A NOTE ON KATO'S UNIQUENESS CRITERION FOR SCHRÖDINGER OPERATOR SELF-ADJOINT EXTENSIONS

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1. Introduction. Kato [2] has shown local square integrability with boundedness at ∞ of the potential coefficient function to be a sufficient condition for the Schrödinger operator in $L_2(R_n)$ to have a unique self-adjoint extension in case dimension $n = 3$. His statement is for $n = 3p$, thus with p factors R_3, but with the condition on V stated separately for each R_3 factor as is natural for application to quantum mechanics; this in essence amounts to $n = 3$ from our standpoint. Using the Young-Titchmarsh theorem on Fourier transforms, we generalize Kato's argument to general dimension $n \geq 1$. We show the connection of the resulting criterion with our earlier construction [1] of a self-adjoint extension as the inverse of a modified Green function integral operator. We also give a variational characterization of the spectrum here.

2. Uniqueness condition. Let $V(\boldsymbol{x})$ be a given, real-valued, measurable function over $\boldsymbol{x} \in R_n$, euclidean n-space. We consider the following additional conditions upon V, using the notation $(\boldsymbol{x} \cdot \boldsymbol{y}) = \sum_{j=1}^{n} x_j y_j$ and $|\boldsymbol{x}| = \sqrt{(\boldsymbol{x} \cdot \boldsymbol{x})}$ for $\boldsymbol{x}$ and $\boldsymbol{y} \in R_n$, and also denoting n dimensional Lebesque measure on R_n by μ_n.

CONDITION I. *For some $b < +\infty$ let $V(\boldsymbol{x})$ be essentially bounded* ($A = [\text{ess sup} \mid V(\boldsymbol{x}) \mid] < +\infty$) *over* $\{\boldsymbol{x} \in R_n \mid |\boldsymbol{x}| \geq b\}$, *and let*

$$(1) \qquad \int_{\{x \mid |\boldsymbol{x}| \leq b\}} |V(\boldsymbol{x})|^{(1/2)(n+\rho)} d\mu_n(\boldsymbol{x}) = M_\rho < +\infty$$

for some $\rho > 0$ satisfying also $n + \rho \geq 2$.

CONDITION II. *Let $V(\boldsymbol{x})$ satisfy Condition* I *with in addition $n + \rho = 4$ in* (1) *if dimension $n < 4$.*

Condition II is our generalization of Kato's uniqueness criterion, our following Theorem T. 1 in the special case $n = 3$ thus being due to Kato [2]. Following Kato, we define $\mathscr{D}_1 \subseteq L_2(R_n)$ as the linear manifold of Hermite functions, polynomials in the coordinates x_j multiplied by $\exp(-1/2 \mid \boldsymbol{x} \mid^2)$. Assuming Condition II), clearly the pointwise product $Vu \in L_2(R_n)$ for all $u \in \mathscr{D}_1$. Hence

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$$(2) \qquad [H_1 u](\boldsymbol{x}) = -\nabla^2 u(\boldsymbol{x}) + V(\boldsymbol{x})u(\boldsymbol{x})$$

with $\nabla^2 = \sum_{j=1}^{n} (\partial^2/\partial x_j^2)$ the Laplacian, defines H_1 as a linear operator in $L_2(R_n)$ with dense domain $\mathscr{D}_1$. Also the easily established Green's identity for u and $w \in \mathscr{D}_1$ shows that H_1 is symmetric (see [3], p. 28–41, p. 48–50 for terminology and theorems used hereafter).

Next for $u \in L_2(R_n)$ we have existent (see [4]) the Fourier-Plancherel transform $\hat{u} \in L_2(R_n)$ defined by

$$(3) \qquad \hat{u}(\boldsymbol{y}) = \lim_{N\to\infty}\left(\frac{1}{2\pi}\right)^{n/2}\int_{\{\boldsymbol{x}\,|\,|\boldsymbol{x}|\le N\}} e^{-i(\boldsymbol{x}\cdot\boldsymbol{y})}u(\boldsymbol{x})d\mu_n(\boldsymbol{x}) \,,$$

with the limit in the $L_2(R_n)$ norm sense over $\boldsymbol{y} \in R_n$; similarly

$$(4) \qquad u(\boldsymbol{x}) = \lim_{N\to\infty}\left(\frac{1}{2\pi}\right)^{n/2}\int_{\{\boldsymbol{y}\,|\,|\boldsymbol{y}|\le N\}} e^{i(\boldsymbol{x}\cdot\boldsymbol{y})}\hat{u}(\boldsymbol{y})d\mu_n(\boldsymbol{y}) \,,$$

with the limit also in the $L_2(R_n)$ norm sense. In terms of (3) and (4), define $\mathscr{D}$ as the set of $u \in L_2(R_n)$ such that $|\boldsymbol{y}|^2\hat{u}(\boldsymbol{y})$ is also in $L_2(R_n)$ over $\boldsymbol{y}$. Define T as a linear operation in $L_2(R_n)$ with domain $\mathscr{D}$ by $Tu = w$, $\hat{w}(\boldsymbol{y}) = |\boldsymbol{y}|^2\hat{u}(\boldsymbol{y})$ for $u \in \mathscr{D}$, $w \in L_2(R_n)$ existing uniquely for such u since (3) and (4) define a unitary operator and its inverse on $L_2(R_n)$.

We may now state the main theorem of this section as follows. Actually, since Condition II will be seen at the end of the next section to imply Condition S stated there, this theorem is a consequence of Stummel's theorem ([5], Th 4.2), p. 171), except for an awkward but essentially trivial change of basic domain. Also our proof is rather different, being much closer to Kato's original argument. See also [6].

THEOREM T.1. *Let V satisfy condition* II. *Then the pointwise product $[Vu](\boldsymbol{x}) = V(\boldsymbol{x})u(\boldsymbol{x})$ has $Vu \in L_2(R_n)$ for $u \in \mathscr{D}$, and $Hu = Tu + Vu$ for $u \in \mathscr{D}$ has H to be a self-adjoint operator in $L_2(R_n)$ with dense domain $\mathscr{D}$. Furthermore, $\mathscr{D}_1 \subseteq \mathscr{D}$, $H_1 \subseteq H$, and H is the unique self-adjoint extension of H_1.*

Here $\mathscr{D}_1 \subseteq \mathscr{D}$, and hence $\mathscr{D}$ is dense, follows clearly from the fact ([4], p. 81, Theorem 57) that $S^*(\mathscr{D}_1) \subseteq \mathscr{D}_1$, where S denotes the unitary operator from $L_2(R_n)$ onto itself given by (4), $S\hat{u} = u$, and where $S^*u = \hat{u}$ in (3) represents the adjoint and inverse S^*. Thus $T_1 u = -V^2 u$ for $u \in \mathscr{D}_1$ has $T_1 u = Tu$ for $u \in \mathscr{D}_1$ from $[S^*(T_1 u)](\boldsymbol{y}) = |\boldsymbol{y}|^2\hat{u}(\bar{\boldsymbol{y}})$ by integration by parts; hence $T_1 \subseteq T$. Thus $H_1 \subseteq H$ follows from the following lemma (Lemma 4 of Kato [2]), which represents the heart of our argument.

LEMMA T. 2. *Let V satisfy Condition* II. *Then for $u \in \mathscr{D}$ follows both $Vu \in L_2(R_n)$ and the $L_2(R_n)$ norm inequality*

$$(5) \qquad \| Vu \| \leq \alpha \| Tu \| + \beta \| u \|$$

for some α and β positive and finite, for which α may be chosen as small as desired with β depending on α.

To prove this lemma, we will first establish that $\mathscr{D} \subseteq L_{r'}(R_n)$ with $r' = 2(n+\rho)/(n-4+\rho)$ and $\rho > 0$ given in Condition II if dimension $n \geq 4$, and that $\mathscr{D} \subseteq L_\infty(R_n)$ if $n = 1, 2$, or 3. For this purpose we start, for $u \in \mathscr{D}$ and arbitrary $\omega > 0$ and with $\rho > 0$ as in Condition II, with the Schwarz-Hölder estimate

$$(6) \int_{R_n} | \hat{u}(\boldsymbol{y}) |^{2(n+\rho)/(n+\rho+4)} d\mu_n(\boldsymbol{y})$$
$$\leq \left[\int_{R_n} |\hat{u}(\boldsymbol{y})|^2 (1+\omega^4 |\boldsymbol{y}|^4) d\mu_n(\boldsymbol{y})\right]^{1/p} \left[\int_{R_n} (1+\omega^4 |\boldsymbol{y}|^4)^{-(p'/p)} d\mu_n(\boldsymbol{y})\right]^{1/p'}$$
$$= [\| u \|^2 + \omega^4 \| Tu \|^2]^{1/p} \left[\sigma_n \int_0^\infty \frac{r^{n-1}}{(1+\omega^4 r^4)^{(n+\rho)/4}} dr\right]^{1/p'}$$
$$= [\| u \|^2 + \omega^4 \| Tu \|^2]^{1/p} C_{n,\rho} \omega^{-(n/p')}$$
$$= C_{n,\rho} [\omega^{-(4n/n+\rho)} \| u \|^2 + \omega^{4\rho/(n+\rho)} \| Tu \|^2]^{1/p}$$

where

$$C_{n,\rho} = \left[\sigma_n \int_0^\infty \frac{t^{n-1}}{(1+t^4)^{(n+\rho)/4}} dt\right]^{1/p'} < +\infty ,$$

where $\sigma_n = 2\pi^{n/2}[\Gamma(n/2)]^{-1}$ is $n-1$ dimensional "area" measure of the unit spherical shell in R_n, where $1/p + 1/p' = 1$ with $2 = p[2(n+\rho)/(n+\rho+4)]$ and thus $1 < p = 1 + 4/(n+\rho) \leq 2$, $p'/p = 1/(p-1) = (n+\rho)/4$, $-np/p' = -4n/(n+\rho)$, and $4 - np/p' = 4\rho/(n+\rho)$.

