A CONVERSE OF HELLY'S THEOREM ON CONVEX SETS

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1. Introduction. Helly's well known theorem on convex sets states that families of compact convex sets in Euclidean n-space E^n, have the following property:

Property $\mathfrak{H}$: *If every* $n+1$ *of the sets have a point in common, then there exists a point common to all sets of the family.*

If a family of compact sets in E^n has property $\mathfrak{H}$ this, clearly, does not imply that the sets are convex. The purpose of this short note is to show that (loosely speaking) if the family possesses property $\mathfrak{H}$ not accidentally but by virtue of the geometric structure of its sets, then all the sets of the family are convex. The proof of our result is rather simple, but as far as we are aware no theorems converse to Helly's have been noticed before.

In order to state our result briefly we make the following definition:

DEFINITION. A family of sets K_α in E^n is said to have property $\mathfrak{A}\mathfrak{H}$ if every family $\{K'_\alpha\}$, with $K'_\alpha = T_\alpha K_\alpha$ an affine[1] transform of K_α, possesses property $\mathfrak{H}$.

We may now formulate our result.

THEOREM. *Let* $\{K_\alpha\}$ *be a family of more than* $n+1$ *compact sets in* E^n, *all having linear dimension* n *(that is, no* K_α *lies in a hyperplane). If the family has property* $\mathfrak{A}\mathfrak{H}$ *then all sets* K_α *are convex.*

2. Proof. We shall show that if one of the sets of the family, say K_0, is not convex then the family cannot have property $\mathfrak{A}\mathfrak{H}$.

Since K_0 is closed and its linear dimension is n, there exist $n+1$ points $P_1, \cdots, P_{n+1} \in K_0$ forming the vertices of a simplex whose interior contains points not belonging to K_0. Let P_0 be such a point.

[1] By an affine transformation we understand a nonsingular one.

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The family contains more than $n+1$ sets. Let $K_1, K_2, \cdots, K_{n+1}$ be $n+1$ arbitrary sets of the family different[2] from K_0. For $i = 1, \cdots, n+1$ let $\hat{K}_i$ denote the convex hull of K_i. Let $Q_0^{(i)}$ be any extreme point of $\hat{K}_i$ (that is, $Q_0^{(i)} \in \hat{K}_i$ and it is not an interior point of any segment contained in $\hat{K}_i$). Since K_i is compact and not empty there exists such a point and, moreover, $Q_0^{(i)} \in K_i$. Let π_i be a hyperplane containing $Q_0^{(i)}$ and such that all other points of $\hat{K}_i$ are in one of the *open* half-spaces, say π_i^+, determined by it (such a plane π_i exists since $Q_0^{(i)}$ is an extreme point of $\hat{K}_i$). As K_i has linear dimension n there exist points $Q_1^{(i)}, \cdots, Q_n^{(i)} \in K_i$, such that the $n+1$ points $Q_0^{(i)}, Q_1^{(i)}, \cdots, Q_n^{(i)}$ form the vertices of a simplex. Let T_i be an affine transformation sending $Q_0^{(i)}$ into P_0 and $Q_1^{(i)}, \cdots, Q_n^{(i)}$ into the n points P_j with $1 \le j \le n+1$ and $j \ne i$.

Let T_0 denote the identity transformation. Also, if the family contains more then $n+2$ sets, associate with every set K_β, different from the $n+2$ sets already considered, an affine transformation T_β such that $T_\beta K_\beta$ contains the $n+1$ points $P_1, \cdots, P_{n+1}$ (this is possible since the linear dimension of K_β is n).

Put $K'_\alpha = T_\alpha K_\alpha$ (for $K_\alpha = K_i$, $i = 0, 1, \cdots, n+1$ as well as for $K_\beta = K_\alpha$). Now every $n+1$ of the sets K'_α have a point in common. Indeed, the sets $K'_1, \cdots, K'_{n+1}$ have the point P_0 in common, while any other collection of $n+1$ sets K'_α must omit at least one of these sets, say K'_i and then P_i belongs to all the K'_α in the collection. On the other hand we shall prove that there is no point common to all K'_α. This will be done by showing that $\bigcap_{i=1}^{n+1} K'_i = \phi$ (the void set).

