NOTE ON THE MULTIPLICATION FORMULAS FOR THE JACOBI ELLIPTIC FUNCTIONS

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1. Introduction. For t an odd integer it is well known [**4**, vol. 2, p. 197] that

$$sn\, tx = \frac{sn\, x \cdot G_1^{(t)}(z)}{G_0^{(t)}(z)} \qquad (z = sn^2 x), \tag{1.1}$$

where

$$\begin{aligned} G_0^{(t)} &= 1 + a_{01} z + a_{02} z^2 + \cdots + a_{0t'} z^{t'}, \\ G_1^{(t)} &= t + a_{11} z + a_{12} z^2 + \cdots + a_{1t'} z^{t'} \qquad (t' = (t^2-1)/2), \end{aligned} \tag{1.2}$$

and the a_{ij} are polynomials in $u = k^2$ with rational integral coefficients. If we define

$$\beta_m(t) = \beta_m(t, u)$$

by means of

$$\frac{sn\, tx}{t\, sn\, x} = \sum_{m=0}^{\infty} \beta_{2m}(t) \frac{x^{2m}}{(2m)!} \qquad (\beta_{2m+1}(t) = 0), \tag{1.3}$$

it follows from (1.1) and (1.2) that $t\beta_{2m}(t)$ is a polynomial in u with integral coefficients for all m and all odd t. We shall show that

$$\beta_{2m}(t) = H_m(t) - \sum_{\substack{p-1 \mid 2m \\ p \mid t}} \frac{1}{p} A_p^{2m/(p-1)}(u), \tag{1.4}$$

where $H_m(t) = H_m(t, u)$ denotes a polynomial in u with integral coefficients,

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the summation in the right member is over all (odd) primes p such that $(p-1) \mid 2m$ and $p \mid t$; finally $A_p(u)$ is defined [**4**, vol. 1, p. 399] by means of

$$(1.5) \qquad sn\, x = sn(x, u) = \sum_{m=0}^{\infty} A_{2m+1}(u) \frac{x^{2m+1}}{(2m+1)!} ,$$

so that $A_{2m+1}(u)$ is a polynomial in u with integral coefficients. We show also that

$$(1.6) \qquad t \sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} \beta_{m+s(p-1)}(t) A_p^{r-s}(u) \equiv 0 \pmod{(p^m, p^r)},$$

where p is an arbitrary odd prime and $r \geq 1$; by (1.6) we understand that the left member is a polynomial in u every coefficient of which is divisible by the indicated power of p.

The proof of these formulas depends upon the results of [**2**]; for a theorem analogous to (1.4), see [**1**].

2. Proof of (1.4). Put

$$(2.1) \qquad \frac{x}{sn\, x} = \sum_{m=0}^{\infty} \beta_{2m} \frac{x^{2m}}{(2m)!} .$$

Then β_{2m} is a polynomial in u with rational coefficients; indeed [**2**, Theorem 2],

$$(2.2) \qquad p\beta_{2m} \equiv \begin{cases} -A_p^{2m/(p-1)}(u) & ((p-1) \mid 2m) \\ 0 & ((p-1) \nmid 2m). \end{cases} \pmod p$$

In the next place, if we write

$$\frac{sn\, tx}{t\, sn\, x} = \frac{sn\, tx}{tx} \frac{x}{sn\, x} ,$$

and make use of (1.3), (1.5), and (2.1), it follows that

$$(2.3) \qquad \beta_{2m}(t) = \sum_{s=0}^{m} \binom{2m}{2s} \beta_{2m-2s} A_{2s+1}(u) \frac{t^{2s}}{2s+1} .$$

As already observed, $t\beta_{2m}(t)$ has integral coefficients; thus the denominator of $\beta_{2m}(t)$ is a divisor of t. Now let p denote a prime divisor of t, and assume $p^e \mid (2s+1)$, $e \geq 1$. Then

$$2s+1 \geq p^e \geq 3^e \geq e+2, \qquad 2s \geq e+1.$$

Thus not only is $t^{2s}/(2s+1)$ integral (mod p) but it is divisible by p. Since by (2.2) the denominator of β_{2m} contains p to at most the first power it therefore follows that the product

$$\beta_{2m-2s}\, t^{2s}/(2s+1) \tag{2.4}$$

is integral (mod p) when $p \mid (2s+1)$.

Suppose next that $p \nmid (2s+1)$, where $s \geq 1$. It is again clear that (2.4) is integral (mod p) since p occurs in the denominator of β_{2m-2s} at most once while it occurs in t^{2s} at least twice. Thus as a matter of fact (2.4) is divisible by p in this case.