Now if dimension $n = 1, 2$, or 3, then $n + \rho = 4$ in Condition II and (6) yields for $u \in \mathscr{D}$

$$(7) \qquad \left(\operatorname*{ess\,sup}_{\boldsymbol{x} \in R_n} | u(\boldsymbol{x}) |\right) \leq \left(\frac{1}{2\pi}\right)^{n/2} \int_{R_n} | \hat{u}(\boldsymbol{y}) | d\mu_n(\boldsymbol{y})$$
$$\leq \left(\frac{1}{2\pi}\right)^{n/2} C_{n,\rho} [\omega^{-n} \| u \|^2 + \omega^{4-n} \| Tu \|^2]^{1/2}$$

using also (4) with convergence almost (μ_n) everywhere for a subsequence from $L_2(R_n)$ norm convergence.

Now if dimension $n \geq 4$, then in (6) define $r = 2(n+\rho)/(n+\rho+4) = 2/p$, and hence $1 < r < 2$ from $1 < p = 1 + 4/(n+\rho) < 2$. Now $1/r + 1/r' = 1$ has

$$r' = \frac{1}{1-\frac{1}{r}} = \frac{1}{1-\left(\frac{1}{2}+\frac{2}{n+\rho}\right)} = \frac{2(n+\rho)}{n-4+\rho}\,.$$

Hence the Young-Hausdorff-Titchmarsh theorem ([4], Theorem 74), p. 96), generalizing with negligible changes in proof from R_1 to R_n, using subsequences convergent almost everywhere to show that the known existent $L_2(R_n)$ and $L_{r'}(R_n)$ norm limits in (4) must agree, yields in (6) for $u \in \mathscr{D}$ if $n \geq 4$

$$(8)\qquad \left[\int_{R_n} |\, u(\boldsymbol{x})\,|^{r'} d\mu_n(\boldsymbol{x})\right]^{1/r'}$$
$$\leq \left(\frac{1}{2\pi}\right)^{n(1/2-1/r\,)} \left[\int_{R_n} R_n \,|\, \hat{u}(\boldsymbol{y})\,|^{r}\, d\mu_n(\boldsymbol{y})\right]^{1/r}$$
$$\leq \left(\frac{1}{2\pi}\right)^{n(1/2-1/r')} (c_{n,\rho})^{1/r}[\omega^{-4n/(n+\rho)}\, \|\, u\,\|^2 + \omega^{4\rho/(n+\rho)}\, \|\, Tu\,\|^2]^{1/2}$$

Thus we see if dimension $n = 1, 2,$ or 3 that (7) with Condition II, $n+\rho = 4$, yields for $u \in \mathscr{D}$

$$(9)\qquad \|\, Vu\,\|^2 \leq \left(\frac{1}{2\pi}\right)^n (c_{n,\rho})^2 M_\rho[\omega^{4-n}\, \|\, Tu\,\|^2 + \omega^{-n}\, \|\, u\,\|^2 + A^2\, \|\, u\,\|^2$$

over all $\omega > 0$. Thus, since $\sqrt{|\, a\,|^2 + |\, b\,|^2} \leq |\, a\,| + |\, b\,|$, (5) follows with α arbitrarily small as desired for Lemma T. 2, since $4 - n \geq 1$ here.

If dimension $n \geq 4$, then we use (8), Condition II, and over the $|\,\boldsymbol{x}\,| \leq b$ portion of the integral a Schwarz-Hölder estimate with $2\tilde{r} = r' = 2(n+\rho)/(n-4+\rho) > 2$ from $1 < r < 2$, $1/\tilde{r} + 1/\tilde{r}' = 1$, and thus

$$2\tilde{r} = \frac{2}{1-\left(\frac{n-4+\rho}{n+\rho}\right)} = \frac{2(n+\rho)}{4} = \frac{1}{2}(n+\rho)\,.$$

Hence, if $n \geq 4$, for $u \in \mathscr{D}$

$$(10)\qquad \|\, Vu\,\|^2 \leq (M_\rho)^{4/(n+\rho)}\left(\frac{1}{2\pi}\right)^{n(1-(2/r'))}(c_{n,\rho})^{2/r}$$
$$\times\ [\omega^{4\rho/(n+\rho)}\, \|\, Tu\,\|^2 + \omega^{-4n/(n+\rho)}\, \|\, u\,\|^2] + A^2\, \|\, u\,\|^2$$

for all $\omega > 0$. Thus again (5) follows in this case $n \geq 4$ with α arbitrarily small, since $\omega^{4\rho/(n+\rho)} \to 0$ as $\omega \to 0^+$. Thus the proof of Lemma T. 2 is complete.

Returning to the proof of our Theorem T. 1, from the remarks preceding Lemma T. 2 we see this lemma permits H to be defined on $\mathscr{D}$ dense, and $H_1 \subseteq H$ from $T_1 \subseteq T$. Also T is self-adjoint with domain $\mathscr{D}$.

For by definition S^*TS is a purely multiplicative operator, $[S^*TS\hat{u}](\boldsymbol{y}) = |\boldsymbol{y}|^2\hat{u}(\boldsymbol{y})$, with the natural domain of all $\hat{u} \in L_2(R_n)$ such that $|\boldsymbol{y}|^2 \hat{u}(\boldsymbol{y})$ is in $L_2(R_n)$. It is well known and easy to see that this makes S^*TS self-adjoint, and hence so is T since S is unitary.

Next $(Tu, u) = \int_{R_n} |\boldsymbol{y}|^2 |\hat{u}(\boldsymbol{y})|^2 d\mu_n(\boldsymbol{y}) > 0$ for $\|u\| > 0$ shows that the spectrum of T is confined to $[0, +\infty]$. Hence $(T + \lambda^2 I)^{-1}$ is for real $\lambda > 0$ a bounded Hermitian operator on $L_2(R_n)$ with range $\mathscr{D}$, $(T+\lambda^2 I)\mathscr{D} = L_2(R_n)$ following from the spectral theorem for self-adjoint T. Thus (much as in Kato [2], Lemma 5)), from (5), we have for all $u \in L_2(R_n)$

$$\begin{aligned} \text{(11)} \qquad \| V(T + \lambda^2 I)^{-1}u \| &\leq \alpha \| T(T + \lambda^2 I)^{-1}u \| + \beta \|(T + \lambda^2 I)^{-1}u \| \\ &\leq \left(\alpha + \frac{\beta}{\lambda^2}\right)\| u \| , \end{aligned}$$

since $\| T(T + \lambda^2 I)^{-1} \| < 1$ and $\| (T + \lambda^2 I)^{-1} \| \leq 1/\lambda^2$ are clear from the spectral representation of T. Thus choosing $\alpha < 1/2$ in (5), and then λ sufficiently positive so that $\frac{\beta}{\lambda^2} < \frac{1}{2}$, we see from (11) that the operator $\tilde{V}$ defined on $\mathscr{D}$ by $[\tilde{V}u](\boldsymbol{x}) = V(\boldsymbol{x})u(\boldsymbol{x})$ satisfies

$$\text{(12)} \qquad \| \tilde{V}(T + \lambda^2 I)^{-1} \| \leq \left(\alpha + \frac{\beta}{\lambda^2}\right) < 1 .$$

Hence $I + \tilde{V}(T + \lambda^2 I)^{-1}$ is a bounded linear operator on $L_2(R_n)$ with range $L_2(R_n)$, since

$$[I + \tilde{V}(T + \lambda^2 I)^{-1}]^{-1} = I + \sum_{p=1}^{\infty} (-1)^p [\tilde{V}(T + \lambda^2 I)^{-1}]^p$$

also exists bounded. Thus, for λ large so (12) holds,

$$\text{(13)} \qquad H + \lambda^2 I = T + \lambda^2 I + \tilde{V} = [I + \tilde{V}(T + \lambda^2 I)^{-1}](T + \lambda^2 I)$$

takes $\mathscr{D}$ onto $L_2(R_n)$, since $T + \lambda^2 I$ has already been seen to do so. Since $T = T^*$ has been shown and since $\tilde{V}$ is obviously symmetric, it follows that $H = T + \tilde{V}$ and $H + \lambda^2 I$ are symmetric, $H + \lambda^2 I \subseteq (H + \lambda^2 I)^*$. But $(H + \lambda^2 I)\mathscr{D} = L_2(R_n)$ in (13) thus makes $H + \lambda^2 I = (H + \lambda^2 I)^* = H^* + \lambda^2 I$, $H = H^*$, and hence H is self-adjoint (see [3], p. 35).

In order to complete the proof of Theorem T. 1, it remains only to show that the self-adjoint extension H of H_1 is the unique self-adjoint extension. Since here $H_1 \subseteq H_1^{**} \subseteq H = H^* \subseteq H_1^*$ is well-known [3], and likewise $H_1^{**} \subseteq \tilde{H} \subseteq H_1^*$ for any other self-adjoint extension $\tilde{H}$, since $H = H_1^{**}$ will make $H_1^* = (H_1^*)^{**} = (H_1^{**})^* = H^* = H = H_1^{**}$, and since $H_1^{**} = \bar{H}_1$ the closure of H_1, it suffices for this uniqueness to show $H \subseteq \bar{H}_1$.

In order to do so, we first (Lemma 1), Kato [2]) notice that orthogonality of nonzero $u_0 \in L_2(R_n)$ to $(I + T_1)u = (I + T)u$ for all $u \in \mathscr{D}_1$ would require $\hat{u}_0$ to be orthogonal to all $(1 + |\boldsymbol{y}|^2)\hat{u}(\boldsymbol{y})$; equivalently, since $S^*(\mathscr{D}_1) \subseteq \mathscr{D}_1$ and $S(\mathscr{D}_1) \subseteq \mathscr{D}_1$ makes $S^*(\mathscr{D}_1) = \mathscr{D}_1 = S(\mathscr{D}_1)$, this would require $\hat{u}_0(\boldsymbol{y})(1 + |\boldsymbol{y}|^2)\exp(-(1/4)|\boldsymbol{y}|^2)$, an element of $L_2(R_n)$, to be orthogonal to all polynomials in y_j multiplied by $\exp(-(1/4)|y|^2)$. But the density of $\mathscr{D}_1$ in $L_2(R_n)$ and a change of scale by the factor $\sqrt{2}$ shows this to be impossible. Hence $(I + T)\mathscr{D}_1$ is dense in $L_2(R_n)$.

