CESÀRO PARTIAL SUMS OF HARMONIC SERIES EXPANSIONS

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1. Introduction. Let the harmonic function $v(r, \theta)$ have the sine series expansion

$$(1.1) \qquad v(r, \theta) = \sum_1^\infty a_\nu r^\nu \sin \nu\theta ,$$

convergent for $0 \leqq r < 1$ and suppose that $v(r, \theta)$ is non-negative for $0 < \theta < \pi$. Denote the nth partial sum of (1.1) by

$$(1.2) \qquad S_n^{(0)}(r, \theta) = \sum_1^n a_\nu r^\nu \sin \nu\theta ,$$

and the nth Cesàro partial sum of order k, by

$$(1.3) \qquad S_n^{(k)}(r, \theta) = \sum_1^n C_k^{n+k-\nu} a_\nu r^\nu \sin \nu\theta , \qquad k = 1, 2, \cdots .$$

It was shown by Fejér [2, p. 61] and Szász [8] that when $v(r, \theta) \geqq 0$ for $0 < \theta < \pi$, $0 < r < 1$, then $S_n^{(0)}(r, \theta)$ is also non-negative for all n when $0 < \theta < \pi$, $0 < r \leqq 1/4$, and the constant $1/4$ is sharp. Fejér [2] showed that the functions $S_n^{(3)}(1, \theta)$ are also non-negative for all n, $0 < \theta < \pi$. In addition, Szász [8] showed that there exists an $R_n^{(0)}$, depending upon n only, so that $S_n^{(0)}(r, \theta) \geqq 0$ for $0 < r \leqq R_n^{(0)}$, $0 \leqq \theta \leqq \pi$, but not always for $r > R_n^{(0)}$, and that

$$(1.4) \qquad R_n^{(0)} = 1 - 3\frac{\log n}{n} + \frac{\log \log n}{n} + O(1/n) .$$

In this paper we shall extend the results of Szász to Cesàro partial sums of integral order k, $k = 1, 2, 3$. For $k = 3$ the theorem obtained is a sharpened form of the theorem of Fejér [2]. We prove the following:

THEOREM 1. *Let the harmonic series expansion*

$$v(r, \theta) = \sum_1^\infty a_\nu r^\nu \sin \nu\theta$$

be convergent for $0 \leqq r < 1$ *and let* $v(r, \theta) \geqq 0$ *for* $0 < \theta < \pi$, $0 < r < 1$. *Then for* $k = 0, 1, 2, 3$ *there exists a positive number* $R_n^{(k)}$ *depending upon* n *only, so that*

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$$(1.5)\qquad S_n^{(k)}(r,\theta) \geqq 0 \quad \text{for} \quad 0 \leqq r \leqq R_n^{(k)}, \quad 0 \leqq \theta \leqq \pi ,$$

but not always for $r > R_n^{(k)}$, *and that*

$$(1.6)\qquad R_n^{(k)} = 1 - (3-k)\frac{\log n}{n} + \frac{\log\log n}{n} - \frac{g_k + o(1)}{n}, \quad k = 0, 1, 2,$$

where

$$g_k = \log\left[\left\{1 + \frac{k+1}{6}(1 + (-1)^k)\right\}\mu\right],$$

and where

$$\mu = \begin{cases} 1 \text{ for } n \text{ even} \\ \max\limits_{\pi \leqq h \leqq 3\pi/2} |\sin h|h|| = 0.217\cdots \text{ for } n \text{ odd}; \end{cases}$$

and

$$(1.7)\qquad R_{2n}^{(3)} = 1, \quad R_{2n-1}^{(3)} > 1, \qquad n = 1, 2, \cdots,$$

$$(1.8)\qquad \limsup_{n\to\infty} (2n-1)(R_{2n-1}^{(3)} - 1) \leqq \alpha_0 = 1.07\cdots$$

where α_0 *is the positive root of the equation*

$$(1.9)\qquad 3 - \alpha - 3\mu e^{\alpha} = 0 .$$

Moreover, $R_n^{(k)}$ *is the largest* r *for which* $\psi_n^{(k)}(r,\theta)$ *is non-negative for all* θ, *where* $\psi_n^{(k)}(r,\theta)$ *is defined for* $k = 0, 1, 2, 3$ *by the equations* (2.18), (2.19), (2.20), *and* (2.21).

Since $v(r,\theta)$ in (1.1) may be regarded as the imaginary part of the analytic function

$$f(z) = \sum_1^\infty a_\nu z^\nu , \quad z = re^{i\theta} , \qquad r < 1, \ a_\nu \text{ real},$$

the property $v(r,\theta) \geqq 0$ for $0 < \theta < \pi$ may be interpreted by saying that $f(z)$ is typically-real in the unit circle, that is $\Im f(z) > 0$ for $\Im z > 0$, and $\Im f(z) < 0$ for $\Im z < 0$, $|z| < 1$. In this case

$$(1.10)\qquad F(z) = \int_0^z \frac{f(z)}{z} dz$$

is schlicht and convex in the direction of the imaginary axis for $|z| < 1$. For from (1.10) we have

$$(1.11)\qquad \frac{\partial}{\partial\theta} \Re F(re^{i\theta}) = -\Im zF'(z) = -\Im f(z) < 0$$

for $|z| < 1$, $0 < \theta < \pi$.

DEFINITION. Let $\mathscr{F}_r$ and $\mathscr{F}_r^*$ denote the families of functions

$$F(z) = z + b_2z^2 + \cdots + b_nz^n + \cdots \tag{1.12}$$

which are regular, real on the real axis, schlicht and convex in the direction of the imaginary axis in $|z| < r$, and in $|z| \leqq r$ respectively.

With the help of Theorem 1 we then obtain the following.

THEOREM 2. *Let*

$$F(z) = z + b_2z^2 + \cdots + b_nz^n + \cdots \tag{1.13}$$

be a member of the family $\mathscr{F}_1$. *Then for* $n = 1, 2, 3, \cdots$ *the* nth *Cesàro partial sum of order one of* (1.13) *is a member of* $\mathscr{F}_{1/2}^*$. *The radius* $1/2$ *cannot be replaced by a larger number. Also the* nth *Cesàro partial sum of order* k, $k = 0, 1, 2, 3$, *is a member of* $\mathscr{F}_{\rho_k}^*$ *where*

$$\rho_0 = 1 - 3n^{-1}\log n + n^{-1}\log\{(3/4 - \varepsilon)\log n\}\ , \qquad n > n_0(\varepsilon),$$

$$\rho_1 = 1 - 2n^{-1}\log n + n^{-1}\log\{(1 - \varepsilon)\log n\}\ , \qquad n > n_1(\varepsilon),$$

$$\rho_2 = 1 - n^{-1}\log n + n^{-1}\log\{1/2 - \varepsilon)\log n\}\ , \qquad n > n_2(\varepsilon),$$

$$\rho_3 \begin{cases} = 1, & n \text{ even}, \\ > 1, & n \text{ odd}, \end{cases}$$

and where $\varepsilon > 0$ *is arbitrarily small and* $n_k(\varepsilon)$, $k = 0, 1, 2$, *are positive integers depending only upon* ε. *The radii* ρ_k *are sharp to within* $O(1/n)$.