It remains to consider the term $s = 0$ in (2.3). Clearly we have proved that

$$p\beta_{2m}(t) \equiv p\beta_{2m} \qquad (\mathrm{mod}\ p). \tag{2.5}$$

Comparing (2.5) with (2.2) we may state:

THEOREM 1. *If t is an arbitrary odd integer then* (1.4) *holds.*

We remark that the residue of $A_p(u)$ is determined [2, §6] by

$$\begin{aligned} A_p(u) &\equiv (-1)^{\frac{1}{2}(p-1)} F\left(\frac{1}{2}, \frac{1}{2}; 1; u\right) \\ &\equiv (-1)^{\frac{1}{2}(p-1)} \sum_{j=0}^{\frac{1}{2}(p-1)} \binom{\frac{1}{2}(p-1)}{j}^2 u^j \qquad (\mathrm{mod}\ p). \end{aligned} \tag{2.6}$$

Here F denotes the hypergeometric function.

3. Some corollaries. By means of Theorem 1 a number of further results are readily obtained. By H_{2m} will be understood an unspecified polynomial in u with integral coefficients.

Since β_{2m}, as defined by (2.1), is integral (mod 2) we have first:

THEOREM 2. *If t is divisible by the denominator of β_{2m}, then*

$$\beta_{2m}(t) = H_{2m} + \beta_{2m}. \tag{3.1}$$

If t is prime to the denominator of β_{2m}, then $\beta_{2m}(t)$ has integral coefficients.

THEOREM 3. *If t_1, t_2 are relatively prime and odd, then*

$$\beta_{2m}(t_1 t_2) = H_{2m} + \beta_{2m}(t_1) + \beta_{2m}(t_2). \tag{3.2}$$

If t is a power of a prime we get:

THEOREM 4. *If p is an odd prime and $r \geq 1$ we have*

$$\beta_{2m}(p^r) = H_{2m} + \beta_{2m}(p). \tag{3.3}$$

Using (3.2) and (3.3) we get also:

THEOREM 5. *The following identity holds:*

$$\beta_{2m}(t) = H_{2m} + \sum_{p \mid t} \beta_{2m}(p), \tag{3.4}$$

where the summation is over all prime divisors of t.

We have also:

THEOREM 6. *If a is an arbitrary integer, then the product*

$$a(a^m - 1)\beta_{2m}(t) \tag{3.5}$$

has integral coefficients.

4. A related result. It follows from (1.1) and (1.2) that, for t odd,

$$sn\ tx = \sum_{r=0}^{\infty} C_{2r+1} sn^{2r+1} x, \tag{4.1}$$

where the C_{2r+1} are polynomials in u with integral coefficients. Clearly we have

$$\beta_{2m}(t) = \frac{1}{t} \sum_{r=0}^{m} A_{2m}^{(2r)} C_{2r+1}, \tag{4.2}$$

where the $A_{2m}^{(2r)}$ are defined by

$$sn^{2r} x = \sum_{m=0}^{\infty} A_{2m}^{(2r)} \frac{x^{2m}}{(2m)!}, \tag{4.3}$$

and like the C's are polynomials with integral coefficients.

We shall now prove the following property of the C's.

THEOREM 7. *For t odd we have*

$$(2m+1)C_{2m+1} \equiv 0 \pmod{t} \qquad (m = 0, 1, 2, \cdots), \tag{4.4}$$

where (4.4) *indicates that every coefficient in* $(2m+1)C_{2m+1}$ *is divisible by* t.

Proof. Differentiating (4.1) with respect to x, we get

$$t \frac{cn\, tx\, dn\, tx}{cn\, x\, dn\, x} = \sum_{m=0}^{\infty} (2m+1) C_{2m+1} sn^{2m} x. \tag{4.5}$$

Now we have, in addition to (1.1),

$$\frac{cn\, tx}{cn\, t} = \frac{G_2^{(t)}(z)}{G_0^{(t)}(z)}, \quad \frac{dn\, tx}{dn\, x} = \frac{G_3^{(t)}(z)}{G_0^{(t)}(z)} \qquad (z = sn^2 x), \tag{4.6}$$

where G_2 and G_3 are polynomials in z of the same form as G_0. By means of (1.1) and (4.6) it is evident that (4.5) implies

$$t \sum_{m=0}^{\infty} H_m^{(t)} z^m = \sum_{m=0}^{\infty} (2m+1) C_{2m+1} z^m, \tag{4.7}$$

where the H_m are polynomials in u with integral coefficients. Clearly (4.4) is an immediate consequence of (4.7).

Kronecker [5, p. 439] has proved a similar result in connection with the transformation of prime order of $sn\, x$. For a result like Theorem 7 for the Weierstrass $\wp$-function, see [3].