Thus given $u \in \mathscr{D}$ and $\delta > 0$ there exists $u_1 \in \mathscr{D}_1$ such that

$$\begin{aligned}\delta &> \|(I + T)u - (I + T)u_1\| = \|(I + S^*TS)(\hat{u} - \hat{u}_1)\| \\ &= \left[\int_{R_n} (1 + |\boldsymbol{y}|^2)^2 |\hat{u}(\boldsymbol{y}) - \hat{u}_1(\boldsymbol{y})|^2 d\mu_n(\boldsymbol{y})\right]^{1/2} \\ &\geq (\max \|u - u_1\|, \|T(u - u_1)\|) .\end{aligned}$$

Thus by (5),

$$\begin{aligned}\|Hu - H_1u_1\| &= \|H(u - u_1)\| \leq \|T(u - u_1)\| + \|V(u - u_1)\| \\ &\leq (1 + \alpha)\|T(u - u_1)\| + \beta\|u - u_1\| < (1 + \alpha + \beta)\delta .\end{aligned}$$

Hence the graph of H is contained in the closure of the graph of H_1, $H \subseteq \overline{H}_1$, and H is the unique self-adjoint extension of H_1 as desired. Thus Theorem T. 1 is completely proved.

3. **Connection with other conditions.** We will show in this section that Condition I, which is always implied by (and for $n \geq 4$ coincides with) Condition II, implies our earlier one (Condition III, see eq. 19) for the construction of a self-adjoint extension as the inverse of a modified Green function integral operator. In fact, it is easy to verify for $V(\boldsymbol{x}) = |\boldsymbol{x}|^{-\eta}$ that Condition I and Condition III are each equivalent to $0 \leq \eta < (\min 2, n)$, so that in this sense they have the same strength. We remark that Condition I is the natural one, used in a forthcoming joint paper, for an asymptotic formula for the distribution of eigenvalues of the bottom part of the Schrödinger operator spectrum. Finally we will show, as noted before T. 1, that

$$\text{Condition II} \Rightarrow \text{Condition S} \Rightarrow \text{Condition III} .$$

In order to give this connection with the modified Green function, we need to introduce the fundamental singularity ${}_nK_\omega(r)$ for $-\nabla^2 + \omega^2 I$ with constant $\omega > 0$. This may be defined (see [1], p. 555) uniquely by the requirements that ${}_nK_\omega(r)$ be continuous over $r > 0$, that ${}_nK_\omega(|\boldsymbol{x}|) \in L_1(R_n)$ over $\boldsymbol{x}$, and that $[\omega^2 + |\boldsymbol{y}|^2]^{-1} = \int_{R_n} {}_nK_\omega(|\boldsymbol{x}|)e^{i(\boldsymbol{x}\cdot\boldsymbol{y})}d\mu_n(\boldsymbol{x})$ over $y \in R_n$. Such ${}_nK_\omega(r) > 0$ over $r > 0$ and $\omega > 0$. We define

$$_n\tilde{K}_\omega(r) = M_n r^{-(n-2)} \exp\left(-\frac{\omega r}{4}\right) \quad \text{for} \quad n \geq 3 ,$$

$$_2\tilde{K}_\omega(r) = M_2[1 + \sqrt{\omega r}]^{-1}[1 + \ln(1 + (\omega r)^{-1})]e^{-\omega r},$$

and

$$_1\tilde{K}_\omega(r) = (2\omega)^{-1}e^{-\omega r} = {}_1K_\omega(r) ,$$

with M_n the least possible real constant having $_nK_\omega(r) \leq {}_n\tilde{K}_\omega(r)$ over all $r > 0$ and $\omega > 0$, such positive finite M_n always existing. Finally define for $\omega > 0$,

$$\overline{|V|}_\omega = \operatorname*{ess\,sup}_{x \in R_n} \int_{R_n} {}_n\tilde{K}_\omega(|\boldsymbol{x} - \boldsymbol{y}|) \, | V(\boldsymbol{y}) | \, d\mu_n(\boldsymbol{y}) . \tag{14}$$

THEOREM T. 3. *Let V satisfy Condition* I. *Then $\overline{|V|}_\omega < +\infty$ for all $\omega > 0$ and*

$$\lim_{\omega \to +\infty} \overline{|V|}_\omega = 0 . \tag{15}$$

Moreover for all $\omega > 0$

$$\lim_{p \to +\infty} \overline{|V - V_p|}_\omega = 0 , \tag{16}$$

where $V_p(\boldsymbol{x}) = V(\boldsymbol{x})$ if $|V(\boldsymbol{x})| \leq p$, $V_p(\boldsymbol{x}) = p$ if $V(\boldsymbol{x}) > p$, and $V_p(\boldsymbol{x}) = -p$ if $V(\boldsymbol{x}) < -p$.

The proof is rather elementary, using for $n \geq 2$ the Schwarz-Hölder inequality with $r = (1/2)(n + \rho) > 1$ and $1/r + 1/r' = 1$, and hence

$$r' = \frac{1}{1 - \dfrac{2}{n + \rho}} = \frac{n + \rho}{n - 2 + \rho} .$$

Thus Condition I yields in (14) for $n \geq 2$, the Schwarz-Hölder inequality being used on the $|\boldsymbol{y}| \leq b$ portion, and also $_n\tilde{K}_\omega(t/\omega) = \omega^{n-2}\, {}_n\tilde{K}_1(t)$ and $(n - 2)(n + \rho)/(n - 2 + \rho) - n = -2\rho/(n - 2 + \rho) < 0$,

$$\begin{aligned} \overline{|V|}_\omega \leq (M_\rho)^{1/r} \omega^{-2\rho/(n+\rho)} &\left[\sigma_n \int_0^\infty \{{}_n\tilde{K}_1(t)\}^{(n+\rho)/(n-2+\rho)} \, t^{n-1} dt \right]^{1/r'} \\ &+ A\omega^{-2}\sigma_n \int_0^\infty {}_n\tilde{K}_1(t) t^{n-1} dt . \end{aligned} \tag{17}$$

In (17) the second integral is obviously finite, and so is the first for $n = 2$. For $n > 2$ we see in the first integral that only the portion $0<t<1$ is in doubt, and here we have to consider the integrand factor t raised to the exponent

$$-(n - 2)\frac{n + \rho}{n - 2 + \rho} + n - 1 = \frac{2\rho}{n - 2 + \rho} - 1 > -1 .$$

Thus the first integral in (17) is also finite for $n > 2$ as well as for $n=2$, and (17) shows $\overline{|V|}_\omega < +\infty$ for all $\omega > 0$ and also that (15) follows for $n \geq 2$.

Finally for (16), taking $p > A$ so that $V(\boldsymbol{x}) - V_p(\boldsymbol{x}) = 0$ almost (μ_n) everywhere over $|\boldsymbol{x}| \geq b$ by Condition I, we see that in place of (17) we have, with $c_n < +\infty$ by the finiteness of the first integral in (17), for $n \geq 2$

$$(18) \qquad \overline{|V - V_p|}_\omega \leq c_n \omega^{-2\rho/(n+\rho)} \left[\int_{\{x \mid |x| \leq b\}} |V(\boldsymbol{x}) - V_p(\boldsymbol{x})|^{(1/2)(n+\rho)} d\mu_n(\boldsymbol{x}) \right]^{1/x} .$$

Since $\lim_{p \to \infty} |V(\boldsymbol{x}) - V_p(\boldsymbol{x})| = 0$ for all $\boldsymbol{x} \in R_n$, and since $|V(\boldsymbol{x}) - V_p(\boldsymbol{x})| \leq |V(\boldsymbol{x})|$, we see Condition I and dominated convergence in (18) yields (16) as desired for $n \leq 2$.

Finally consider $n = 1$, ${}_1\tilde{K}_\omega(r) = {}_1K_\omega(r) = (2\omega)^{-1}e^{-\omega r}$. Notice that Condition I with $1 + \rho \geq 2$ clearly implies itself with ρ replaced by $\rho' = 1$. Thus in place of (17) and (18) we have for $n = 1$

$$(17)' \qquad \overline{|V|}_\omega \leq M_1(2\omega)^{-1} + A\omega^{-2} ,$$

$$(18)' \qquad \overline{|V - V_p|}_\omega \leq (2\omega)^{-1} \int_{\{x \mid |x| \leq b\}} |V(\boldsymbol{x}) - V_p(\boldsymbol{x})| \, d\mu_n(\boldsymbol{x}) ,$$

which clearly yield (15) and (16) in the same way as above. Thus the proof of Theorem T.3 is complete.

Now consider the following condition on V. As stated in Corollary T.4 immediately thereafter, this condition is implied by Condition I, as we see from (15) above.

CONDITION III. *There exists some* $\omega, 0 < \omega < +\infty$, *such that*

$$(19) \qquad \overline{|V|}_\omega < 1 .$$

COROLLARY T.4. *If Condition* I *is satisfied, then so is Condition* III.