2. Preliminary formulas. Before we proceed to the proof of Theorem 1 we shall mention several formulas which will be needed. The following sums are easily calculated:

$$S(z) = z + 2z^2 + \cdots + nz^n + \cdots = z(1 - z)^{-2}\ , \tag{2.1}$$

$$S_n^{(0)}(z) = z + 2z^2 + \cdots + nz^n = \{z - (n + 1)z^{n+1} + nz^{n+2}\}(1 - z)^{-2}\ , \tag{2.2}$$

$$S_n^{(k)}(z) = S_1^{(k-1)}(z) + S_2^{(k-1)}(z) + \cdots + S_n^{(k-1)}(z)\ , \qquad k = 1, 2, \cdots, \tag{2.3}$$

$$S_n^{(k)}(z) = C_k^{n+k-1}z + 2C_k^{n+k-2}z^2 + \cdots + nC_k^kz^n\ , \tag{2.4}$$

$$S_n^{(1)}(z) = \{nz - (n + 2)z^2 + (n + 2)z^{n+2} - nz^{n+3}\}(1 - z)^{-3}\ , \tag{2.5}$$

$$S_n^{(2)}(z) = \frac{1}{2!}\{n(n + 1)z - 2n(n + 3)z^2 + (n + 2)(n + 3)z^3 - 2(n + 3)z^{n+3} + 2nz^{n+4}\}(1 - z)^{-4}\ , \tag{2.6}$$

$$S_n^{(3)}(z) = \frac{1}{3!}\{n(n + 1)(n + 2)z - 3n(n + 1)(n + 4)z^2$$

$$+ 3n(n+3)(n+4)z^3 - (n+2)(n+3)(n+4)z^4 + 6(n+4)z^{n+4} - 6nz^{n+5}\}(1-z)^{-5}\,,$$

$$(2.8)\quad S_n^{(k)}(z) = \frac{1}{k!}\left\{\sum_{m=1}^{k+1}(-1)^{m-1}\prod_{p=0}^{k-m}(n+p)\prod_{q=k+3-m}^{k+1}(n+q)z^m + (-1)^{k-1}k\,!\,(n+k+1)z^{n+k+1} + (-1)^k k\,!\,nz^{n+k+2}\right\}(1-z)^{-k-2}\,, \qquad k = 0, 1, \cdots,$$

where $\prod_{p=i}^{j}(n+p)$ is defined to be 1 if $i > j$.

Let

$$(2.9)\qquad f(z) = \sum_1^\infty a_\nu z^\nu\,, \qquad a_1 > 0,\ a_\nu \text{ real},$$

be regular and typically-real in $|z| < 1$, which is to say that $v(r, \theta) = \Im f(re^{i\theta})$ is non-negative for $0 \leqq \theta \leqq \pi$, $0 < r < 1$, and $f(z)$ is real on the real axis. As I have shown elsewhere [3] the function $f(z)$ may be represented by the Stieltjes integral

$$(2.10)\qquad f(z) = \frac{a_1}{\pi}\int_0^\pi P(z, \phi)d\alpha(\phi)\,, \qquad |z| < 1,$$

where $\alpha(\phi)$ is a non-decreasing real function of ϕ in the interval $[0, \pi]$, and where $P(z, \phi)$ is the typically-real, schlicht and star-like function

$$(2.11)\qquad P(z, \phi) \equiv z(1 - 2z\cos\phi + z^2)^{-1} = \sum_1^\infty \frac{\sin\nu\phi}{\sin\phi}z^\nu\,.$$

For $\phi = 0$ we have $P(z, 0) = S(z)$ where $S(z)$ is defined as in (2.1). From (2.10) we have immediately that

$$(2.12)\qquad S_n^{(k)}(r, \theta) = \frac{a_1}{\pi}\int_0^\pi \Im P_n^{(k)}(re^{i\theta}, \phi)d\alpha(\phi)$$

where $P_n^{(k)}(re^{i\theta}, \phi)$ is the nth Cesàro partial sum of order k of the power series for $P(z, \phi)$ given in (2.11),

$$(2.13)\qquad \Im P_n^{(k)}(re^{i\theta}, \phi) = \sum_{\nu=1}^{n} C_k^{n+k-\nu} r^\nu \frac{\sin\nu\phi}{\sin\phi}\sin\nu\theta\,.$$

By a lemma of L. Fejér [8], [9], [4], it follows that

$$(2.14)\qquad \Im P_n^{(k)}(re^{i\theta}, \phi) \geqq 0\,, \qquad 0 \leqq \phi \leqq \pi,\ 0 \leqq \theta \leqq \pi,$$

if, and only if,

$$(2.15)\qquad \Im S_n^{(k)}(re^{i\theta}) = \Im P_n^{(k)}(re^{i\theta}, 0) = \sum_{\nu=1}^{n} \nu C_k^{n+k-\nu} r^\nu \sin\nu\theta \geqq 0, \quad 0 \leqq \theta \leqq \pi.$$

Thus, the behavior of the Cesàro partial sums of the Koebe function

$z(1-z)^{-2}$ determines the extremes to which the Cesàro partial sums of the series expansion of an arbitrary typically-real function $f(z)$ exhibit their properties. Therefore, in order to prove Theorem 1 for the imaginary part of an arbitrary typically-real function $f(z)$ we may confine ourselves to proving these results merely for the Koebe function $S(z) = z(1-z)^{-2}$. For this function the partial sums

$$S_n^{(0)}(z) = \{z - (n+1)z^{n+1} + nz^{n+2}\}(1-z)^{-2} \tag{2.16}$$

are known to be schlicht and star-like with respect to the origin in [1]

$$|z| \leqq 1 - 3n^{-1}\log n\,, \quad n > n_0\,,$$

and *à fortiori* typically-real in the same radius.

