A NOTE ON ADDITIVE FUNCTIONS

H. DELANGE AND H. HALBERSTAM

1. A real valued function $f(n)$, defined on the set of natural numbers, is called *additive* if $f(mn)=f(m)+f(n)$ whenever $(m, n)=1$, and *strongly additive* if also $f(p^\alpha)=f(p)$ for p prime and $\alpha=2, 3, \cdots$. We define

$$(1) \qquad A_n=\sum_{p<n} f(p)/p\ , \qquad B_n=\sum_{p<n} f^2(p)/p\ ,$$

and we assume throughout that

$$(2) \qquad B_n \to \infty\ , \qquad n \to \infty\ .$$

Additive functions for which $B_n=O(1)$ have already been discussed thoroughly in Erdös and Wintner [4]. They proved the following theorem:

Define

$$f'(p)=\begin{cases} 1 \text{ for } |f(p)|>1\ , \\ f(p) \text{ for } |f(p)|\leqq 1\ . \end{cases}$$

Then the additive function $f(n)$ *possesses a distribution function if, and only if, the series*

$$\sum_p f'(p)/p \qquad \text{and} \qquad \sum_p \{f'(p)\}^2/p$$

converge.

Moreover, it follows from a general result of P. Lévy [10] that this distribution function is continuous if, and only if, the series $\sum_{f(p)\neq 0} f(p)/p$ diverges. Surveys of this subject are given in Kac [7] and Kubilyus [9]. A comprehensive account is being prepared by H. N. Shapiro.

Our knowledge of functions subject to (2) is not as complete. Outstanding is the result of Erdös and Kac [3] which states that if

$$(3) \qquad f(p)=O(1)\ ,$$

the distribution of

$$\frac{f(m)-A_n}{B_n^{1/2}}\ , \qquad m \leqq n\ ,$$

is asymptotically Gaussian. In a recent note H. N. Shapiro [11] has shown that the theorem of Erdös and Kac remains true even when (3) is replaced by

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$$\lim_{n\to\infty} B_n^{-1} \sum_{\substack{p<n \\ |f(p)|>\varepsilon B_n^{1/2}}} f^2(p)/p = 0 \qquad \text{for every } \varepsilon>0 . \tag{4}$$

Since (4) is essentially the Lindeberg condition which is necessary and sufficient for the central limit theorem to hold, one is led to conjecture that (4) is not only the sufficient but also the necessary condition for the truth of the theorem of Erdös and Kac. However, it seems very difficult to establish the necessity (see Kubilyus [8] and Tanaka [12]).

Associated with such questions about the distributions of additive arithmetic functions is a number of 'moment' problems, which, if solved, lead to results of independent interest. Thus, for example, the following result is suggested by, and includes, the theorem of Erdös and Kac.

THEOREM 1. *Let $f(m)$ be strongly additive and subject to* (2) *and*

$$f(p)=o(B_p^{1/2}) . \tag{5}$$

Then we have for each fixed $k=1, 2, 3, \cdots$

$$\lim_{n\to\infty} \frac{\sum_{m=1}^{n} (f(m)-A_n)^k}{nB_n^{k/2}} = (2\pi)^{-1/2} \int_{-\infty}^{\infty} \omega^k e^{-\omega^2/2}\, d\omega .$$

(For proofs see Delange [1], [2], Halberstam [5], [6].)

The purpose of the present communication is to indicate briefly a proof that Theorem 1 remains true even when (5) is replaced by the weaker pair of conditions (4) and

$$f(p)=O(B_p^{1/2}) . \tag{5a}$$

That (5a) alone does not suffice can be seen readily from the case $f(p) =\log p$, which determines a very different kind of distribution. On the other hand, (4) alone would also be inadequate, as can be seen from the following example.

Let $p_1, p_2, \cdots, p_j, \cdots$ be an increasing sequence of primes with the property that the number of primes which belong to this sequence and do not exceed x is $o(\log\log x)$. Now take

$$f(p)=\begin{cases} (p_j)^{1/2} & \text{if } p=p_j , \\ 1, & \text{if } p \text{ does not belong to the sequence.} \end{cases}$$

Then $B_n \sim (\log\log n)$ and condition (4) is satisfied. However,

$$\sum_{m\leqq p_j} (f(m)-A_{p_j})^4 \geqq (f(p_j)-A_{p_j})^4 \sim p_j^2$$

whereas, if Theorem 1 were true in this case, we should have

$$\sum_{m \leqq p_j} (f(m)-A_{p_j})^4 \sim 3p_j(\log\log p_j)^2 .$$

The most general formulation of Theorem 1 remains an open question. The theorem shows, incidentally, that although the method of moments is in many ways more tractable for determining the distributions of given functions, it is not as wide in scope as the method evolved by Erdös and Kac.