Condition III is our earier condition in [1] mentioned above. For our modified Green function, consider the formulae

$$(20) \qquad G_\omega(\boldsymbol{x}, \boldsymbol{y}) = {}_nK_\omega(|\boldsymbol{x} - \boldsymbol{y}|) + \sum_{p=1}^{\infty} (-1)^p \int_{R_n} \int_{R_n} \cdots \int_{R_n} {}_nK_\omega(|\boldsymbol{x} - {}_1\boldsymbol{z}|) V({}_1\boldsymbol{z}) {}_nK_\omega(|{}_1\boldsymbol{z} - {}_2\boldsymbol{z}|) V({}_2\boldsymbol{z}) \cdots V({}_p\boldsymbol{z}) {}_nK_\omega(|{}_p\boldsymbol{z} - \boldsymbol{y}|) d\mu_n({}_1\boldsymbol{z}) \cdots d\mu_n({}_p\boldsymbol{z}) ,$$

$$(21) \qquad [G_\omega u](\boldsymbol{x}) = \int_{R_n} G_\omega(\boldsymbol{x}, \boldsymbol{y}) u(\boldsymbol{y}) d\mu_n(\boldsymbol{y}) .$$

By virtue of our earlier work ([1], p. 560, 567, Lemma 3.4, Theorem 3.5, Theorem 4.5), we have the following theorem, using $|\overline{V}|_\omega \leq |\overline{V}|_{\omega'}$ from ${}_n\tilde{K}_\omega(r) \leq {}_n\tilde{K}_{\omega'}(r)$ in (14) for $\omega \geq \omega'$.

THEOREM T.5. *Let the conditions of Theorem* T.3 *hold and let* ω_1, $0 < \omega_1 < +\infty$, *be chosen so that* (19) *holds. Then for* $\omega \geq \omega_1$ *the right side of* (20) *converges almost* $(\mu_n \times \mu_n)$ *everywhere as a definition of* $G_\omega(\boldsymbol{x}, \boldsymbol{y})$, $G_\omega(\boldsymbol{x}, \boldsymbol{y}) = G_\omega(\boldsymbol{y}, \boldsymbol{x})$ *almost* $(\mu_n \times \mu_n)$ *everywhere, in* (21) *the right side exists finite almost* (μ_n) *everywhere and is in* $L_2(R_n)$ *for* $u \in L_2(R_n)$, *and the operator* G_ω *on* $L_2(R_n)$ *so defined is bounded Hermitian* $\|G_\omega\| \leq \omega^{-2}(1 - |\overline{V}|_\omega)^{-1}$. *Moreover the operator* H_2 *defined by*

$$H_2 = G_\omega^{-1} - \omega^2 I \tag{22}$$

exists as a self-adjoint operator in $L_2(R_n)$ *independent of* $\omega \geq \omega_1$.

Now under Condition I here, which is less than Condition II if $n \leq 3$, the linear manifold $\mathscr{N} = \{u \in L_2(R_n) \mid Vu \in L_2(R_n)\}$ need no longer contain $\mathscr{D}_1$, and hence H_1 may not exist as an operator in $L_2(R_n)$. Thus define $\tilde{\mathscr{D}}_1 = \mathscr{N} \cap \mathscr{D}_1$, and as in (2)

$$[\tilde{H}_1 u](\boldsymbol{x}) = -\nabla^2 u(\boldsymbol{x}) + V(\boldsymbol{x})u(\boldsymbol{x}) \tag{23}$$

for $u \in \tilde{\mathscr{D}}_1$; thus $\tilde{H}_1$ satisfies $(\tilde{H}_1 u, w) = (u, \tilde{H}_1 w)$ for $u, w \in \tilde{\mathscr{D}}_1$. Note $\tilde{\mathscr{D}}_1 = \mathscr{D}_1$ and $\tilde{H}_1 = H_1$ if $n \geq 4$, Condition I and II coinciding. Hence, after proving the following theorem, $H_2 = H$ follows for $n \geq 4$.

THEOREM T.6. *Let* V *satisfy Condition* I. *Then the self-adjoint operator* H_2 *defined by* (22), *known existent by Corollary* T.4 *and Theorem* T.5, *is an extension of* $\tilde{H}_1$, $\tilde{H}_1 \subseteq H_2$

We note here that $\tilde{\mathscr{D}}_1$ need not be dense in $L_2(R_n)$ if $n \leq 3$, although $\tilde{H}_1$ will not be a very respectable operator from the Hilbert space viewpoint if $\tilde{\mathscr{D}}_1$ is not dense, in particular not being symmetric. This theorem is the same as our earlier one ([1]), Theorem 5.3, p. 572) except for change in the initial domain from $\mathscr{G}_0 = \mathscr{N} \cap \mathscr{G}$ there to $\tilde{\mathscr{D}}_1 = \mathscr{N} \cap \mathscr{D}_1$ here. Merely sketching the proof, we first see

$$(u, \varphi) = \int_{R_n} [G_\omega u](\boldsymbol{x})\{(\omega^2 + V(\boldsymbol{x}))\overline{\varphi(\boldsymbol{x})} - \nabla^2\overline{\varphi(\boldsymbol{x})}\}\, d\mu_n(\boldsymbol{x}) \tag{24}$$

follows for $\varphi \in \mathscr{D}_1$ and $u \in L_1(R_n) \cap L_2(R_n)$, the proof being unchanged from the earlier one ([1], Theorem 5.1, p. 568) for φ having continuous second partials and vanishing outside a bounded set. Taking $\varphi \in \tilde{\mathscr{D}}_1 = \mathscr{N} \cap \mathscr{D}_1$ in (24) and using the facts that G_ω is bounded Hermitian and that $L_1(R_n) \cap L_2(R_n)$ is dense in $L_2(R_n)$, we obtain from (23)

$$G_\omega(\omega^2 I + \tilde{H}_1)\varphi = \varphi \tag{25}$$

for $\varphi \in \tilde{\mathscr{D}}_1$. Thus $\tilde{\mathscr{D}}_1 \subseteq$ (range of G_ω) = (domain of G_ω^{-1}), and $\omega^2 I + \tilde{H}_1 \subseteq G_\omega^{-1}$, $\tilde{H}_1 \subseteq G_\omega^{-1} - \omega^2 I = H_2$ as desired, proving T.6.

THEOREM T.7. *Let V satisfy Condition* I *and define* $h_1 = \lim_{r\to\infty}(\text{ess inf}_{|x|\geq r} V(x))$. *Then this limit exists satisfying* $-A \leq h_1 \leq A$ *and the spectrum Σ of the self-adjoint operator H_2 defined by* (22), *known existent by Corollary* T.4 *and Theorem* T.5, *has* $(-\infty, h_1) \cap \Sigma$ *to consist of pure point spectra with* $(-\infty, h) \cap \Sigma$ *finite and having a finite dimensional eigenspace for all* $h < h_1$. *If also* $h_0 = [\operatorname*{ess\,inf}_{x\in R_n} V(x)] > -\infty$, *then* $(-\infty, h_0) \cap \Sigma$ *is empty.*

Since (19) and (16) follow from Condition I for large ω by Theorem T.3, this theorem follows from our earlier one ([1], Theorem 6.4, p. 579).

Finally we finish this section by proving in the following Theorems T.8 and T.9 the implications asserted before, namely II $\Rightarrow$ S $\Rightarrow$ III. Since Condition S, as noted before Theorem T.1, implies the conclusion of that theorem, from II $\Rightarrow$ S we have an alternate proof of Theorem T.1. For knowledge of this work of Stummel [5] we are indebted to the referee. Although Theorems T.8 and T.9 seem of sufficient interest to record, their proofs are simple exercises in the use of the Schwarz-Hölder inequality.

We start by stating Stummel's Condition S.

CONDITION S.

$$\left\{\sup_{x\in R_n}\int_{\{y\,|\,|x-y|\leq 1\}} |V(y)|^2\,|x-y|^{-\gamma}d\mu_n(y)\right\} < +\infty \tag{26}$$

for some real γ satisfying $\gamma > n-4$ and $\gamma \geq 0$.

THEOREM T.8. *If V satisfies Condition* II, *then it also satisfies Condition* S.

THEOREM T.9. *If V satisfies Condition* S, *then equation* (15) *and hence Condition* III *are satisfied by V.*

To prove T.8 first, Condition II clearly yields (26) with $\gamma = 0$, which thus takes care of the trivial case $1 \leq n < 4$.

Now consider dimension $n \geq 4$. Then for the $\rho > 0$ in (1) of the given Condition II, we may choose real γ to satisfy

$$n - 4\left(\frac{n}{n+\rho}\right) > \gamma > n - 4 \tag{27}$$

and must then verify (26). Take $p = (1/4)(n+\rho) > 1$ and then $1/p + 1/p' = 1$, for which

$$n[(n+\rho)-4] > (n+\rho)\gamma\,, \qquad n+\rho > 4+(n+\rho)\frac{\gamma}{n}\,,$$

$$\left(1-\frac{\gamma}{n}\right)(n+\rho) > 4\,, \quad p=(1/4)(n+\rho) > \left(1-\frac{\gamma}{n}\right)^{-1},$$

$$1-\frac{1}{p'}=\frac{1}{p}<1-\frac{\gamma}{n}\,,$$

and hence $\gamma p' < n$. Thus for (26) we have the Schwarz-Hölder estimate

$$\begin{aligned}(28)\qquad &\int_{\{y \mid |x-y|\leq 1\}} |V(y)|^2\,|x-y|^{-\gamma} d\mu_n(y) \\ &\leq \left[\int_{\{y\mid |x-y|\leq 1\}} |V(y)|^{2p} d\mu_n(y)\right]^{1/p}\left[\sigma_n\int_0^1 r^{-\ p'} r^{n-1} dr\right]^{1/p'}\end{aligned}$$

with $2p=(1/2)(n+\rho)$, $\gamma p' < n$, and σ_n as in (6). Thus the second factor on the right of (28) is a finite constant, Condition II assures that the first factor is bounded over $x \in R_n$, and (27) and (28) yield (26) for Condition S. This completes the proof of T.8.