From formulas (2.2), (2.5), (2.6), (2.7), on letting $z = re^{i\theta}$ we obtain by simple, straightforward but long computations the following additional formulas which we shall need.

$$k!\,|1-z|^{2k+4}\Im S_n^{(k)}(re^{i\theta}) = r\sin\theta\,\psi_n^{(k)}(r,\theta)\,, \quad k = 0, 1, 2, 3, \tag{2.17}$$

where

$$\begin{aligned}\psi_n^{(0)}(r,\theta) = 1 - r^2 - (n+1)r^{n+2}\frac{\sin(n-1)\theta}{\sin\theta} \\ + r^{n+1}(2n+2+nr^2)\frac{\sin n\theta}{\sin\theta} - r^n(n+1+2nr^2)\frac{\sin(n+1)\theta}{\sin\theta} \\ + nr^{n+1}\frac{\sin(n+2)\theta}{\sin\theta}\,,\end{aligned} \tag{2.18}$$

$$\begin{aligned}\psi_n^{(1)}(r,\theta) = \{n + 6r^2 - (n+2)r^4\} - \{(n+2)r - nr^3\}\frac{\sin 2\theta}{\sin\theta} \\ - (n+2)r^{n+4}\frac{\sin(n-1)\theta}{\sin\theta} + \{(3n+6)r^{n+3} + nr^{n+5}\}\frac{\sin n\theta}{\sin\theta} \\ - \{(3n+6)r^{n+2} + 3nr^{n+4}\}\frac{\sin(n+1)\theta}{\sin\theta} \\ + \{(n+2)r^{n+1} + 3nr^{n+3}\}\frac{\sin(n+2)\theta}{\sin\theta} - nr^{n+2}\frac{\sin(n+3)\theta}{\sin\theta}\,,\end{aligned} \tag{2.19}$$

$$\begin{aligned}\psi_n^{(2)}(r,\theta) = \{n(n+1) + (2n^2+18n)r^2 \\ - (2n^2-6n-36)r^4 - (n^2+5n+6)r^6\} \\ - r\{2n^2 + 6n + (16n+24)r^2 - (2n^2+6n)r^4\}\frac{\sin 2\theta}{\sin\theta} \\ + r^2\{n^2+5n+6-(n^2+n)r^2\}\frac{\sin 3\theta}{\sin\theta} \\ - (2n+6)r^{n+6}\frac{\sin(n-1)\theta}{\sin\theta} + (8n+24+2nr^2)r^{n+5}\frac{\sin n\theta}{\sin\theta}\end{aligned} \tag{2.20}$$

$$- (12n + 36 + 8nr^2)r^{n+4}\frac{\sin (n+1)\theta}{\sin\theta}$$

$$+ (8n + 24 + 12nr^2)r^{n+3}\frac{\sin (n+2)\theta}{\sin\theta}$$

$$- (2n + 6 + 8nr^2)r^{n+2}\frac{\sin (n+3)\theta}{\sin\theta} + 2nr^{n+3}\frac{\sin (n+4)\theta}{\sin\theta},$$

$$\begin{aligned}(2.21)\quad \varphi_n^{(3)}(r,\theta) = {} & \{n(n+1)(n+2) + 5n(n+1)(n+8)r^2 + 60n(n+4)r^4 \\ & - 5(n+3)(n^2-16)r^6 - (n+2)(n+3)(n+4)r^8\} \\ & - \{3n(n+1)(n+4)r + 5(n^3 + 15n^2 + 32n)r^3 \\ & - 5(n+4)(n^2 - 7n - 12)r^5 - 3n(n+3)(n+4)r^7\}\frac{\sin 2\theta}{\sin\theta} \\ & + \{3n(n+3)(n+4)r^2 + 30(n+2)^2r^4 \\ & - 3n(n+1)(n+4)r^6\}\frac{\sin 3\theta}{\sin\theta} \\ & - \{(n+2)(n+3)(n+4)r^3 - n(n+1)(n+2)r^5\}\frac{\sin 4\theta}{\sin\theta} \\ & - (6n+24)r^{n+8}\frac{\sin (n-1)\theta}{\sin\theta} \\ & + \{(30n+120)r^{n+7} + 6nr^{n+9}\}\frac{\sin n\theta}{\sin\theta} \\ & - \{(60n+240)r^{n+6} + 30nr^{n+8}\}\frac{\sin (n+1)\theta}{\sin\theta} \\ & + \{(60n+240)r^{n+5} + 60nr^{n+7}\}\frac{\sin (n+2)\theta}{\sin\theta} \\ & - \{(30n+120)r^{n+4} + 60nr^{n+6}\}\frac{\sin (n+3)\theta}{\sin\theta} \\ & + \{(6n+24)r^{n+3} + 30nr^{n+5}\}\frac{\sin (n+4)\theta}{\sin\theta} \\ & - 6nr^{n+4}\frac{\sin (n+5)\theta}{\sin\theta}.\end{aligned}$$

3. Proof of Theorem 1 for $k = 1$. We proceed now to the proof of Theorem 1. For $k = 0$ Theorem 1 follows from the theorem of Szász [8]. For $k = 1$ we have $\Im S_n^{(1)}(re^{i\theta}) \geqq 0$ for $0 < \theta < \pi$ provided $\varphi_n^{(1)}(r,\theta) \geqq 0$ for all θ and $r \leqq R_n^{(1)}$. From (2.19) we must determine the largest r for which

$$(3.1)\quad \varphi_n^{(1)}(r,\theta) = \{n + 6r^2 - (n+2)r^4\} - \{(n+2)r - nr^3\}\frac{\sin 2\theta}{\sin\theta}$$

$$
\begin{aligned}
&-(n+2)r^{n+4}\frac{\sin(n-1)\theta}{\sin\theta}+\{(3n+6)r^{n+3}+nr^{n+5}\}\frac{\sin n\theta}{\sin\theta}\\
&-\{(3n+6)r^{n+2}+3nr^{n+4}\}\frac{\sin(n+1)\theta}{\sin\theta}\\
&+\{(n+2)r^{n+1}+3nr^{n+3}\}\frac{\sin(n+2)\theta}{\sin\theta}-nr^{n+2}\frac{\sin(n+3)\theta}{\sin\theta}
\end{aligned}
$$

is non-negative for all θ. We rewrite $\psi_n^{(1)}(r,\theta)$ in the form

$$
\psi_n^{(1)}(r,\theta)=A+B(1-\cos\theta)-r^n\sum_{j=0}^{4}(-1)^j C_j\frac{\sin(n-1+j)\theta}{\sin\theta} \tag{3.2}
$$

where

$$
\begin{aligned}
A&=n+6r^2-(n+2)r^4-(2n+4)r+2nr^3 \\
&=n(1-r)^3(1+r)-2r(2+r)(1-r)^2, \\
B&=(2n+4)r-2nr^3\,, \\
C_0&=(n+2)r^4\,, \qquad C_1=(3n+6)r^3+nr^5\,, \\
C_2&=(3n+6)r^2+3nr^4\,, \qquad C_3=(n+2)r+3nr^3\,, \qquad C_4=nr^2\,.
\end{aligned} \tag{3.3}
$$

Let

$$
\begin{aligned}
&r=e^{-\varepsilon},\quad \varepsilon=\frac{2\log n}{n}-\frac{\log\log n}{n}+\frac{p}{n}\,,\quad r^n=\frac{\log n}{n^2}e^{-p}\,,\\
&1-r=1-2\frac{\log n}{n^2}+\frac{\log\log n}{n}-\frac{p}{n}+O\left(\left(\frac{\log n}{n}\right)^2\right).
\end{aligned} \tag{3.4}
$$