This last assertion is established as follows. (i) Since

$$K_i \subset Q_i^{(0)} \cup \pi_i^+$$

we have

$$K'_i \subset P_0 \cup T_i \pi_i^+$$

for $i = 1, \cdots, n+1$, and, therefore,

$$\bigcap_{i=1}^{n+1} K'_i \subset P_0 \cup \left(\bigcap_{i=1}^{n+1} T_i \pi_i^+ \right).$$

[2] Different means labelled differently. The theorem applies also to families in which one set appears several times, for example, to a family consisting of $n+2$ identical sets.

(ii) For $i = 1, \cdots, n+1$ let C_i denote the closed polyhedral cone with vertex at P_0 and edges obtained by prolongation of the n directed segments $\overrightarrow{P_j P_0}$ $(1 \leq j \leq n+1,\ j \neq i)$. Since

$$P_j \in K_i' \cap T_i \pi_i^+$$

for these j, we have $C_i \cap T_i \pi_i^+ = \phi$; also, since P_0 is an interior point of the simplex with vertices $P_1, \cdots, P_{n+1}$, we have $\bigcup_{i=1}^{n+1} C_i = E^n$, the whole space. Therefore

$$\bigcap_{i=1}^{n+1} T_i \pi_i^+ = \bigcup_{i=1}^{n+1} \left(C_i \cap \left(\bigcap_{i=1}^{n+1} T_i \pi_i^+\right)\right) \subset \bigcup_{i=1}^{n+1} (C_i \cap T_i \pi_i^+) = \phi.$$

(iii) Combining this with the result of (i) we have

$$\bigcap_{i=1}^{n+1} K_i' \subset P_0.$$

Thus P_0 is the only common point of $K_1', \cdots, K_{n+1}'$, but $P_0 \notin K_0' = K_0$, hence there is no common point to the $n+2$ sets $K_0', K_1', \cdots, K_{n+1}'$. Q.e.d.

3. Generalizations. We indicate two stronger versions of the theorem of § 1.

3.1. Similarly to the way we defined property $\mathfrak{AH}$, we can define the weaker property $\mathfrak{A}^+\mathfrak{H}$ by restricting the affine transformations T_α in the definition, to those for which the determinant of the non-translational part is positive. Only minor modifications are required in the proof in order to show that the theorem of § 1 remains valid if property $\mathfrak{AH}$ is replaced by property $\mathfrak{A}^+\mathfrak{H}$.

3.2. Let K be a closed set in E^n having linear dimension n, and such that its complement in E^n contains a nondegenerate cone. It can then be shown that there exist points $Q_0, Q_1, \cdots, Q_n \in K$ forming the vertices of a simplex, and having the further property that Q_0 is the only point belonging to K in the closed cone having Q_0 as vertex and whose edges are the prolongations of $\overrightarrow{Q_i Q_0}$, $i = 1, \cdots, n$. Using this fact the proof of § 2 easily yields the theorem of § 1 with the assumption of *compactness* weakened to: *every K_α is closed and its complement contains a nondegenerate cone.*

3.3. For $n \geq 2$ both 3.1 and 3.2 can be carried out simultaneously. That this cannot be done for $n = 1$ is shown by the following example: A family of

3 sets, one consisting of two points and the other two being two equally directed closed half lines.

4. Remarks. It might be interesting to consider the necessity of the various assumptions made in the theorem.

4.1. It is natural to ask whether property $\mathfrak{AH}$ could be weakened in that we would allow not all affine transformations but only some transformations of a special kind. As shown in 3.1 it is possible to do something in this direction; however not much more can be done as is seen from examples that follow.