Now for Theorem T.9 it suffices to prove that Condition S implies $\lim_{\omega\to+\infty}|\overline{V}|_\omega=0$, since equation (15) yields the conclusion of Corollary T.4 as noted there. Considering first the general case $n\geq 4$, and taking $\beta=n-2-\gamma/2<n-2-(n-4)(1/2)=n/2$ so that $2\beta<n$ from $\gamma>n-4\geq 0$, the Schwarz inequality yields

$$\begin{aligned}(29)\qquad &\int_{\{y\mid |x-y|\leq 1\}} |V(y)|\frac{e^{-\omega|x-y|}}{|x-y|^{n-2}} d\mu_n(y) \\ &\leq \left[\int_{\{y\mid |x-y|\leq 1\}} |V(y)|^2\,|x-y|^{-\gamma} d\mu_n(y)\right]^{1/2}\left[\sigma_n\int_0^1 e^{-\omega r} r^{n-1-2\beta} dr\right]^{1/2}.\end{aligned}$$

On the right here the second factor is $\leq[\omega^{-(n-2\beta)}\sigma_n\Gamma(n-2\beta)]^{1/2}\to 0$ as $\omega\to+\infty$ since $n-2\beta>0$; the first factor is independent of ω and bounded over $x\in R_n$ according to Condition S. Hence we see that the left side of (29) converges to zero uniformly over $x\in R_n$ as $\omega\to+\infty$ for $n\geq 4$.

In order to estimate $|\overline{V}|_\omega$, we must also consider the left side of (29) with the range of integration replaced by its complement in R_n. For this we define

$B(j)=\{x\in R_n \mid\mid x_i-2j_i(n)^{-1/2}\mid\leq(n)^{-1/2}$ for $1\leq i\leq n\}$, $j=(j_1,j_2,\cdots,j_n)$ for integer j_i, and also $r(j)=\inf_{x\in B(j)}|x|$. Noting that $B(0)\subseteq\{x \mid\mid x\mid\leq 1\}$ makes $\{x\mid\mid x\mid>1\}\subseteq\bigcup_{j\neq 0}B(j)$, we see with n ≥ 4

$$(30)\qquad \int_{\{y \mid |x-y|>1\}} |V(y)| \frac{e^{-\omega|x-y|}}{|x-y|^{n-2}} d\mu_n(y)$$
$$\leq \int_{\{z \mid |z|>1\}} e^{-\omega|z|} |V(x-z)| \, d\mu_n(z)$$
$$\leq \sum_{j\neq 0} e^{-\omega r(j)} \int_{B(j)} |V(x-z)| \, d\mu_n(z)$$
$$\leq \left(\frac{2}{\sqrt{n}}\right)^{n/2} \left[\sup_{x\in R_n} \int_{\{y \mid |x-y|\leq 1\}} |V(y)|^2 \, d\mu_n(y)\right]^{1/2} \left\{\sum_{j\neq 0} e^{-\omega r(j)}\right\} .$$

Since $|x-y|^{-\gamma} \geq 1$ in (26), we see that Condition S assures that the first factor on the far right side of (30) is a finite constant. Moreover, we see that the second factor

$$\left\{\sum_{j\neq 0} e^{-\omega r(j)}\right\} \to 0 \quad \text{as} \quad \omega \to +\infty ,$$

using

$$r(j) \geq \left(\sup_{x\in B(j)} |x|\right) - 2$$

to estimate the portion of this sum where

$$r(j) \geq 3 \quad \text{by} \quad \left(\frac{2}{\sqrt{n}}\right)^{-n} \sigma_n \int_3^\infty e^{-\omega(r-2)} r^{n-1} dr$$

which $\to 0$ by dominated convergence, and using $r(j) \geq 1/\sqrt{n} > 0$ for $j \neq 0$ to estimate the remaining finite sum portion. Thus the left side of (30) converges to zero uniformly over $x \in R_n$ as $\omega \to +\infty$, which when combined with the same conclusion about (29) proved above yields $|\overline{V}|_\omega \to 0$ and completes the proof of T.9 for dimension $n \geq 4$.

For dimension $n < 4$, we see Condition S becomes just (26) with $\gamma = 0$. Hence $\int_{R_n} [{}_n\tilde{K}_\omega(|x|)]^2 d\mu_n(x) = c_n\omega^{-(4-n)}$, easily seen with $c_n < +\infty$ for $n < 4$ from the definition preceding (14), gives in place of (29)

$$(31)\qquad \left[\sup_{x\in R_n} \int_{\{y \mid |x-y|\leq 1\}} |V(y)| \, {}_n\tilde{K}_\omega(|x-y|) d\mu_n(y)\right] = 0\left(\omega^{-(4-n)1/2}\right)$$

as $\omega \to +\infty$. Also (30) still shows the integral over the complimentary region to converge to zero uniformly over $x \in R_n$ as $\omega \to +\infty$ if $n = 3$, and a very similar computation gives the same result if $n = 1$ or 2. Hence $\lim_{\omega\to+\infty} |\overline{V}|_\omega = 0$ follows from Condition S when dimension $n < 4$ as well as when $n \geq 4$, and the proof of T.9 is complete.

4. Variational characterization of the spectrum. In this section we will show (see T.13 following) that a variational characterization of the spectrum, well-known at least for continuous V and bounded domains, also holds for H_2 with V subject only to Condition I. This is rather easy to obtain ([2], p. 209, eq. (23)) under Condition II, and the major effort in our argument amounts to showing that Condition I, which is weaker for $1 \leq n \leq 3$, actually suffices.

We start with the following theorem, where by the L_1 sense of the Fourier transform $\hat{u}$ for $u \in L_1(R_n)$ we mean (3) with no limit and $\int_{\{x \mid |x| \leq N\}}$ replaced by the oridinary Lebesque integral $\int_{R_n}$. Notice that if $u \in L_1(R_n) \cap L_2(R_n)$, then by taking subsequences we may be sure that the two definitions of $\hat{u}(\boldsymbol{y})$ are equal almost (μ_n) everywere. Hereafter $\| u \|_r$ denotes the $L_r(R_n)$ norm of u, and $\| u \|$ or $\| u \|_2$ the $L_2(R_n)$ norm.

THEOREM T.10. *Let V satisfy Condition* I *and let λ_0 be in the point spectrum of H_2, defined by* (22), *with eigenvector $u_0 \in \mathscr{D}_2 = G_\omega(L_2(R_n))$, $H_2 u_0 = \lambda_0 u_0$ and $\| u_0 \| = 1$. Then $Vu_0 \in L_1(\{\boldsymbol{x} \mid |\boldsymbol{x}| \leq b\})$ and over $\boldsymbol{y} \in R_n$*

$$|\boldsymbol{y}|^2 \hat{u}_0(\boldsymbol{y}) + \psi_0(\boldsymbol{y}) = \lambda_0 \hat{u}_0(\boldsymbol{y}) \tag{32}$$

where $\psi_0 = \hat{f}_0 + \hat{g}_0$, $\hat{f}_0$ is the L_1 sense transform of $f_0(\boldsymbol{x}) = V(\boldsymbol{x})u_0(\boldsymbol{x})\chi_b(\boldsymbol{x})$ with $\chi_b(\boldsymbol{x})$ the characteristic function of $\{\boldsymbol{x} \in R_n \mid |\boldsymbol{x}| \leq b\}$, and $\hat{g}_0$ is the usual L_2 transform of $g_0 = Vu_0 - f_0$.

If $(n + \rho) \geq 4$, then Condition II follows from Condition I, $H_2 = H$ and $u_0 \in \mathscr{D}_2 = \mathscr{D}$ by Theorems T.1 and T.6, $Vu_0 \in L_2(R_n)$ by Lemma T.2 and hence $\varepsilon L_1(\{\boldsymbol{x} \mid |\boldsymbol{x}| \leq b\})$, ψ_0 exists as defined and $= \widehat{Vu_0}$ defined in the usual L_2 sense, and (32) follows from $Hu_0 = \lambda_0 u_0$ and the definition of H.

The proof of T. 10 thus being complete for $(n + \rho) \geq 4$ and hence for $n \geq 4$, we now consider the remaining case $2 \leq n + \rho < 4$, for which $1 \leq n \leq 3$. Since $G_\omega u_0 = (\lambda_0 + \omega^2)^{-1} u_0$ with $\lambda_0 + \omega^2 > 0$ for $\omega \geq \omega_1$ follows from (22) and $H_2 u_0 = \lambda_0 u_0$, we see ([1], (3.5), (3.6), and (3.21), p. 558 and 562) by using the Schwarz inequality that u_0 is essentially bounded, $u_0 \in L_\infty(R_n)$ and $\|u_0\|_\infty = \operatorname{ess\,sup}_{x \in R_n} |u_0(x)| < +\infty$. Thus by Condition I, $Vu_0 \in L_r(\{\boldsymbol{x} \mid |\boldsymbol{x}| \leq b\}) \subseteq L_1(\{\boldsymbol{x} \mid |\boldsymbol{x}| \leq b\})$ with $r = \frac{1}{2}(n + \rho)$ satisfying $1 \leq r < 2$, and ψ_0 exists as defined.