Then

$$
A\cong 16\frac{(\log n)^3}{n^2}\,,\qquad B\cong 8\log n\,.
$$

For fixed k we have

$$
r^k=e^{-k\varepsilon}=1-k\varepsilon+\frac{k^2}{2}\varepsilon^2+O(\varepsilon^3)\,,
$$

so that

$$
\begin{aligned}
C_0&=(n+2)-(4n+8)\varepsilon+8n\varepsilon^2+O(\varepsilon^3 n)\\
C_1&=(4n+6)-(14n+18)\varepsilon+26n\varepsilon^2+O(\varepsilon^3 n)\\
C_2&=(6n+6)-(18n+12)\varepsilon+30n\varepsilon^2+O(\varepsilon^3 n)\\
C_3&=(4n+2)-(10n+2)\varepsilon+14n\varepsilon^2+O(\varepsilon^3 n)\\
C_4&=n\qquad\quad -2n\varepsilon\quad +2n\varepsilon^2+O(\varepsilon^3 n)\,.
\end{aligned} \tag{3.5}
$$

To obtain an asymptotic estimate for $\psi_n^{(1)}(r,\theta)$ in (3.2) we shall make use of the following lemma.

LEMMA 1. *Let α_j, $j = 0, 1, \cdots, 5$, be constants. Let n be a positive integer and let*

$$S = \sum_{j=0}^{5} (-1)^j \alpha_j \frac{\sin (n-1+j)\theta}{\sin \theta}$$

$$= \frac{\sin (n-1)\theta}{\sin \theta} \sum_{j=0}^{5} (-1)^j \alpha_j \cos j\theta + \cos (n-1)\theta \sum_{j=1}^{5} (-1)^j \alpha_j \frac{\sin j\theta}{\sin \theta} .$$

(a) *If* $\sum_{j=0}^{5} (-1)^j \alpha_j = 0$, *then*

$S = \{n(1-\cos\theta)+1\} \cdot \max_j |\alpha_j| \cdot O(1)$ *as* $n \to \infty$.

(b) *If in addition to* (a), $\sum_{j=1}^{5} (-1)^j j^2 \alpha_j = 0$, *then*

$S = \{n(1-\cos\theta)^2+1\} \cdot \max_j |\alpha_j| \cdot O(1)$ *as* $n \to \infty$.

(c) *If in addition to* (a) *and* (b), $\sum_{j=1}^{5} (-1)^j j a_j = 0$, *then*

$S = \{n(1-\cos\theta)^2+(1-\cos\theta)\} \cdot \max_j |\alpha_j| \cdot O(1)$ *as* $n \to \infty$.

The lemma is easily obtained by considering the limits

$$\lim_{\theta\to 0} \frac{\sum_{j=0}^{5}(-1)^j \alpha_j \cos j\theta}{1-\cos\theta}, \qquad \lim_{\theta\to 0} \frac{\sum_{j=1}^{5}(-1)^j \alpha_j \sin j\theta}{\sin\theta - \frac{1}{2}\sin 2\theta} .$$

From (3.2) and (3.5) we obtain

$$\varphi_n^{(1)}(r, \theta) = A + B(1-\cos\theta) - r^n[D_0 - D_1\varepsilon + D_2\varepsilon^2 - D_3\varepsilon^3] \tag{3.6}$$

where

$$\begin{aligned} D_0 &= (n+2)\frac{\sin(n-1)\theta}{\sin\theta} - (4n+6)\frac{\sin n\theta}{\sin\theta} + (6n+6)\frac{\sin(n+1)\theta}{\sin\theta} \\ &\qquad - (4n+2)\frac{\sin(n+2)\theta}{\sin\theta} + n\frac{\sin(n+3)\theta}{\sin\theta} \\ &= 4n\frac{\sin(n+1)\theta}{\sin\theta}(1-\cos\theta)^2 \\ &\qquad - 4\frac{\sin(n-1)\theta}{\sin\theta}(1+2\cos\theta)(1-\cos\theta)^2 \\ &\qquad - 4\cos(n-1)\theta(1-2\cos\theta)(1-\cos\theta) \\ &= 4n\frac{\sin(n+1)\theta}{\sin\theta}(1-\cos\theta)^2 + (1-\cos\theta)O(n) . \end{aligned} \tag{3.7}$$

$$(3.8)\quad D_1 = (4n+8)\frac{\sin(n-1)\theta}{\sin\theta} - (14n+18)\frac{\sin n\theta}{\sin\theta} + (18n+12)\frac{\sin(n+1)\theta}{\sin\theta} - (10n+2)\frac{\sin(n+2)\theta}{\sin\theta} + 2n\frac{\sin(n+3)\theta}{\sin\theta}$$

$$= \{n(1-\cos\theta)+1\}\cdot n\cdot O(1)\,.$$

$$(3.9)\quad D_2 = 8n\frac{\sin(n-1)\theta}{\sin\theta} - 26n\frac{\sin n\theta}{\sin\theta} + 30n\frac{\sin(n+1)\theta}{\sin\theta} - 14n\frac{\sin(n+2)\theta}{\sin\theta} + 2n\frac{\sin(n+3)\theta}{\sin\theta}$$

$$= \{n(1-\cos\theta)+1\}\cdot n\cdot O(1)\,.$$

$$(3.10)\quad D_3 = \sum_{j=0}^{4} O(n)\frac{\sin(n-1+j)\theta}{\sin\theta} = O(n^2)\,.$$

$$(3.11)\quad D_0 - D_1\varepsilon + D_2\varepsilon^2 - D_3\varepsilon^3 = 4n\frac{\sin(n+1)\theta}{\sin\theta}(1-\cos\theta)^2 + (1-\cos\theta)O(n) - \{n^2(1-\cos\theta)+n\}O(1)\frac{\log n}{n} + \{n^2(1-\cos\theta)+n\}O(1)\left(\frac{\log n}{n}\right)^2 + O(n^2)\left(\frac{\log n}{n}\right)^3$$

$$= 4n\frac{\sin(n+1)\theta}{\sin\theta}(1-\cos\theta)^2 + (1-\cos\theta)O(n\log n) + O(\log n)$$

$$(3.12)\quad \psi_n^{(1)}(r,\theta) = 16\frac{(\log n)^3}{n^2} + 8\log n(1-\cos\theta) - \frac{\log n}{n^2}e^{-p}\left[4n\frac{\sin(n+1)\theta}{\sin\theta}(1-\cos\theta)^2 + (1-\cos\theta)O(n\log n) + O(\log n)\right]$$

$$= 16\frac{(\log n)^3}{n^2}(1+o(1)) + 8\log n(1+o(1))(1-\cos\theta) - 4\frac{\log n}{n}e^{-p}\frac{\sin(n+1)\theta}{\sin\theta}(1-\cos\theta)^2\,.$$

Thus the essential part of $\psi_n^{(1)}(r,\theta)$ is the expression

$$(3.13)\qquad 4(1-\cos\theta)\log n\left[2 - \frac{e^{-p}}{n}\,\frac{\sin(n+1)\theta}{\sin\theta}(1-\cos\theta)\right].$$

When n is even, the minimum of the square bracket in (3.13) is reached for $\theta=\pi$. Thus $1-e^{-p}$ must be non-negative. If p denotes a bounded function of n, $p(n)$, we then have $\lim_{n\to\infty} p(2n) = 0$.