Returning to (4.2) we recall [2, § 2] that

$$(4.8)\qquad A_{2m}^{(2r)} \equiv 0 \qquad (\mathrm{mod}\,(2r)!) \qquad (m = 0, 1, 2, \cdots).$$

We rewrite (4.2) in the form

$$(4.9)\qquad \beta_{2m}(t) = \sum_{r=0}^{m} \frac{(2r)!}{2r+1}\,\frac{A_{2m}^{(2r)}}{(2r)!}\,\frac{(2r+1)C_{2r+1}}{t}.$$

By (4.4) and (4.8) the last two fractions in the right member of (4.9) have integral coefficients; also $(2r)!/(2r+1)$ is integral unless $2r+1$ is prime. Consequently (4.9) becomes

$$(4.10)\qquad \beta_{2m}(t) = H_{2m} - \sum_{\substack{p-1 \mid 2m \\ p \mid t}} \frac{1}{p} A_{2m}^{(p-1)} \frac{pC_p}{t}.$$

Comparing (4.10) with (1.4) we get:

THEOREM 8. *If the prime p divides t, then*

$$(4.11)\qquad \frac{pC_p}{t} \equiv 1 \quad (\mathrm{mod}\ p).$$

Hence if $p^e \mid t$, $p^{e+1} \nmid t$ *it follows that*

$$(4.12)\qquad C_p \equiv \frac{t}{p} \qquad (\mathrm{mod}\ p^e).$$

5. Proof of (1.6). Again using (5.1) we have

$$(5.1)\qquad \frac{sn\ tx}{sn\ x} = \sum_{i=0}^{\infty} C_{2i+1}\, sn^{2i} x.$$

Now it is proved in [2, Theorem 4] that the coefficients $A_{2m}^{(2i)}$ defined by (4.3) satisfy

$$(5.2)\qquad \sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} A_p^{(r-s)b/(p-1)} A_{2m+sb}^{(2i)} \equiv 0 \quad (\mathrm{mod}\,(p^{2m}, p^{er})),$$

where $p^{e-1}(p-1) \mid b$. Hence using (1.3) and (5.1) we get:

THEOREM 9. *If* $p^{e-1}(p-1)\,|\,b$, *then*

$$(5.3)\qquad t\sum_{s=0}^{r}(-1)^{r-s}\binom{r}{s}A_p^{(r-s)b/(p-1)}\beta_{2m+sb}(t)\equiv 0 \pmod{(p^{2m},p^{er})},$$

For $b = p-1$, (5.3) evidently reduces to (1.6).

It is of some interest to compare Theorem 9 with the results of [2, § 7]. If we take $r=1$, (5.3) becomes

$$t\{\beta_{2m+b}(t)-A_p^{b/(p-1)}\beta_{2m}(t)\}\equiv 0 \qquad (\mathrm{mod}\,(p^{2m},\,p^{e})).$$

If we put

$$\beta_{2m}(t)=\sum_i \beta_{2m,i}u^i$$

and recall that, by (2.6),

$$A_p(0)\equiv(-1)^{\frac{1}{2}(p-1)} \pmod{p}$$

we get exactly as in the proof of [2, Theorem 6].

THEOREM 10. *Let* $p^{e-1}(p-1)\,|\,b$ *and* $p^{j-1}\le i<p^j$. *Then*

$$(5.4)\qquad \beta_{2m+b,\,i}\equiv(-1)^{\frac{1}{2}b}\beta_{2m,\,i} \quad (\mathrm{mod}\,(p^{2m},\,p^{e-j})).$$

6. An elementary analogue of $\beta_{2m}(t)$. It may be of interest to say a word about the numbers $\phi_m(t)$ defined by

$$(6.1)\qquad \frac{e^{tx}-1}{t(e^x-1)}=\sum_{m=0}^{\infty}\phi_m(t)\frac{x^m}{m!},$$

where t is now an arbitrary integer. Clearly (6.1) implies that

$$t\phi_m(t)=S_m(t)=\sum_{s=0}^{t-1}s^m.$$

By a theorem of Staudt (see for example [6, p. 143]),

$$\phi_m(t) = G + \sum_{p \mid t} \phi_m(p), \tag{6.2}$$

where G is an integer. Moreover,

$$p\phi_m(p) = \begin{cases} -1 & \pmod p \quad (p-1 \mid m) \\ 0 & \pmod p \quad (p-1 \nmid m). \end{cases} \tag{6.3}$$

It follows [6, p. 153] that

$$\phi_{2m}(t) = G - \sum_{\substack{p-1 \mid 2m \\ p \mid t}} \frac{1}{p}. \tag{6.4}$$

Thus Staudt's theorems (6.2) and (6.4) may be viewed as elementary analogues of (3.4) and (1.4).

Formulas like (6.2) and (6.4) hold also for the numbers $\psi_{2m}(t)$ occurring in

$$\frac{\sin tx}{t \sin x} = \sum_{m=0}^{\infty} \psi_{2m}(t) \frac{x^{2m}}{(2m)!}.$$

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