Now, $L_1 \cap L_2$ being dense in L_2, there exists a sequence $u'_k \in L_1(R_n) \cap L_2(R_n)$ such that the L_2 norm $\|u_0 - u'_k\|_2 \to 0$. Hence as above, $u_k = (\lambda_0 + \omega^2) G_\omega u'_k$ has $u_k \in L_\infty \cap L_2$ and both $\|u_0 - u_k\|_2 \to 0$ and also $\|u_0 - u_k\|_\infty \to 0$. Actually ([1], Lemma 4.1, p. 565), u_k and $Vu_k \in L_1(R_n)$ also, and

$$(|\boldsymbol{y}|^2 - \lambda_0)\hat{u}_k(\boldsymbol{y}) + \psi_k(\boldsymbol{y}) = (\lambda_0 + \omega^2)\{\hat{u}'_k(\boldsymbol{y}) - \hat{u}_k(\boldsymbol{y})\} \tag{33}$$

with $\psi_k = \widehat{Vu_k}$ in the L_1 sense. Defining f_k and g_k from u_k analogously to f_0 and g_0 from u_0, $\psi_k = \hat{f}_k + \hat{g}_k$ defined in the L_1 sense. Moreover,

$$\|\hat{f}_0 - \hat{f}_k\|_\infty \leq (2\pi)^{-n/2} \|f_0 - f_k\|_1 \leq (2\pi)^{-n/2} \|V\|_{1,b} \|u_0 - u_k\|_\infty \to 0$$

with

$$\| V \|_{1,b} = \int_{\{x \mid |x| \le b\}} | V(x) | \, d\mu_n(x) \, ,$$

and $\| g_0 - g_k \|_2 \le A \| u_0 - u_k \|_2 \to 0$ by using Condition I. Thus, after taking subsequences, we may assume almost (μ_n) everywhere that

$$\psi_k(y) = \hat{f}_k(y) + \hat{g}_k(y) \to \hat{f}_0(y) + \hat{g}_0(y) = \psi_0(y) \, , \quad \hat{u}_k(y) \to \hat{u}_0(y) \, ,$$

and $\hat{u}'_k(y) \to \hat{u}_0(y)$, since $\| \hat{u}_k - \hat{u}_0 \|_2 = \| u_k - u_0 \|_2 \to 0$ and $\| \hat{u}'_k - \hat{u}_0 \|_2 = \| u'_k - u_0 \|_2 \to 0$. Thus (33) yields (32), and the proof of theorem T. 10 is complete.

We next give some approximation lemmas.

LEMMA T. 11. *Let V satisfy Condition* I *with $n + \rho \ge 4$, and hence Condition* II *also; let $u_0 \in \mathscr{D}$. Then there exists a sequence of $u_k \in \mathscr{D}$, satisfying simultaneously $\| u_0 - u_k \| \to 0$, $\| T(u_0 - u_k) \| \to 0$, $\| V(u_0 - u_k) \| \to 0$ for these $L_2(R_n)$ norms.*

This was proved in the last two paragraphs of § 2. In the following we denote $(z \cdot \xi) = \sum_{j=1}^{n} z_j \bar{\xi}_j$, $|z| = \sqrt{(z \cdot z)}$ for z and $\xi \in C_n$, unitary n space. $\mathscr{D}_2 = G_\omega(L_2(R_n))$ for $\omega \ge \omega_1$ is the domain of H_2 as usual.

LEMMA T. 12. *Let V satisfy Condition* I *with $2 \le n + \rho < 4$ and let $u_0 \in \mathscr{D}_2$ satisfy $H_2 u_0 = \lambda_0 u_0$ and $\| u_0 \| = 1$. Then $|y| \hat{u}_0(y) \in L_2(R_n)$ and $u_0 \in L_\infty(R_n)$ and $\hat{u}_0 \in L_1(R_n)$, and there exists a sequence of $u_k \in \mathscr{D}_1$ such that simultaneously $\| u_0 - u_k \|_2 \to 0$, $\| u_0 - u_k \|_\infty \to 0$,*

$$\int_{R_n} | V(x) | \, | u_0(x) - u_k(x) |^2 d\mu_n(x) \to 0 \, ,$$

and

$$\int_{R_n} | \nabla_{\mathrm{gen}} u_0(x) - \nabla u_k(x) |^2 d\mu_n(x) \to 0 \, ,$$

where ∇ denotes the ordinary gradient differential operator and $\nabla_{\mathrm{gen}} u$ the C_n vector valued function whose components are in $L_2(R_n)$ and have the components of $i y \hat{u}(y)$ as their L_2 sense Fourier transforms.

To prove T. 12, first notice $2 \le n + \rho < 4$ makes $1 \le n \le 3$, and hence, as shown in proving T. 10, $u_0 \in L_\infty(R_n)$ and $f_0 \in L_r(R_n)$ with $r = \frac{1}{2}(n + \rho)$, $1 \le r < 2$. Thus, using the Young-Hausdorff-Titchmarsh theorem as in (8), the L_1 sense $\hat{f}_0 \in L_{r'}(R_n)$ with

$$r' = \frac{1}{1 - 1/r} = \frac{n + \rho}{n + \rho - 2} > 2 \, , \text{ and } r' = \infty$$

if $n + \rho = 2$.

Next notice that for $0 < \nu < 2$ we have from (32)

$$\left(\frac{|\boldsymbol{y}|}{1+|\boldsymbol{y}|}\right)^{2-\nu}|\boldsymbol{y}|^{\nu}\hat{u}_0(\boldsymbol{y}) = (1+|\boldsymbol{y}|)^{-2+\nu}[\lambda_0\hat{u}_0(\boldsymbol{y}) - \hat{g}_0(\boldsymbol{y})] - (1+|\boldsymbol{y}|)^{-2+\nu}\hat{f}_0(\boldsymbol{y}) \,. \tag{34}$$

Thus we may conclude $|\boldsymbol{y}|^{\nu}\hat{u}_0(\boldsymbol{y}) \in L_2(R_n)$, as desired, whenever this holds for both terms on the right of (34). The first term is obviously in $L_2(R_n)$. For the second term we use $\hat{f}_0 \in L_{r'}(R_n)$ and the Schwarz-Hölder inequality with

$$2\alpha = r' = \frac{n+\rho}{n+\rho-2} > 2\,, \quad \alpha' = \frac{1}{1-1/\alpha} = \frac{n+\rho}{4-(n+\rho)}$$

holding even for $n+\rho=2$, for which $\alpha = \infty$ and $\alpha' = 1$. Thus, with σ_n as in (6),

$$\int_{R_n}\frac{|\hat{f}_0(\boldsymbol{y})|^2}{(1+|\boldsymbol{y}|)^{2(2-\nu)}}d\mu_u(\boldsymbol{y}) \le ||\hat{f}_0||_{r'}^2\left[\sigma_n\int_0^{\infty}\frac{t^{n-1}}{(1+t)^{2(2-\nu)\alpha'}}dt\right]^{1/\alpha'} < +\infty \tag{35}$$

provided that

$$n < (2-\nu)2\alpha' = \frac{2(n+\rho)(2-\nu)}{4-(n+\rho)}\,.$$

This last inequality is equivalent to

$$2-\nu > \frac{[4-(n+\rho)]n}{2(n+\rho)}\,,$$

and this to

$$\nu < \frac{4\rho+n(n+\rho)}{2(n+\rho)} = \frac{n}{2}+\frac{2\rho}{n+\rho}\,.$$

We see for our $n = 1, 2$, or 3, $\rho > 0$, $2 \le n+\rho < 4$, that this last inequality is always satisfied for $\nu = 1$ and for $\nu = \nu_1 = n/2 + \rho/(n+\rho)$. Note $n/2 < \nu_1 < 2$. Thus we have shown $|\boldsymbol{y}|\hat{u}_0(\boldsymbol{y})$ and $|\boldsymbol{y}|^{\nu_1}\hat{u}(\boldsymbol{y})$ to be $\in L_2(R_n)$.

Next for any finite set of $\nu_p > 0$, define

$$[L\hat{u}](\boldsymbol{y}) = \left(1+\sum_p|\boldsymbol{y}|^{\nu_p}\right)\hat{u}(\boldsymbol{y})\,.$$

As in the last two paragraphs of § 2, $L\mathscr{D}_1$ is dense in $L_2(R_n)$, since any $\hat{u} \in L_2(R_n)$ has

$$\left(1+\sum_p|\boldsymbol{y}|^{\nu_p}\right)\exp\left(-\tfrac{1}{4}|\boldsymbol{y}|^2\right)\hat{u}(\boldsymbol{y}) \in L_2(R_n)$$

and therefore is not orthogonal to all $Q(\boldsymbol{y}) \exp(-\frac{1}{2}|\boldsymbol{y}|^2)$ with polynomial Q, and thus $\hat{u}$ cannot be orthogonal to $L\mathscr{D}_1$. Hence, for any $u \in L_2(R_n)$ such that $L\hat{u} \in L_2(R_n)$ there exists (since $\mathscr{D}_1$ transforms onto $\mathscr{D}_1$) a sequence $u_k \in \mathscr{D}_1$ such that $||L(\hat{u} - \hat{u}_k)||_2 \to 0$, and thus simultaneously

$$\int_{R_n} |\boldsymbol{y}|^{2\nu_p} |\hat{u}(\boldsymbol{y}) - \hat{u}_k(\boldsymbol{y})|^2 d\mu_n(\boldsymbol{y}) \to 0$$

for the finite set of ν_p as well as $||u - u_k||_2 = ||\hat{u} - \hat{u}_k||_2 \to 0$. Applying this result to $u_0 \in L_2(R_n)$ with the finite set $\{1, \nu_1\}$ of ν's, since $|\boldsymbol{y}|\hat{u}_0(\boldsymbol{y})$ and $|\boldsymbol{y}|^{\nu_1}\hat{u}_0(\boldsymbol{y})$ were shown to be in $L_2(R_n)$, there thus exists a sequence of $u_k \in \mathscr{D}_1$ such that simultaneously

$$||u_0 - u_k||_2 = ||\hat{u} - \hat{u}_k||_2 \to 0 \ , \quad \int_{R_n} |\boldsymbol{y}|^2 |\hat{u}_0(\boldsymbol{y}) - \hat{u}_k(\boldsymbol{y})|^2 d\mu_n(\boldsymbol{y}) \to 0 \ ,$$

and

$$\int_{R_n} |\boldsymbol{y}|^{2\nu_1} |\hat{u}_0(\boldsymbol{y}) - \hat{u}_k(\boldsymbol{y})|^2 d\mu_n(\boldsymbol{y}) \to 0$$

with

$$\nu_1 = \frac{n}{2} + \frac{\rho}{n+\rho} > \frac{n}{2} \ .$$

From the second limit statement just proved, and from $|\boldsymbol{y}|\hat{u}_0(\boldsymbol{y}) \in L_2(R_n)$, we see that $\boldsymbol{\nabla}_{\mathrm{gen}} u_0$ exists as defined and that, since $i\boldsymbol{y}\hat{u}_k(\boldsymbol{y})$ clearly has its components the L_2 transforms of the C_n vector valued function $\boldsymbol{\nabla} u_k(\boldsymbol{x})$,

$$\int_{R_n} |\boldsymbol{\nabla}_{\mathrm{gen}} u_0(\boldsymbol{x}) - \boldsymbol{\nabla} u_k(\boldsymbol{x})|^2 d\mu_n(\boldsymbol{x}) = \int_{R_n} |\boldsymbol{y}|^2 |\hat{u}_0(\boldsymbol{y}) - \hat{u}_k(\boldsymbol{y})|^2 d\mu_n(\boldsymbol{y}) \to 0 \ .$$