If n is odd, we let $-\mu = -0.217\cdots =$ the absolute minimum of $\sin h/h$, which occurs in $\pi < h < \frac{3\pi}{2}$. If c_0 is a sufficiently large constant it is easily seen that the square bracket in (3.13) is positive for

$$0 < \frac{c_0}{n+1} < \theta < \pi - \frac{c_0}{n+1}$$

and that its minimum occurs in the interval $\pi - \{(c_0)/n+1\} < \theta < \pi$ for large odd values of n. Let $\theta = \pi - \{(h)/n+1\}$. Then for n odd

$$\left[2 - \frac{e^{-p}}{n}\frac{\sin(n+1)\theta}{\sin\theta}(1-\cos\theta)\right] = 2\left(1 + e^{-p}\frac{\sin h}{h}\right) + O(1/n^2)$$
$$= 2(1 - \mu e^{-p}) + O(n^{-2}) .$$

It follows that

$$\lim_{n\to\infty} p(2n+1) = \log\mu = -1.527\cdots .$$

It follows from the discussion above that we have

$$R_n^{(1)} = 1 - 2\frac{\log n}{n} + \frac{\log\log n}{n} + \frac{\beta}{n} + o(1/n) \tag{3.14}$$

where $\beta = 0 = -\log 1$, if n is even, and where

$$\beta = -\log\left\{\max_{\pi\leqq h\leqq\frac{3\pi}{2}}\left|\frac{\sin h}{h}\right|\right\} = -\log\mu = 1.527\cdots$$

when n is odd. This completes the proof of Theorem 1 for the case $k = 1$.

We note that for $0 \leqq r \leqq 1/2$, $\psi_n^{(1)}(r, \theta) \geqq 0$ for all n and all θ. Indeed, when $r = 1/2$, we obtain from (3.1) that $\psi_n^{(1)}(1/2, \theta) \geqq 0$ provided

$$\begin{aligned}(3.15)\quad &(30n+44)\sin\theta - (12n+32)\sin 2\theta - (2n+4)2^{-n}\sin(n-1)\theta\\ &+(13n+24)2^{-n}\sin n\theta - (30n+48)2^{-n}\sin(n+1)\theta\\ &+(28n+32)2^{-n}\sin(n+2)\theta - 8n\cdot 2^{-n}\sin(n+3)\theta\\ &\geqq 0,\ 0<\theta<\pi .\end{aligned}$$

Since $|\sin k\theta/\sin\theta| \leqq k$, $k = 1, 2, \cdots$, (3.15) is satisfied if

$$(6n-20)2^n \geqq 73n^2 + 192n + 108 . \tag{3.16}$$

It is easily verified that (3.17) is true for $n > 7$. For $1 \leqq n \leqq 7$ the author has verified that $\psi_n(1/2, \theta) \geqq 0$. The calculations are simple but somewhat tedious, and will be omitted.

4. Proof of Theorem 1 for $k = 2$. From (2.20) we have

$$\psi_n^{(2)}(r, \theta) = P_n(r, \theta) - r^n \sum_{j=0}^{5} (-1)^j C_j \frac{\sin (n - 1 + j)\theta}{\sin \theta} \tag{4.1}$$

where

$$\begin{aligned} P_n(r, \theta) = {} & \{n(n + 1) + (2n^2 + 18n)r^2 - (2n^2 - 6n - 36)r^4 \\ & \qquad - (n^2 + 5n + 6)r^6\} \\ & - \{(2n^2 + 6n)r + (16n + 24)r^3 - (2n^2 + 6n)r^5\} \frac{\sin 2\theta}{\sin \theta} \\ & + \{(n^2 + 5n + 6)r^2 - (n^2 + n)r^4\} \frac{\sin 3\theta}{\sin \theta} \\ = {} & A + B(1 - \cos \theta) + C(1 - \cos \theta)^2 , \end{aligned} \tag{4.2}$$

$$\begin{aligned} A = {} & n^2 + n - (4n^2 + 12n)r + (5n^2 + 33n + 18)r^2 - (32n + 48)r^3 \\ & - (5n^2 - 3n - 36)r^4 + (4n^2 + 12n)r^5 - (n^2 + 5n + 6)r^6 \\ = {} & -(1 - r)^6(n^2 + 5n + 6) + (1 - r)^5(2n^2 + 18n + 36) \\ & - (1 - r)^4(12n + 54) + 24(1 - r)^3 , \end{aligned} \tag{4.3}$$

$$\begin{aligned} B = {} & (4n^2 + 12n)r - (8n^2 + 40n + 48)r^2 + (32n + 48)r^3 \\ & + (8n^2 + 8n)r^4 - (4n^2 + 12n)r^5 \\ = {} & -r(1 - r)^4(4n^2 + 12n) + r(1 - r)^3(8n^2 + 40n) \\ & - r(1 - r)^2(16n - 48) - 48r(1 - r) . \end{aligned} \tag{4.4}$$

$$\begin{aligned} C &= (4n^2 + 20n + 24)r^2 - (4n^2 + 4n)r^4 \\ C_0 &= (2n + 6)r^6 \\ C_1 &= (8n + 24)r^5 + 2nr^7 \\ C_2 &= (12n + 36)r^4 + 8nr^6 \end{aligned} \tag{4.5}$$

$$\begin{aligned} C_3 &= (8n + 24)r^3 + 12nr^5 \\ C_4 &= (2n + 6)r^2 + 8nr^4 \\ C^5 &= \qquad\qquad 2nr^3 . \end{aligned} \tag{4.6}$$

