CORRECTON TO "EQUIVALENCE AND PERPENDICULARITY OF GAUSSIAN PROCESSES"

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It has been kindly pointed out to me by D. Lowdenslager that, as it stands, the argument in [1] only works when $\boldsymbol{L}_2(\mu)$ and $\boldsymbol{L}_2(\nu)$ are separable. In particular, the theorem of von Neumann from [2], which is used there, only holds in separable Hilbert spaces. Our theorem nevertheless holds in the non-separable case; an argument will be supplied here enabling one to go from the separable to the general case. We retain notation and terminology of [1].

For any *countable* subset $\boldsymbol{C}$ of $\boldsymbol{L}$, let $\mathscr{S}_C$ be the σ-subalgebra of $\mathscr{S}$ generated by $\boldsymbol{C}$, $\boldsymbol{L}_C$ the linear subspace of $\boldsymbol{L}$ spanned by $\boldsymbol{C}$, and μ_C, ν_C the restrictions of μ, ν to $\mathscr{S}_C$. $\bigcup_C \mathscr{S}_C$ is a σ-algebra contained in $\mathscr{S}$, and, since each $x \in \boldsymbol{L}$ is in some $\boldsymbol{L}_C$, each x in $\boldsymbol{L}$ is measurable with respect to $\bigcup_C \mathscr{S}_C$. Therefore $\mathscr{S} = \bigcup_C \mathscr{S}_C$. Now, suppose, under the assumptions of the theorem of [1], that μ and ν are not equivalent. Then there is some set in $\mathscr{S}$ with μ-measure 0 and ν-measure > 0 (or vice versa). This set is in some $\mathscr{S}_C$. So μ_C and ν_C are not equivalent. By the separable case of the theorem, they are mutually perpendicular, i.e., there is some set in $\mathscr{S}_C$ with μ-measure 0 and ν-measure 1. Thus μ and ν are mutually perpendicular.

Next we show that $\mu \sim \nu$ implies that the correspondence $x^\nu \xrightarrow{T} x^\mu$ between equivalence classes of functions has the property that T extends to an equivalence operator between the linear subspaces $\bar{\boldsymbol{L}}_\mu$ and $\bar{\boldsymbol{L}}_\nu$ of $\boldsymbol{L}_2(\mu)$, $\boldsymbol{L}_2(\nu)$ generated by $\boldsymbol{L}$. Assume, then, that $\mu \sim \nu$. By using the separable case, we easily see that T and T^{-1} are bounded. An argument on p. 704 of [1] still works to show that the extension of T to an operator from $\bar{\boldsymbol{L}}_\mu$ onto $\bar{\boldsymbol{L}}_\nu$ still has the property that, given ξ in $\bar{\boldsymbol{L}}_\mu$, there is an $\mathscr{S}$-measurable x such that $x^\mu = \xi$ and $x^\nu = T\xi$. Write $T^* T$ as $\int \lambda \, d\, F(\lambda)$. Let $E_n = F\left(1 + \frac{1}{n}\right) - F\left(1 - \frac{1}{n}\right)$, $n = 2, 3, 4, \cdots$ Let $E = \bigcap_n E_n$. I now assert $(I - E)\, \bar{\boldsymbol{L}}_\mu$ is separable. For otherwise $(I - E_n)\, \bar{\boldsymbol{L}}_\mu$ would be inseparable for some n, and one could therefore find a countable orthonormal infinite set $\xi_1, \xi_2, \cdots$ of elements of $\bar{\boldsymbol{L}}_\mu$ for which $\|(T^* T - I)\, \xi_i\| \geqq \frac{1}{n} \|\xi_i\|$, all i. Let $\boldsymbol{H}$ be the Hilbert space spanned by the ξ_i. Let $\tilde{\boldsymbol{L}}$ be the set of μ-measurable functions x on S such that $x^\mu \in \boldsymbol{H}$. Let $\tilde{\mathscr{S}}$ be the σ-algebra spanned by them. Let $\tilde{\mu}$, $\tilde{\nu}$ be the completions of μ and ν, restricted to $\tilde{\mathscr{S}}$. Then the Hilbert spaces $\bar{\tilde{\boldsymbol{L}}}_{\tilde{\mu}}$, $\bar{\tilde{\boldsymbol{L}}}_{\tilde{\nu}}$ are isometric to $\boldsymbol{H}$ and $T(\boldsymbol{H})$, respectively, in a natural way. Therefore they are separable, and, since $\tilde{\mu} \sim \tilde{\nu}$, the operator $\tilde{T}$ induced by the correspondence $\tilde{x}^{\mu} \longrightarrow \tilde{x}^{\nu}$ is an equivalence operator. But T is unitarily equivalent to $T|\boldsymbol{H}$, and $T|\boldsymbol{H}$ was constructed so as *not* to be an equivalence operator, giving a contradiction.

To show T is an equivalence operator, it suffices to show this for $T|(I - E)\ \bar{\boldsymbol{L}}_{\mu}$. Since $(I - E)\ \bar{\boldsymbol{L}}_{\mu}$ is separable, we can reduce to the separable case exactly as in the last five sentences of the previous paragraph, with $(I - E)\ \bar{\boldsymbol{L}}_{\mu}$ playing the role played there by $\boldsymbol{H}$ to show that T is an equivalence operator.

Finally, suppose that, for $x \in \boldsymbol{L}$, $x^{\mu} = 0 \Longleftrightarrow x^{\nu} = 0$, and that the one-to-one operator T from $\boldsymbol{L}_{\mu}$ to $\boldsymbol{L}_{\nu}$ induced thereby extends to an equivalence operator from $\bar{\boldsymbol{L}}_{\mu}$ to $\bar{\boldsymbol{L}}_{\nu}$. It must be shown that $\mu \sim \nu$. If μ is not equivalent to ν, then as shown in the first paragraph (and using the notation established there) there is some countable subset $\boldsymbol{C}$ of $\boldsymbol{L}$ such that μ_C and ν_C are not equivalent. But the operator T_C induced by sending x^{μ} to x^{ν} for $x \in \boldsymbol{L}_C$ is precisely the restriction of T to those elements in $\boldsymbol{L}_{\mu}$ which come from $\boldsymbol{L}_C$. Now, the restriction of T to a subspace is again an equivalence operator, so T_C extends to an equivalence operator from $(\overline{\boldsymbol{L}_C})_{\mu}$ to $(\overline{\boldsymbol{L}_C})_{\nu}$, which contradicts the separable case of the theorem.

Also, in reviewing [1], E. Nelson noticed that Lemma 1 is misstated. It should read "positive" instead of "self-adjoint," and, in (b), "$A^2 - I$" rather than "$(A - I)^2$."

Bibliography

1. J. Feldman, *Equivalence and perpendicularity of Gaussian processes,* Pacific J. Math., Vol. 8 No. 4, 1958.

2. J. von Neumann, *Charakterisierung des spektrums eines integral-operatoren,* Actualites Sci. Ind. 229, Paris, 1935.