ON CERTAIN SINGULAR INTEGRALS

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1. Introduction. The purpose of this paper is to consider a modification of the Hilbert transform and the singular integrals treated by Calderon and Zygmund in [1] and [3], and to use the results to generalize some standard results on fractional integration. In the one dimensional case the Hilbert transform of a function $f(x)$ is essentially the integral $\int_{-\infty}^{\infty} \frac{f(x-t)}{t} dt$. In the one dimensional case the transform to be considered will be a convolution with $\frac{1}{|t|^{1+i\gamma}}$ instead of with $\frac{1}{t}$. Throughout this paper γ will denote a real number not zero. As in the Hilbert transform case there is trouble with the definition; for the Hilbert transform this is solved by taking a Cauchy value at the origin. The obvious extension of this method was used by Thorin [6] when he considered a transform of the type

$$\lim_{\varepsilon \to 0} \int_{\varepsilon}^{\infty} \frac{f(x-t) - f(x+t)}{t^{1+i\gamma}} dt .$$

Here and subsequently ε will always be greater than 0 and the limits in ε will be one sided. In this case, however, obtaining cancellation by taking a Cauchy value is unnecessary; the kernel already has sufficient oscillations to accomplish this. The integral $\lim_{\varepsilon \to 0} \int_{\varepsilon}^{\infty} \frac{f(x-t)}{t^{1+i\gamma}} dt$ will not, in general, exist, but by using some suitable summation procedure, it may be given meaning. Starting with two such methods, it is shown that this transform has the usual singular integral properties. Specifically, for functions in a Lebesgue L^p class $1 < p < \infty$, it is shown that the summation procedure converges in L^p and that the resulting transformation is bounded in L^p. For $p = 1$ substitute results are obtained. Furthermore, for functions in $L^p, 1 \le p < \infty$, the summation procedure is shown to converge pointwise almost everywhere.

Carried along simultaneously with the preceding is the n dimensional extension of the sort considered by Calderón and Zygmund for the Hilbert transform. In Euclidean n space, E^n, let $x = (x_1, x_2 \cdots x_n)$, $|x| = (x_1^2 + \cdots x_n^2)^{\frac{1}{2}}$ and $dx = dx_1 \cdots dx_n$. The transforms to be considered are of the form

$$\int_{E^n} \frac{f(x-t)}{|t|^{n+i\gamma}} \Omega(t) \, dt .$$

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where $\Omega(t) = \Omega\left(\frac{t}{|t|}\right)$ is integrable on the unit sphere, and the integral in the neighborhood of the origin is again defined by a suitable summation method. In this case, unlike the Calderón and Zygmund results, the integral of $\Omega(t)$ on the unit sphere need not be zero. Again for functions in L^p, $1 < p < \infty$, the summation procedure converges in L^p, pointwise almost everywhere, and the resulting transformation is bounded in L^p. Substitute results for L^1 including pointwise convergence are also proved although for some it must be assumed that $\Omega(t)$ satisfies a continuity condition. The method used to obtain all these results is first to reduce the summation definition to one more closely resembling the Cauchy value definition of ordinary singular integrals. After this, lemmas similar to some lemmas in [1] make the methods of [1] and [3] applicable to these transformations.

In the last section the preceding results and an interpolation theorem of Stein [4] are used to prove the following theorem.

Let p, q, and λ be positive numbers such that $1 < p < q < \infty$ and $\frac{1}{p} = \frac{1}{q} + \lambda$. Let $f(x)$ be in L^p in E^n and let $\Omega(t) = \Omega\left(\frac{t}{|t|}\right)$ be in L^s, $s = \frac{1}{1-\lambda}$, on the unit sphere. Then the integral

$$D_\lambda(f) = \int_{E^n} \frac{\Omega(t) f(x-t)}{|t|^{n(1-\lambda)}}\, dt$$

exists for almost all x and $\|D_\lambda(f)\|_q \leq A\|f\|_p$ where A is independent of f.