2. **We suppose throughout this section** that (4) and (5a) hold. First of all, we rewrite (4) as

$$\lim_{n\to\infty} \phi(n, \varepsilon)=0 \qquad \text{for every } \varepsilon>0 , \tag{6}$$

where

$$\phi(n, \varepsilon)=B_n^{-1} \sum_{\substack{p<n \\ |f(p)|>\varepsilon B_n^{1/2}}} f^2(p)/p . \tag{7}$$

To simplify subsequent arithmetic we choose $\varepsilon<1/2$ and keep it fixed; then we choose n so large that

$$\phi(n, \varepsilon)<\frac{1}{2}\varepsilon \tag{8}$$

as is possible by (6). We set

$$\alpha_n=n^{1/(3k)} \tag{9}$$

and observe that in view of (9) and the well-known relation

$$\sum_{p<y} p^{-1}=\log\log y+c+o(1) \tag{10}$$

where c is an absolute constant,[1]

$$\sum_{\alpha_n \leqq p<n} p^{-1}=O(1) . \tag{11}$$

We define

$$A_y^*= \sum_{\substack{p<y \\ |f(p)|\leqq\varepsilon B_n^{1/2}}} f(p)/p , \qquad B_y^*= \sum_{\substack{p<y \\ |f(p)|\leqq\varepsilon B_n^{1/2}}} f^2(p)/p \tag{12}$$

and

$$f^*(m)= \sum_{\substack{p<\alpha_n, p|m \\ |f(p)|\leqq\varepsilon B_n^{1/2}}} f(p) . \tag{13}$$

By (7) and (12)

[1] The constants implied by the use of the O-notation depend throughout on at most k.

$$B_n^* = B_n(1-\phi(n,\varepsilon))$$

and this combines with (11) to give

$$B_{\alpha_n}^* = B_n(1+O(\varepsilon^2+\phi(n,\varepsilon)))\ . \tag{14}$$

LEMMA 1. $\quad A_n = A_{\alpha_n}^* + O(B_n^{1/2}\{\varepsilon+\varepsilon^{-1}\phi(n,\varepsilon)\})\ .$

Proof. By (1)

$$A_n = \sum_{\substack{p<\alpha_n \\ |f(p)|\leqq \varepsilon B_n^{1/2}}} f(p)/p + \sum_{\substack{\alpha_n \leqq p<n \\ |f(p)|\leqq \varepsilon B_n^{1/2}}} f(p)/p + \sum_{\substack{p<n \\ |f(p)|>\varepsilon B_n^{1/2}}} f(p)/p\ .$$

The first sum on the right is $A_{\alpha_n}^*$ by (12) with $y=\alpha_n$, the second sum is $O(\varepsilon B_n^{1/2})$ by (11), and the third is less than

$$\varepsilon^{-1}B_n^{-1/2} \sum_{\substack{p<n \\ |f(p)|>\varepsilon B_n^{1/2}}} f^2(p)/p = B_n^{1/2}\varepsilon^{-1}\phi(n,\varepsilon)$$

by (7). Hence the result.

LEMMA 2. *If $r \leqq k$, then*

$$\sum_{m=1}^{n} (f(m)-f^*(m))^{2r} = O(nB_n^r\{\varepsilon+\varepsilon^{-1}\phi(n,\varepsilon)\})\ .$$

Proof. By (13) and the definition of $f(m)$

$$f(m)-f^*(m) = \sum_{\substack{p<n,\, p|m \\ |f(p)|>\varepsilon B_n^{1/2}}} f(p) + \sum_{\substack{\alpha_n \leqq p<n,\, p|m \\ |f(p)|\leqq \varepsilon B_n^{1/2}}} f(p) = \sum_{\substack{p|m \\ p\in \mathscr{E}_n}} f(p)$$

where $\mathscr{E}_n$ is the set of those primes less than n which satisfy either

$$\text{(i)}\quad |f(p)| > \varepsilon B_n^{1/2}$$

or

$$\text{(ii)}\quad |f(p)| \leqq \varepsilon B_n^{1/2},\ p \geqq \alpha_n\ .$$

Then the sum of Lemma 2 is

$$O\Bigg(\sum_{\nu=1}^{2r} \sum_{\substack{r_1+\cdots+r_\nu=2r \\ r_1\geqq\cdots\geqq r_\nu\geqq 1}} \sum_{p_1,\cdots,p_\nu}'' |f^{r_1}(p_1)\cdots f^{r_\nu}(p_\nu)| \sum_{\substack{m=1 \\ (p_1\cdots p_\nu)|m}}^{n} 1\Bigg)$$

$$= O\Bigg(\sum_{\nu=1}^{2r} \{\max_{p\leqq n} |f(p)|^{2r-\nu}\} \sum_{p_1,\cdots,p_\nu}'' \left[\frac{n}{p_1\cdots p_\nu}\right] |f(p_1)\cdots f(p_\nu)|\Bigg)$$

where $\sum''$ indicates that the summation is carried out over all sets of distinct prime numbers $p_1, p_2, \cdots, p_\nu$ with $p_i \in \mathscr{E}$ $(i=1, 2, \cdots, \nu)$, and $[y]$ stands for the integer part of y. Using (5a), (i) and (ii) this expression is

