 0).$$

## REFERENCES

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