Letting

$$r = e^{-\varepsilon} , \quad \varepsilon = \frac{\log n}{n} - \frac{\log \log n}{n} + \frac{q}{n} ,$$

$$r^n = \frac{\log n}{n} e^{-q} , \quad 1 - r = 1 - \frac{\log n}{n} + \frac{\log \log n - q}{n} + O\left(\left(\frac{\log n}{n}\right)^2\right) ,$$

we obtain

$$A \cong 2\frac{(\log n)^5}{n^3} , \qquad B \cong 8\frac{(\log n)^3}{n} , \qquad C \cong 8n \log n , \tag{4.7}$$

$$(4.8)\quad \begin{aligned} C_0 &= (2n+6) - (12n+36)\varepsilon + (36n+108)\varepsilon^2 \\ &\qquad - (72n+216)\varepsilon^3 + 108n\varepsilon^4 + O(n\varepsilon^5) \\ C_1 &= (10n+24) - (54n+120)\varepsilon + (149n+300)\varepsilon^2 \\ &\qquad - (281n+500)\varepsilon^3 + \frac{4901}{12}n\varepsilon^4 + O(n\varepsilon^5) \\ C_2 &= (20n+36) - (96n+144)\varepsilon + (240n+288)\varepsilon^2 \\ &\qquad - (416n+384)\varepsilon^3 + 560n\varepsilon^4 + O(n\varepsilon^5) \\ C_3 &= (20n+24) - (84n+72)\varepsilon + (186n+108)\varepsilon^2 \\ &\qquad - (286n+108)\varepsilon^3 + \frac{679}{2}n\varepsilon^4 + O(n\varepsilon^5) \\ C_4 &= (10n+6) - (36n+12)\varepsilon + (68n+12)\varepsilon^2 \\ &\qquad - (88n+8)\varepsilon^3 + \frac{260}{3}n\varepsilon^4 + O(n\varepsilon^5) \\ C_5 &= 2n - 6n\cdot\varepsilon + 9n\cdot\varepsilon^2 - 9n\cdot\varepsilon^3 + \frac{27}{4}n\varepsilon^4 + O(n\varepsilon^5)\ . \end{aligned}$$

We now write

$$(4.9)\quad \phi_n^{(2)}(r,\theta) = A + B(1-\cos\theta) + C(1-\cos\theta)^2 - r^n\cdot\sum_{j=0}^{5}(-1)^j D_j\varepsilon^j\ .$$

From (4.1), (4.2), and (4.8), we find

$$\begin{aligned} D_0\cdot\sin\theta &= (2n+6)\sin(n-1)\theta - (10n+24)\sin n\theta \\ &\qquad - (20n+36)\sin(n+1)\theta + (20n+24)\sin(n+2)\theta \\ &\qquad\qquad - (10n+6)\sin(n+3)\theta + 2n\sin(n+4)\theta\ , \end{aligned}$$

$$D_0 = \left[-8n\cdot\frac{\cos(2n+3)\dfrac{\theta}{2}}{\cos\dfrac{\theta}{2}} + 24\frac{\sin(n+1)\theta}{\sin\theta}\right](1-\cos\theta)^2.$$

$$\begin{aligned} D_1\cdot\sin\theta &= (12n+36)\sin(n-1)\theta - (54n+120)\sin n\theta \\ &\qquad + (96n+144)\sin(n+1)\theta - (84n+72)\sin(n+2)\theta \\ &\qquad\qquad + (36n+12)\sin(n+3)\theta - 6n\sin(n+4)\theta\ . \end{aligned}$$

By Lemma 1, we obtain

$$\begin{aligned} D_1 &= [n(1-\cos\theta)^2 + (1-\cos\theta]\cdot[O(n)+O(1)]\cdot O(1) \\ &= [n^2(1-\cos\theta)^2 + n(1-\cos\theta)]\cdot O(1)\ . \end{aligned}$$

$$\begin{aligned} D_2\cdot\sin\theta &= (36n+108)\sin(n-1)\theta - (149n+300)\sin n\theta \\ &\qquad + (240n+288)\sin(n+1)\theta - (186n+108)\sin(n+2)\theta \\ &\qquad\qquad + (68n+12)\sin(n+3)\theta - 9n\sin(n+4)\theta\ , \end{aligned}$$

$$D_2 = [n^2(1-\cos\theta)^2 + n(1-\cos\theta)]\cdot O(1) + [n(1-\cos\theta) + (1-\cos\theta)]\cdot O(1)\,.$$

$$D_3\cdot\sin\theta = (72n+216)\sin(n-1)\theta - (281n+500)\sin n\theta + (416n+384)\sin(n+1)\theta - (286n+108)\sin(n+2)\theta + (88n+8)\sin(n+3)\theta - 9n\sin(n+4)\theta\,,$$

$$D_3 = \{n(1-\cos\theta)+1\}\cdot n\cdot O(1) = [n^2(1-\cos\theta)+n]O(1)\,.$$

$$D_4\cdot\sin\theta = 108n\sin(n-1)\theta - \frac{4901}{12}n\sin n\theta + 560n\sin(n+1)\theta - \frac{679}{2}\sin(n+2)\theta + \frac{260}{3}n\sin(n+3)\theta - \frac{27}{4}n\sin(n+4)\theta\,,$$

$$D_4 = \{n(1-\cos\theta)+1\}\cdot n\cdot O(1) = [n^2(1-\cos\theta)+n]\cdot O(1)\,.$$

$$D_5 = O(n)\cdot\sum_{j=0}^{5}\left|\frac{\sin(n-1+j)\theta}{\sin\theta}\right| = O(n^2)\,.$$

$$\text{(4.10)}\qquad D_0 - D_1\varepsilon + D_2\varepsilon^2 - D_3\varepsilon^3 + D_4\varepsilon^4 - D_5\varepsilon^5$$
$$= \left[-8n\frac{\cos(2n+3)\theta/2}{\cos\theta/2} + 24\frac{\sin(n+1)\theta}{\sin\theta}\right](1-\cos\theta)^2$$
$$+ [n^2(1-\cos\theta)^2 + n(1-\cos\theta)]O(1)\frac{\log n}{n}$$
$$+ [n^2(1-\cos\theta)^2 + n(1-\cos\theta)]O(1)\left(\frac{\log n}{n}\right)^2$$
$$+ [n^2(1-\cos\theta)+n]O(1)\left(\frac{\log n}{n}\right)^3$$
$$+ [n^2(1-\cos\theta)+n]O(1)\left(\frac{\log n}{n}\right)^4 + O(n^2)\left(\frac{\log n}{n}\right)^5$$
$$= \left[-8n\frac{\cos(2n+3)\theta/2}{\cos\theta/2} + 24\frac{\sin(n+1)\theta}{\sin\theta} + O(n\log n)\right](1-\cos\theta)^2$$
$$+ (1-\cos\theta)\cdot O(\log n) + O\left(\frac{(\log n)^3}{n^2}\right)\,.$$