$$O\Big(n\sum_{\nu=1}^{2r} B_n^{r-\frac{1}{2}\nu}\sum_{s=0}^{\nu}\Big\{\sum_{\substack{p<n\\ |f(p)|>\varepsilon B_n^{1/2}}}|f(p)|/p\Big\}^s\Big\{\sum_{\substack{\alpha_n\leqq p<n\\ |f(p)|\leqq\varepsilon B_n^{1/2}}}|f(p)|/p\Big\}^{\nu-s}\Big)\,,$$

which, as in the proof of Lemma 1, becomes

$$O\Big(n\sum_{\nu=1}^{2r} B_n^{r-\frac{1}{2}\nu}\sum_{s=0}^{\nu}\{B_n^{1/2}(\varepsilon^{-1}\phi)\}^s\{B_n^{1/2}\varepsilon\}^{\nu-s}\Big)=O\Big(nB_n^r\sum_{\nu=1}^{2r}\sum_{s=0}^{\nu}(\varepsilon^{-1}\phi)^s\varepsilon^{\nu-s}\Big)$$
$$=O(nB_n^r\{\varepsilon^{-1}\phi+\varepsilon\})\ ;$$

here we have used the restrictions on the magnitudes of ε and ϕ imposed at the beginning of § 2 (see inequality (8)).

Next we set

$$M_k(n)=\sum_{m=1}^{n}(f(m)-A_n)^k\ ,\qquad M_r^*(n)=\sum_{m=1}^{n}(f^*(m)-A_{\alpha_n}^*)^r\ .$$

Then

$$M_k(n)=\sum_{m=1}^{n}\{(A_{\alpha_n}^*-A_n)+(f(m)-f^*(m))+(f^*(m)-A_{\alpha_n}^*)\}^k\ ,$$

so that by Lemmas 1 and 2 and Cauchy's inequality

$$\begin{aligned}
&M_k(n)-M_k^*(n)\\
&\quad=O\Big(\sum_{\substack{r_1+r_2+r_3=k\\ r_3\leqq k-1}}|A_n-A_{\alpha_n}^*|^{r_1}\sum_{m=1}^{n}|f(m)-f^*(m)|^{r_2}|f^*(m)-A_{\alpha_n}^*|^{r_3}\Big)\\
&\quad=O\Big(\sum_{\substack{r_1+r_2+r_3=k\\ r_3\leqq k-1}}B_n^{r_1/2}\{\varepsilon+\varepsilon^{-1}\phi\}^{r_1}\{\sum_{m=1}^{n}(f(m)-f^*(m))^{2r_2}\}^{1/2}\{M_{2r_3}^*(n)\}^{1/2}\Big)\\
&\quad=O\Big(n^{1/2}\sum_{r\leqq k-1}B_n^{(k-r)/2}\{\varepsilon+\varepsilon^{-1}\phi\}^{1/2}\{M_{2r}^*(n)\}^{1/2}\Big)\,.
\end{aligned}$$

But by the methods of Halberstam [5] or Delange [2] it is a straightforward matter to confirm that for n sufficiently large

$$M_l^*(n)=n(B_{\alpha_n}^*)^{l/2}(2\pi)^{-1/2}\int_{-\infty}^{\infty}\omega^l e^{-\omega^2/2}d\omega\{1+O(\varepsilon)\}\ ,\qquad l\leqq 2k\ ,$$

so that by (14) and (8)

$$M_l^*(n)=nB_n^{l/2}(2\pi)^{-1/2}\int_{-\infty}^{\infty}\omega^l e^{-\omega^2/2}d\omega\{1+O(\varepsilon)\}\ ,\qquad l\leqq 2k\ , \tag{15}$$

and, in particular

$$M_{2r}^*(n)=O(nB_n^r)\ ,\qquad r\leqq k\ .$$

Hence

$$M_k(n)-M_k^*(n)=O(nB_n^{k/2}\{\varepsilon+\varepsilon^{-1}\phi\}^{1/2})\ ;$$

now, whilst still keeping ε fixed, we let n tend to infinity, and obtain

$$\overline{\lim_{n\to\infty}}\left|\frac{M_k(n)}{nB_n^{k/2}}-\frac{M_k^*(n)}{nB_n^{k/2}}\right|=O(\varepsilon^{1/2})\,.$$

Thus, by (15) with $l=k$,

$$\overline{\lim_{n\to\infty}}\left|\frac{M_k(n)}{nB_n^{k/2}}-(2\pi)^{-1/2}\int_{-\infty}^{\infty}\omega^k e^{-\omega^2/2}d\omega\right|=O(\varepsilon^{1/2})\ .$$

Since the left side is entirely independent of ε, and yet the relation is true for every $\varepsilon<1/2$, we have now proved that

$$\lim_{n\to\infty}\frac{M_k(n)}{nB_n^{k/2}}=(2\pi)^{-1/2}\int_{-\infty}^{\infty}\omega^k e^{-\omega^2/2}d\omega$$

for every fixed $k=1, 2, 3, \cdots$.

This concludes the proof of Theorem 1 with condition (5) replaced by the pair of conditions (5a) and (4).

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Université de Clermont, France
Brown University, Providence, R. I. and University of California, Berkeley