The theorem would become false if in defining property $\mathfrak{AH}$ we would have restricted the affine transformations by the extra condition that the determinant of the nontranslational part of the transformation be rational. Indeed, let the family contain $n+2$ sets $S_1, \cdots, S_{n+2}$, each S_i consisting of $n+1$ points forming the vertices of a simplex. Let V_i be the volume of the simplex whose set of vertices is S_i and assume the numbers $V_1, V_2, \cdots, V_{n+2}$ to be rationally independent. We claim that $S_i' = T_i S_i$, $(i = 1, \cdots, n+2)$ with T_i being affine transformations with rational determinants, has property $\mathfrak{H}$. In fact, otherwise we would have $\bigcap_{i=1}^{n+2} S_i' = \phi$ while any $n+1$ of the sets S_i' would have a point in common. This would be possible only if $\bigcup_{i=1}^{n+2} S_i'$ consists of exactly $n+2$ points, and the $n+2$ sets S_i' are all the different subsets of $n+1$ points of $\bigcup_{i=1}^{n+2} S_i'$. We may denote the points by $Q_1, Q_2, \cdots, Q_{n+2}$ in such a way that S_i' consists of all these points except Q_i. Let v_i be the volume of S_i then either (i) one v_i is equal to the sum of the $n+1$ numbers v_j with $i \neq j$ (this happens if one of the points $Q_1, \cdots, Q_{n+2}$ is an interior point of the simplex formed by the other points); or (ii) the sum $v_i + v_j$ of two volumes equals the sum of two of the n remaining v_k, (this happens if the previous case does not occur). But $v_i = |d_i| V_i$ $(i = 1, \cdots, n+2)$ where d_i is the determinant (of the nontranslational part) of T_i, and either (i) or (ii) would imply a rational relation between the V_i contrary to our assumption.

An argument of the same kind shows the existence of $n+2$ sets S_i as above having the property obtained from $\mathfrak{AH}$ by restricting the affine transformations by the condition that the determinants be bounded and bounded away from zero. Such an argument also applies if instead of considering all affine transformations we consider, say, the similarities, that is, those obtained by combinations of translations, stretchings and orthogonal transformations; etc.

It should be noticed that in the above counterexamples the families consist of $n+2$ sets; thus they apply even if in the theorem the assumption "the family has property $\mathfrak{AH}$" is strengthened to "every subfamily of more than $n+1$ sets has property $\mathfrak{AH}$". On the other hand it is easily seen that with this new formulation (but not with the original one) the theorem remains valid if we restrict the consideration to affine transformations with determinants bounded by an arbitrary positive number (or, alternatively, with determinants bounded away from zero).

4.2. In 3.2 we remarked that the assumption of compactness could be weakened; it is, however, impossible to dispense with it altogether. To see this let O_i $(i = 1, 2, \cdots, N, N > n + 1)$ be nonvoid, open and convex sets in E^n. Let O_i^* be a set obtained from O_i by deleting a single point P_i from it. The sets O_i^* are not convex, yet we claim that the family consisting of these N sets has property $\mathfrak{AH}$. Indeed, let T_i $(i = 1, \cdots, N)$ be affine transformations. If every $n + 1$ of the sets $T_i O_i^*$ have a point in common, so do a fortiori every $n + 1$ of the sets $T_i O_i$. But the sets $T_i O_i$ $(i = 1, \cdots, N)$ are convex and, it is well known that finite families of arbitrary convex sets have property $\mathfrak{H}$. Therefore $\bigcap_{i=1}^N T_i O_i \neq \phi$, but $\bigcap_{i=1}^N T_i O_i$ is an open set, hence it must contain other points besides $T_i P_i$ $(i = 1, \cdots, N)$. Since

$$\bigcap_{i=1}^{N} T_i O_i \subset \left(\bigcap_{i=1}^{N} T_i O_i^* \right) \cup \left(\bigcup_{i=1}^{N} T_i P_i \right)$$

it follows that $\bigcap_{i=1}^N T_i O_i^* \neq \phi$, that is, our family has property $\mathfrak{AH}$ as claimed.

It is even impossible, unless some precautions are taken, to replace the word "compact" in the theorem by the word "closed". One has merely to think of the family $\{K_1, \cdots, K_{n+2}\}$ where $K_1, \cdots, K_{n+1}$ are arbitrary sets of linear dimension n and $K_{n+2} = E^n$.

4.3. Finally, it is easy to see that the assumption about the linear dimension of the sets K_α is essential. The simplest example proving this is obtained by considering the case when each K_α consists of n (or fewer) points. The sets K'_α consist also of fewer than $n + 1$ points and a trivial argument shows that the family $\{K'_\alpha\}$ has property $\mathfrak{H}$.

It is also impossible to improve the theorem by dropping the assumption about the linear dimension and replacing the conclusion by "each set is either convex or has linear dimension n". A trivial counterexample is obtained by taking one arbitrary set and all other sets consisting of single points. It is possible to construct more ingenious examples showing, for example, that one cannot replace the assumption that the linear dimension is n by the assumption that the sets contain more than n points.

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