Next for $u \in L_2(R_n)$ having $|\boldsymbol{y}|^{\nu_1}\hat{u}(\boldsymbol{y}) \in L_2(R_n)$,

$$\begin{aligned} (36) \qquad ||\hat{u}||_1 &\le M\Big[\int_{R_n} (1 + |\boldsymbol{y}|^{2\nu_1}) |\hat{u}(\boldsymbol{y})|^2 d\mu_n(\boldsymbol{y})\Big]^{1/2} \\ &\le M||\hat{u}||_2 + M\Big[\int_{R_n} |\boldsymbol{y}|^{2\nu_1} |\hat{u}(\boldsymbol{y})|^2 d\mu_n(\boldsymbol{y})\Big]^{1/2} , \\ M &= \Big[\sigma_n \int_0^\infty \frac{t^{n-1}}{1 + t^{2\nu_1}} dt\Big]^{1/2} < +\infty \ , \end{aligned}$$

using the Schwarz inequality and $2\nu_1 > n$. Thus from $|\boldsymbol{y}|^{\nu_1}\hat{u}_0(\boldsymbol{y}) \in L_2(R_n)$ we conclude $||\hat{u}_0||_1 < +\infty$ and $\hat{u}_0 \in L_1(R_n)$, and likewise $||\hat{u}_0 - \hat{u}_k||_1 \to 0$ follows from

$$\int_{R_n} |\boldsymbol{y}|^{2\nu_1} |\hat{u}_0(\boldsymbol{y}) - \hat{u}_k(\boldsymbol{y})|^2 d\mu_n(\boldsymbol{y}) \to 0$$

shown above. Thus we have

$$||u_0 - u_k||_\infty \le (2\pi)^{-n/2}||\hat{u}_0 - \hat{u}_k||_1 \to 0$$

from the L_1 sense of (4) agreeing here with the L_2 sense as usual. Hence finally

$$\begin{aligned}(37)\qquad &\int_{R_n} |V(\boldsymbol{x})|\,|u_0(\boldsymbol{x}) - u_k(\boldsymbol{x})|^2 d\mu_n(\boldsymbol{x}) \\ &\le ||u_0 - u_k||_\infty^2 \int_{\{\boldsymbol{x}|\ |\boldsymbol{x}|\le b\}} |V(\boldsymbol{x})| d\mu_n(\boldsymbol{x}) + A||u_0 - u_k||_2^2\end{aligned}$$

by Condition I with the right side $\to 0$ as $k \to +\infty$. Thus the proof of Lemma T. 12 is complete.

We now are ready to give our variational characterization of the spectrum Σ of H_2, assuming only Condition I. Define

$$h_1 = \lim_{r\to\infty}\left(\operatorname*{ess\,inf}_{|\boldsymbol{x}|\ge r} V(\boldsymbol{x})\right),$$

and by Theorem T. 7 we know that $\Sigma \cap (-\infty, h)$ for $h < h_1$ consists of a finite set of λ which are each in the point spectrum of H_2 with finite multiplicity. Thus there is uniquely defined a finite or countable set $\{\lambda_p\} = \Sigma \cap (-\infty, h_1)$, $\lambda_p \le \lambda_{p+1}$, and the $\lambda_p = \lambda$ repeat according to the multiplicity of each λ in the point spectrum of H_2. In the statement following, $u \perp S$ means $(u, w) = 0$ for all $w \in S$.

THEOREM T. 13. *Let V satisfy Condition* I *and let $\{\lambda_p\}$, possibly empty, be defined as above. Then each such λ_p satisfies*

$$(38)\qquad \lambda_p = \sup_{\substack{S\subseteq L_2(R_n),\\ \operatorname{card} S<p}}\left\{\inf_{\substack{u\in\mathscr{D}_1\\ ||u||=1,u\perp S}} \int_{R_n} (|\nabla u(\boldsymbol{x})|^2 + V(\boldsymbol{x})|u(\boldsymbol{x})|^2) d\mu_n(\boldsymbol{x})\right\},$$

and such λ_p exists for any integer $p \ge 1$ for which the right side of (38) *is $<h_1$. Moreover, in this statement $\mathscr{D}_1$ may be replaced by $\mathscr{D}_0$, the set of all $u \in L_2(R_n)$ which possess continuous second partials everywhere and such that $u(\boldsymbol{x})$ together with all its partial derivatives of order ≤ 2 is $O([1 + |\boldsymbol{x}|^m]\exp(-\frac{1}{2}|\boldsymbol{x}|^2))$ over $\boldsymbol{x} \in R_n$ for some integer $m > 0$ depending on u.*

For integer $p \ge 1$ define $\tau_p(\mathscr{D}_1)$ as the right side of (38), and similarly $\tau_p(\mathscr{D}_0)$ with $\mathscr{D}_1$ replaced by $\mathscr{D}_0$. $\mathscr{D}_0 \supseteq \mathscr{D}_1$ clearly makes $\tau_p(\mathscr{D}_0) \le \tau_p(\mathscr{D}_1)$. Thus to prove theorem T. 13 we need only show first that any existing λ_p has $\lambda_p \ge \tau_p(\mathscr{D}_1)$, and secondly that $\tau_p(\mathscr{D}_0) < h_1$ has λ_p existing with $\tau_p(\mathscr{D}_0) \ge \lambda_p$.

Now for each λ_p we may choose $\varphi_p \in \mathscr{D}_2$, the domain of H_2, such that $H_2\varphi_p = \lambda_p\varphi_p$ and $(\varphi_p, \varphi_{p'}) = \delta_{p,p'}$, since H_2 is self-adjoint. Thus using T. 10 and multiplying (32) by $\varphi_{p'}(\boldsymbol{y})$ and integrating over R_n we have, since $(\hat{\varphi}_p, \varphi_{p'}) = (\varphi_p, \varphi_{p'}) = \delta_{p,p'}$,

$$(39)\qquad \lambda_p\delta_{p,p'} = \int_{R_n} \{|\boldsymbol{y}|^2\hat{\varphi}_p(\boldsymbol{y})\overline{\varphi_{p'}(\boldsymbol{y})} + \psi_p(\boldsymbol{y})\overline{\hat{\varphi}_{p'}(\boldsymbol{y})}\}\, d\mu_n(\boldsymbol{y}) ,$$

the integral of each term in (39) existing finite in the Lebesque sense. This finiteness is clear if $n+\rho \geq 4$, since then Condition II holds and $\varphi_p \in \mathscr{D}_2 = \mathscr{D}$, $|\boldsymbol{y}|\varphi_p(\boldsymbol{y}) \in L_2(R_n)$ and $\psi_p \in L_2(R_n)$ by T. 2. Otherwise $2 \leq n+\rho < 4$, and T. 12 yields $|\boldsymbol{y}|\hat{\varphi}_p(\boldsymbol{y}) \in L_2(R_n)$ and $\hat{\varphi}_p \in L_1(R_n) \cap L_2(R_n)$; hence $\psi_p = \hat{f}_p + \hat{g}_p$ with $\hat{g}_p \in L_2(R_n)$ and $\hat{f}_p \in L_\infty(R_n)$ from $f_p \in L_1(R_n)$ also makes the second term integral be finite as well as the first. Also Parseval's equality applied to the terms on the right side of (39) yields

$$(40)\qquad \lambda_p\delta_{p,p'} = \int_{R_n} \{(\boldsymbol{\nabla}_{\text{gen}}\varphi_p(\boldsymbol{x}) \cdot \boldsymbol{\nabla}_{\text{gen}}\varphi_{p'}(\boldsymbol{x})) + V(\boldsymbol{x})\varphi_p(\boldsymbol{x})\overline{\varphi_{p'}(\boldsymbol{x})}\}\, d\mu_n(\boldsymbol{x}) ,$$

provided that in addition we show

$$\int_{R_n} \hat{f}_p(\boldsymbol{y})\overline{\hat{\varphi}_{p'}(\boldsymbol{y})}d\mu_n(\boldsymbol{y}) = \int_{R_n} f_p(\boldsymbol{x})\overline{\varphi_p(\boldsymbol{x})}d\mu_n(\boldsymbol{x})$$

in the case $2 \leq n+\rho < 4$, where as usual $f_p(\boldsymbol{x}) = V(\boldsymbol{x})\varphi_p(\boldsymbol{x})\chi_b(\boldsymbol{x})$ as in T. 10. Replacing V by the truncate V_q defined for (16) and defining ${}_qf_p = V_q\varphi_p\chi_b$, then ${}_qf_p \in L_2(R_n)$ and $({}_q\hat{f}_p, \hat{\varphi}_{p'}) = ({}_qf_p, \varphi_{p'})$ follows by Parseval's equality. Clearly Condition I, $\varphi_p \in L_\infty(R_n)$ by T. 12, and dominated convergence over $\{\boldsymbol{x} \mid |\boldsymbol{x}| \leq b\}$ yields $\|f_p - {}_qf_p\|_1 \to 0$ as $q \to +\infty$, and hence also $\|\hat{f}_p - {}_q\hat{f}_p\|_\infty \to 0$. Thus $\varphi_{p'} \in L_\infty(R_n)$ and $\hat{\varphi}_{p'} \in L_1(R_n)$ by T. 12 in our case $2 \leq n+\rho < 4$ gives the desired result

$$(41)\qquad \begin{aligned} \int_{R_n} \hat{f}_p(\boldsymbol{y})\overline{\hat{\varphi}_{p'}(\boldsymbol{y})}d\mu_n(\boldsymbol{y}) &= \lim_{q\to\infty}({}_q\hat{f}_p, \hat{\varphi}_{p'}) = \lim_{q\to\infty}({}_qf_p, \varphi_{p'}) \\ &= \int_{R_n} f_p(\boldsymbol{x})\overline{\varphi_{p'}(\boldsymbol{x})}d\mu_n(\boldsymbol{x}) , \end{aligned}$$

and (40) is completely proved.