From (4.7), (4.9), and (4.10), we have

$$\text{(4.11)}\qquad \psi_n^{(2)}(r,\theta) = \frac{2(\log n)^5}{n^3} + \frac{8(\log n)^3}{n}(1-\cos\theta) + 8n\log n(1-\cos\theta)^2$$
$$- \frac{\log n}{n}e^{-q}\left[-8n\frac{\cos(2n+3)\theta/2}{\cos\theta/2} + 24\frac{\sin(n+1)\theta}{\sin\theta} + O(n\log n)\right](1-\cos\theta)^2$$
$$- \frac{\log n}{n}e^{-q}\left[(1-\cos\theta)O(\log n) + O\left(\frac{(\log n)^3}{n^2}\right)\right]$$

$$= \left[2\frac{(\log n)^5}{n^3} - e^{-q}\cdot O\left(\frac{(\log n)^4}{n^3}\right)\right]$$
$$+ \left[8\frac{(\log n)^3}{n} - e^{-q}\cdot O\left(\frac{(\log n)^2}{n}\right)\right](1-\cos\theta)$$
$$+ 8\log n\left[n + e^{-q}\frac{\cos(2n+3)\theta/2}{\cos\theta/2} - \frac{24e^{-q}}{n}\frac{\sin(n+1)\theta}{\sin\theta}\right.$$
$$\left. + e^{-q}\cdot O(\log n)\right](1-\cos\theta)^2 .$$

From (4.11) it is seen that we must have the quantity $L \geqq 0$ where

$$L = 1 + e^{-q}\left[\frac{\cos(2n+3)\theta/2}{n\cos(\theta/2)} - \frac{24}{n^2}\frac{\sin(n+1)\theta}{\sin\theta}\right]. \tag{4.12}$$

For n even the minimum of L is attained for $\theta = \pi$ and equals

$$1 - e^{-q}\left(\frac{2n+3}{n} + \frac{24(n+1)}{n^2}\right) = 1 - 2e^{-q} + e^{-q}\cdot o\left(\frac{1}{n}\right).$$

Thus if $q = q(n)$, a bounded function of n, we require

$$\lim_{n\to\infty} q(2n) = \log 2 . \tag{4.13}$$

If n is odd, we let $\theta = \pi - (2h)/2n + 3$ and find that

$$L \cong 1 + 2e^{-q}\frac{\sin h}{h} \qquad \text{as } n \to \infty ,$$

and

$$\min_\theta L \cong 1 - 2\mu e^{-q}$$

where

$$\mu = 0.217\cdots = \max_{\pi\leqq h\leqq \frac{3\pi}{2}}\left|\frac{\sin h}{h}\right|, \quad \text{and} \quad \lim_{n\to\infty} q(2n-1) = \log(2\mu) .$$

It follows that we have

$$R_n^{(2)} = 1 - \frac{\log n}{n} + \frac{\log\log n}{n} - \frac{\gamma}{n} + o\left(\frac{1}{n}\right) \tag{4.14}$$

where

$$\gamma = \begin{cases} \log 2, & \text{if } n \text{ is even,} \\ \log(2\mu), & \text{if } n \text{ is odd.} \end{cases}$$

This completes the proof of Theorem 1 for the case $k = 2$.

In the case $k = 0$, which was investigated by Szász [8], if we employ procedures analogous to those above for $k = 1$ and 2, we are led to the expression

$$3\frac{\log n}{n} - \frac{\log n}{n^3}e^{-t} \cdot 2n(1 - \cos\theta)\frac{\cos(2n+1)\theta/2}{\cos\theta/2} \tag{4.15}$$

when

$$r = 1 - 3\frac{\log n}{n} + \frac{\log\log n}{n} - \frac{t}{n}\,.$$

With arguments similar to those used above, we find that the "correct" value of t is $\log(4/3)$ when n is even (as Szász obtained [8]), and $\log(4\mu/3)$ when n is odd, the latter result being new.

5. Proof of Theorem 1 for $k = 3$. The theorem of Fejér [2], quoted in the introduction, states that

$$R_n^{(3)} \geqq 1\,, \qquad n = 1, 2, 3, \cdots. \tag{5.1}$$

We shall give a new and simple proof of (5.1), and also give a demonstration of the sharpened result

$$R_{2n}^{(3)} = 1\,, \quad R_{2n-1}^{(3)} > 1\,, \qquad n = 1, 2, 3, \cdots, \tag{5.2}$$

and

$$\limsup_{n\to\infty}(2n-1)(R_{2n-1}^{(3)} - 1) \leqq \alpha_0 = 1.07\cdots \tag{5.3}$$

where α_0 is the positive root of the equation

$$3 - \alpha - 3\mu e^{\alpha} = 0 \tag{5.4}$$

where

$$\mu = \max_{\pi \leqq h \leqq \frac{3\pi}{2}}\left|\frac{\sin h}{h}\right| = 0.217\cdots.$$

From (2.7) we have

$$\begin{aligned}6(1-z)^5S_{n-2}^{(3)}(z) = {} & n(n-1)(n-2)z - 3(n-1)(n^2-4)z^2 \\ & + 3(n+1)(n^2-4)z^3 - n(n+1)(n+2)z^4 \\ & + 6(n+2)z^{n+2} - 6(n-2)z^{n+3}\,.\end{aligned} \tag{5.5}$$

Letting $z = e^{i\theta}$ in (5.5) we have for $n > 2$

$$\begin{aligned}\Im S_{n-2}^{(3)}(z) = {} & \frac{1}{32\sin^5(\theta/2)}\Big[(n^2-4)\cos\frac{\theta}{2} - n^2\cos\frac{3\theta}{2} \\ & + (n+2)\cos(2n-1)\frac{\theta}{2} - (n-2)\cos(2n+1)\frac{\theta}{2}\Big] \\ = {} & \frac{\sin\theta}{16\sin^4(\theta/2)}\left[n^2 + n\frac{\sin n\theta}{\sin\theta} - 2\left(\frac{\sin n\theta/2}{\sin\theta/2}\right)^2\right].\end{aligned} \tag{5.6}$$

In earlier papers we have shown [5], [6], that

$$n^2 + n\frac{\sin n\theta}{\sin\theta} - 2\left(\frac{\sin n\theta/2}{\sin\theta/2}\right)^2 \geqq 0\,, \tag{5.7}$$

for all integers n and all θ. From (5.6) and (5.7) we have at once that

$$\Im S_n^{(3)}(e^{i\theta}) \geqq 0\,,\quad 0 \leqq \theta \leqq \pi\,,\qquad n = 1, 2, \cdots\,. \tag{5.8}$$

However, the function

$$F(z) = (z^{-1} - z)S_n^{(3)}(z) \tag{5.9}$$

is analytic in $|z| \leqq 1$, and $\Re F(e^{i\theta}) = 2\sin\theta \Im S_n^{(3)}(e^{i\theta}) \geqq 0$. Since the minimum of the harmonic function $\Re F(z)$ in $|z| \leqq 1$ occurs on $|z| = 1$ we have $\Re F(z) > 0$ for $|z| < 1$. From the representation (5.9) it follows from the work of Rogosinski [7] that $S_n^{(3)}(z)$ is typically-real in the unit circle, which is to say that

$$\Im S_n^{(3)}(re^{i\theta}) \geqq 0\,,\qquad 0 \leqq \theta \leqq \pi,\ 0 \leqq r \leqq 1. \tag{5.10}$$

The theorem of Fejér, or inequality (5.1) follows from (5.10) and the remarks made in section two.