For $\Omega(t) = 1$ this is a well known theorem on fractional integrals. See for example [5]. Substitute results are also obtained for $p = 1$ and $q = \infty$ using the proof for the weaker results in [8].

2. Summation.

A summation method for the integral $\lim_{\varepsilon \to 0} \int_\varepsilon^1 f(x)\, dx$ of the form $\lim_{\varepsilon \to 0} \int_0^1 \varphi_\varepsilon(\alpha) d\alpha \int_\alpha^1 f(x)\, dx$ is a *regular* method if

$$\lim_{\varepsilon \to 0} \int_a^1 |\varphi_\varepsilon(\alpha)| d\alpha = 0 \text{ for } a > 0,\ \lim_{\varepsilon \to 0} \int_0^1 \varphi_\varepsilon(\alpha)\, d\alpha = 1 \text{ and } \int_0^1 |\varphi_\varepsilon(\alpha)| d\alpha \leq B\,.$$

LEMMA 1. *If $\lim_{\varepsilon \to 0} \int_\varepsilon^1 f(x) dx$ exists, then any regular method of summation will give the same limit.*

This is a standard fact about these summation methods.

LEMMA 2. *If $\int_\varepsilon^1 f(x, y)\, dy$ converges in L^p norm to $g(x)$ as $\varepsilon \to 0$*

and has a uniformly bounded L^p norm, then any regular summation method will also converge to $g(x)$ in L^p norm.

Let B be a bound for $\int_0^1 |\varphi_\varepsilon(\alpha)|d\alpha$ and C a bound for $\left\|\int_\varepsilon^1 f(x,y)dy\right\|_p$. Then given $\eta > 0$, choose β so that

$$\left\|g(x) - \int_\delta^1 f(x,y)\,dy\right\|_p \leq \frac{\eta}{3B}$$

for $\delta \leq \beta$ and γ so that

$$\int_\beta^1 |\varphi_\varepsilon(\alpha)|d\alpha \leq \frac{\eta}{6C}$$

and

$$\left|\int_0^1 \varphi_\varepsilon(\alpha)\,d\alpha - 1\right| \leq \frac{\eta}{3C}$$

provided that $\varepsilon \leq \gamma$. The existence of β and γ follows from the hypotheses of the lemma.

If $\varepsilon \leq \gamma$, then $\left\|g(x) - \int_0^1 \varphi_\varepsilon(\alpha)d\alpha \int_\alpha^1 f(x,y)dy\right\|_p$

$$= \left\|g(x)\left(1 - \int_0^1 \varphi_\varepsilon(\alpha)\,d\alpha\right) + \int_0^\beta \varphi_\varepsilon(\alpha)\left(g(x) - \int_\alpha^1 f(x,y)\,dy\right)d\alpha \right.$$
$$\left. + \int_\beta^1 \varphi_\varepsilon(\alpha)\left(g(x) - \int_\alpha^1 f(x,y)\,dy\right)d\alpha\right\|_p$$
$$\leqq \|g(x)\|_p \frac{\eta}{3C} + \int_0^\beta |\varphi_\varepsilon(\alpha)| \left\|g(x) - \int_\alpha^1 f(x,y)\,dy\right\|_p d\alpha$$
$$+ \int_\beta^1 |\varphi_\varepsilon(\alpha)| \left\|g(x) - \int_\alpha^1 f(x,y)\,dy\right\|_p d\alpha$$

by use of Minkowski's inequality and Minkowski's integral inequality. Observing that $\|g(x)\|_p$ is also less than or equal to C, this last expression is clearly less than or equal to $\frac{C\eta}{3C} + \frac{B\eta}{3B} + \frac{\eta}{6C}2C = \eta$. Since η was arbitrary, the lemma follows.