Now from (40), for $u = \sum_{j=1}^{p} c_j\varphi_j$ we have

$$(42)\qquad \begin{aligned} \int_{R_n} \{|\boldsymbol{\nabla}_{\text{gen}}u(\boldsymbol{x})|^2 + V(\boldsymbol{x})|u(\boldsymbol{x})|^2\}\, d\mu_n(\boldsymbol{x}) &= \sum_{j=1}^{p} \lambda_j |c_j|^2 \\ &\leq \lambda_p\left[\sum_{j=1}^{p} |c_j|^2\right] = \lambda_p \|u\|^2 . \end{aligned}$$

Next by T. 11 and T. 12, since $\mathscr{D}_2 = \mathscr{D}$ if $n+\rho \geq 4$, for each $\varphi_j \in \mathscr{D}_2$, $1 \leq j \leq p$, we can choose a sequence ${}_k\varphi_j \in \mathscr{D}_1$ having $\|\varphi_j - {}_k\varphi_j\|_2 \to 0$,

$$\int_{R_n} |V(\boldsymbol{x})|\,|\varphi_j(\boldsymbol{x}) - {}_k\varphi_j(\boldsymbol{x})|^2\, d\mu_n(\boldsymbol{x}) \to 0 ,$$

and

$$\int_{R_n} |\boldsymbol{\nabla}_{\text{gen}}\varphi_j(\boldsymbol{x}) - \boldsymbol{\nabla}_k\varphi_j(\boldsymbol{x})|^2 d\mu_n(\boldsymbol{x}) \to 0$$

as $k \to \infty$, and also satisfying $\|\varphi_j - {}_k\varphi_j\|_2 < 1/(3p)$ for all k. This last requirement assures that $|({}_k\varphi_j, {}_k\varphi_{j'}) - \delta_{j,j'}| \leq \theta/p$ for some fixed $\theta < 1$ (actually $\theta = \frac{8}{9}$ here), and hence the set $\{{}_k\varphi_j\}$ over $1 \leq j \leq p$ is linearly independent and thus spans a p dimensional manifold $\mathscr{M}_k$ of $\mathscr{D}_1$. Thus given $S \subseteq L_2(R_n)$ with card $S < p$, the orthogonal projection of S into the subspace $\mathscr{M}_k$ spans at most a $p-1$ dimensional manifold, and hence there exists $u_k \in \mathscr{M}_k$, $\|u_k\| = 1$, $u_k \perp S$. Also

$$u_k = \sum_{j=1}^{p} {}_kc_j\, {}_k\varphi_j$$

has

$$1 = \|u_k\|^2 = \sum_{j,j'=1}^{p} {}_kc_j\, {}_k\bar{c}_{j'}({}_k\varphi_j, {}_k\varphi_{j'}) \geq \sum_{j=1}^{p} |{}_kc_j|^2 - \frac{\theta}{p}\Big(\sum_{j=1}^{p} |{}_kc_j|\Big)^2 \geq (1-\theta)\sum_{j=1}^{p} |{}_kc_j|^2$$

by the Schwarz inequality,

$$\sum_{j=1}^{p} |{}_kc_j|^2 \leq (1-\theta)^{-1} < +\infty ,$$

and hence by taking subsequences we can assume ${}_kc_j \to {}_0c_j$ for some complex ${}_0c_j$ as $k \to +\infty$ for each j, $1 \leq j \leq p$. Thus $u_0 = \sum_{j=1}^{p} {}_0c_j\, \varphi_j$ has $u_k \to u_0$ in each of the three quadratic form norms for which ${}_k\varphi_j \to \varphi_j$ above, using the Minkowski inequality. Hence (42) for u_0 has the left side to be equal the limit as $k \to +\infty$ of the same expression with u_k replacing u_0. Since $u_k \in \mathscr{M}_k \subseteq \mathscr{D}_1$, $\|u_k\| = 1$, and $u_k \perp S$, we thus see that $\tau_p(\mathscr{D}_1) \leq \lambda_p$ holds for existing $\lambda_p < h_1$, which completes the first part of our proof.

In order to complete the proof Theorem T. 13, we must show $\tau_p(\mathscr{D}_0) < h_1$ has $\tau_p(\mathscr{D}_0) \geq \lambda_p$ with λ_p existing. Consider fixed $u_0 \in \mathscr{D}_0$. The truncate V_q, defined as for (16), with $q > A$ satisfies Condition II clearly, and thus defines the self-adjoint ${}_qH$ with domain $\mathscr{D} \supseteq \mathscr{D}_0$ as in T. 1, and ${}_qH \supseteq {}_qH_0$ defined on $\mathscr{D}_0$ by (2) with V_q. Hence by integrating by parts, and using the exponential bounds in the definition of $\mathscr{D}_0$, ${}_qE$ being the spectral measure for ${}_qH$,

$$\begin{aligned}(43)\qquad \int_{R_n} \{|\boldsymbol{\nabla} u_0(\boldsymbol{x})|^2 + V_q(\boldsymbol{x})\,|u_0(\boldsymbol{x})|^2\}\, d\mu_n(\boldsymbol{x}) &= ({}_qHu_0, u_0) \\ &= \int_{-\infty}^{\infty} \lambda d({}_qE(\lambda)u_0, u_0) \\ &= \Big\{\tilde{\sum_{{}_q\lambda_j<h}}{}' {}_q\lambda_j({}_qE(\{{}_q\lambda_j\})u_0, u_0)\Big\} + \int_{\lambda\geq h} \lambda d({}_qE(\lambda)u_0, u_0) \\ &\geq \Big\{\tilde{\sum_{{}_q\lambda_j<h}} {}_q\lambda_j({}_qE(\{{}_q\lambda_j\})u_0, u_0)\Big\} + h\Big\{\|u_0\|^2 - \tilde{\sum_{{}_q\lambda_j<h}} ({}_qE(\{{}_q\lambda_j\})u_0, u_0)\Big\}\end{aligned}$$

for any $h < h_1$, the sum $\sum^{\sim}_{{}_q\lambda_j<h}$ being finite then by T. 7 and here being defined to give one term for each distinct $\lambda \in \Sigma_q$.

Now taking $q \to +\infty$ in (43), by Condition I and dominated convergence the limit of the left side is obtained by replacing V_q by V. On the right side $|\overline{V - V_q}|_\infty \to 0$ by (16) under Condition I, and hence $\| G_\omega - {}_qG_\omega \| \to 0$ ([1], 3.20, p. 561). Defining F_ω as the spectral measure of G_ω and $f_\omega(\lambda) = 1/(\lambda + \omega^2)$, we have ([1], Theorem 4.5, p. 567) $E(B) = F_\omega(f_\omega(B))$ for Borel subsets B of the spectrum Σ of H_2; also the usual loop integral formula

$$F_\omega([a, c]) = \frac{1}{2\pi i}\int (zI - G_\omega)^{-1} dz$$

holds in the weak sense, where C is a rectangular curve in the complex plane with sides parallel to the axes whose interior region intersects the real axis in (a, c), provided both "a" and "c" are at a positive distance from $f_\omega(\Sigma)$. Thus $\| G_\omega - {}_qG_\omega \| \to 0$ implies $\| E(B) - {}_qE(B) \| \to 0$ for any closed interval $B \subseteq (-\infty, h_1)$ whose endpoints are not in $\{\lambda_p\}$. Hence ${}_q\lambda_j \to \lambda_j$ for λ_j existing, and (43) becomes

$$\begin{aligned}(44) \qquad \int_{R_n} \{| \nabla u(x) |^2 + V(x) \,| u(x) |^2\} d\mu_n(x) \\ \geq \Big\{ \sum^{\sim}_{\lambda_j<h} \lambda_j(E(\{\lambda_j\})u, u)\Big\} + h\Big\{\| u \|^2 - \sum^{\sim}_{\lambda_j<h} (E(\{\lambda_j\})u, u)\Big\} \\ = \Big\{ \sum_{\lambda_j<h} \lambda_j \,| (u, \varphi_j) |^2\Big\} + h\Big\{\| u \|^2 - \sum_{\lambda_j<h} | u, \varphi_j) |^2\Big\}\end{aligned}$$

for $u \in \mathscr{D}_0$ and $h < h_1$, the sum $\sum_{\lambda_j<h}$ meaning as usual one term for each index j satisfying $\lambda_j < h$.

Now assume $\tau_p(\mathscr{D}_0) < h_1$ for some integer $p \geq 1$, set $h' = \frac{1}{2}[h_1 + \tau_p(\mathscr{D}_0)]$, and thus $\tau_p(\mathscr{D}_0) < h' < h_1$. Now consider the particular $S = \{\varphi_j \,|\, \lambda_j < h'$ exists and $j < p\} \subseteq L_2(R_n)$, for which (card S) $< p$ clearly. Thus (44) with $\| u \| = 1$, $(u, \varphi_j) = 0$ for $\varphi_j \in S$, and $h = h'$ would give $\tau_p(\mathscr{D}_0) \geq h'$ if either λ_p did not exist or else $\lambda_p \geq h'$, yielding the contradiction $h' \leq \tau_p(\mathscr{D}_0) < h'$. Thus $\lambda_p < h' < h_1$ must exist, and (44) with $\| u \| = 1$, $(u, \varphi_j) = 0$ for $j < p$, and $h = \lambda_p$ gives $\tau_p(\mathscr{D}_0) \geq \lambda_p$ as desired. Thus the proof of Theorem T. 13 is complete.

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