We now attack the problem from an alternative point of view for the case $k = 3$. From (2.17) and (5.6) we write

$$\begin{aligned}\varphi_n^{(3)}(r, \theta) = 384\sin^6\frac{\theta}{2}\Big[(n+2)^2 + (n+2)\frac{\sin(n+2)\theta}{\sin\theta} \\ - 2\left(\frac{\sin(n+2)\theta/2}{\sin\theta/2}\right)^2\Big] \\ + [\varphi_n^{(3)}(r, \theta) - \varphi_n^{(3)}(1, \theta)]\,.\end{aligned} \tag{5.11}$$

Let $r = 1 + \alpha/n$ where $\alpha = \alpha(n) = O(1) > 0$. Then $r^k - 1 = k\alpha/n + O(\alpha^2/n^2)$, $k =$ positive integer independent of n, $r^{n+k} - 1 = e^\alpha - 1 + n^{-1}k\alpha e^\alpha + O(n^{-1}\alpha^2 e^\alpha)$. From (2.21) and (5.11) we then obtain for $r = 1 + \alpha/n$ asymptotically,

$$\begin{aligned}\varphi_n^{(3)}(r, \theta) - \varphi_n^{(3)}(1, \theta) &\cong 28n^2\alpha + 56n^2\alpha\cos\theta - 12n^2\alpha(4\cos^2\theta - 1) \\ &\quad - 2n^2\alpha(4\cos\theta - 8\cos^3\theta) - \frac{(e^\alpha - 1)S}{\sin\theta} \\ &= -128n^2\alpha\sin^6\frac{\theta}{2} - \frac{(e^\alpha - 1)S}{\sin\theta}\,,\end{aligned} \tag{5.12}$$

where

$$\begin{aligned}S &= 6n\sin(n-1)\theta - 36n\sin n\theta + 90n\sin(n+1)\theta - 120n\sin(n+2)\theta \\ &\quad + 90n\sin(n+3)\theta - 36n\sin(n+4)\theta + 6n\sin(n+5)\theta \\ &= -384n\sin(n+2)\theta\sin^6\frac{\theta}{2}\,,\end{aligned}$$

$$(5.13)\qquad \psi_n^{(3)}(r, \theta) - \psi_n^{(3)}(1, \theta) \cong 128 \sin^6\frac{\theta}{2}\left[3n(e^\alpha - 1)\frac{\sin(n+2)\theta}{\sin\theta} - \alpha n^2\right].$$

Since for sufficiently large values of n we can only have

$$\psi_n^{(3)}(r, \theta) \geqq 0 \quad \text{or} \quad r = 1 + n^{-1}\alpha,\ \alpha = \alpha(n) > 0\ ,$$

provided

$$(5.14)\qquad 3\left[(n+2)^2 + (n+2)\frac{\sin(n+2)\theta}{\sin\theta} - 2\left(\frac{\sin(n+2)\theta/2}{\sin\theta/2}\right)^2\right] + 3(e^\alpha - 1)n\frac{\sin(n+2)\theta}{\sin\theta} - n^2\alpha \geqq 0\ ,$$

we see that, when n is even and $\theta = \pi$, we must have

$$(5.15)\qquad -3(e^\alpha - 1)n(n+2) - n^2\alpha \geqq 0\ .$$

(5.15) implies that α is non-positive, contrary to our assumption that $\alpha > 0$. Hence $\alpha = 0$ for n even and sufficiently large. However, it is easily seen that $\alpha = 0$ for all even n by the following argument. Since

$$(5.16)\qquad S_n^{(3)}(z) = \sum_{\nu=1}^{n} \nu C_3^{n+3-\nu} z^\nu\ ,$$

and because of the identity

$$(5.17)\qquad \sum_{\nu=1}^{n} (-1)^\nu \nu^2 (n+1-\nu)(n+2-\nu)(n+3-\nu) = 0\ , \quad n \text{ even},$$

it follows that the derivative of $S_n^{(3)}(z)$ vanishes at $z = -1$ for n even. $S_n^{(3)}(z)$, typically-real in $|z| \leqq 1$, therefore cannot be typically-real in $|z| \leqq r$ for $r > 1$, n even. Thus $\alpha = 0$ for all even n, and $R_{2n}^{(3)} = 1$, $n = 1, 2, \cdots$.

The situation for n odd is not so simple. Fejér has pointed out [2] that $\Im S_n^{(3)}(e^{i\theta}) > 0$, $0 < \theta < \pi$, from which it follows that $\psi_n^{(3)}(1, \theta) > 0$ for all θ with the possible exception of the values $\theta = 0$ and π. When n is odd, however, it is easily seen from (5.6) and (2.17) that $\psi_n^{(3)}(1, \pi) > 0$. From (5.16) it also follows that

$$\lim_{\theta \to 0} \frac{\Im S_n^{(3)}(e^{i\theta})}{\sin\theta} = \sum_{\nu=1}^{n} \nu^2 C_3^{(n+3-\nu)} > 0\ .$$

Consequently $\Im S_n^{(3)}(e^{i\theta}) \cdot \operatorname{cosec}\theta > 0$ for all θ when n is odd, and so $R_{2n-1}^{(3)} > 1$, $n = 1, 2, \cdots$. To obtain an asymptotic upper bound for $R_{2n-1}^{(3)}$ we shall show that (5.14) is not verified, when n is sufficiently large, for all θ when α exceeds $\alpha_0 = 1.07\cdots$, α_0 being the positive root of the equation (5.4).

Letting $\theta = \pi - [h/(n+2)]$, n odd, we find that the left hand side of inequality (5.14) is asymptotically equal to the expression

$$3n^2\left[1+\frac{\sin h}{h}-O(n^{-2})\right]+3n^2(e^\alpha-1)\left(\frac{\sin h}{h}+O(n^{-1})\right)-n^2\alpha\ ,$$

from which (5.4) and (5.3) follow. It should be noticed that the constant α_0 in (5.3) could be replaced by a smaller one. Indeed, for $\theta=x/n$, the left-hand side of (5.14) is asymptotically equal to

$$n^2\left[3-\alpha+3e^\alpha\frac{\sin x}{x}-6\left(\frac{\sin x/2}{x/2}\right)^2\right]. \tag{5.18}$$

Calculation of the smallest positive α for which the expression (5.18) is non-positive for some $x\geqq\pi$ would lead to a smaller constant to replace α_0.

From (2.3) and (5.1) it follows at once that $R_n^{(k)}\geqq 1$ for $k\geqq 3$ and all positive integers n.

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