3. Definitions of the transform. To give meaning to the integral

$$\tilde{f}(x) = \int_0^\infty \frac{f(x-t)}{t^{1+i\gamma}}\,dt$$

it may be written as

$$(S)\int_0^1 \frac{f(x-t)}{t^{1+i\gamma}}\,dt + \int_1^\infty \frac{f(x-t)}{t^{1+i\gamma}}\,dt$$

where the first integral must be obtained by using a suitable method of summation. For this purpose logarithmic Abel summation defined by

$$(S)\int_0^1 g(t)\,dt = \lim_{\varepsilon\to 0}\int_0^1 t^{\varepsilon} g(t)\,dt$$

or logarithmic Cesaro summation defined by

$$(S)\int_0^1 g(t)dt = \lim_{\varepsilon\to 0}\int_\varepsilon^1 g(t)\Big(1 - \frac{\log t}{\log \varepsilon}\Big)dt$$

may be used. Both are regular methods for they may be written as

$$\lim_{\varepsilon\to 0}\int_0^1 \varphi_\varepsilon(\alpha)\,d\alpha\int_\alpha^1 g(t)\,dt$$

where $\varphi_\varepsilon(\alpha) = \varepsilon\alpha^{\varepsilon-1}$ in the case of logarithmic Abel summation and

$$\varphi_\varepsilon(\alpha) = \begin{cases} \dfrac{-1}{\alpha\log\varepsilon} & \varepsilon \le \alpha \le 1 \\ 0 & 0 \le \alpha < \varepsilon \end{cases}$$

for logarithmic Cesaro summation. That these satisfy the necessary conditions is clear from their forms.

In either case $\tilde{f}(x)$ may be written as

$$\begin{aligned}(S)&\int_0^1 \frac{f(x-t)}{t^{1+i\gamma}}dt + \int_1^\infty \frac{f(x-t)}{t^{1+i\gamma}}dt \\ &= (S)\int_0^1 \frac{f(x-t)-f(x)}{t^{1+i\gamma}}\,dt - \frac{f(x)}{i\gamma} + \int_1^\infty \frac{f(x-t)}{t^{1+i\gamma}}\,dt\,.\end{aligned}$$

By the first lemma the existence of this expression can be shown by proving the existence of

$$\lim_{\varepsilon\to 0}\int_\varepsilon^1 \frac{f(x-t)-f(x)}{t^{1+i\gamma}}\,dt\,.$$

Therefore, showing the convergence almost everywhere of the expression

$$\begin{aligned}\lim_{\varepsilon\to 0}&\int_\varepsilon^1 \frac{f(x-t)-f(x)}{t^{1+i\gamma}}dt - \frac{f(x)}{i\gamma} + \int_1^\infty \frac{f(x-t)}{t^{1+i\gamma}}\,dt \\ &= \lim_{\varepsilon\to 0}\left[\int_\varepsilon^\infty \frac{f(x-t)}{t^{1+i\gamma}}dt - \frac{f(x)\varepsilon^{-i\gamma}}{i\gamma}\right]\end{aligned} \tag{3.1}$$

will imply convergence almost everywhere for the original definition of $\tilde{f}(x)$. Furthermore, by Lemma 2 the convergence in L^p norm of (3.1) will imply the convergence in L^p norm of the original definition of $\tilde{f}(x)$.

4. Convergence in L^2 norm. Define

$$K_{N,\varepsilon}(t) = \begin{cases} \dfrac{1}{t^{1+i\gamma}} & \varepsilon \leq t \leq N \\ 0 & \text{elsewhere} \end{cases}, \text{ and let}$$

$$\tilde{f}_{N,\varepsilon}(x) = \int_{-\infty}^{\infty} f(x-t) K_{N,\varepsilon}(t)\,dt - \frac{f(x)\varepsilon^{-i\gamma}}{i\gamma} .$$

Then the transform of $f(x)$ defined before, $\tilde{f}(x) = \lim_{\varepsilon \to 0} \lim_{N \to \infty} \tilde{f}_{N,\varepsilon}(x)$ if this last limit exists. Now if $f(x) \varepsilon L^2$, it is possible to take Fourier transforms and obtain

$$\hat{\tilde{f}}_{N,\varepsilon}(x) = \hat{f}(x)\Big(\hat{K}_{N,\varepsilon}(x) - \frac{\varepsilon^{-i\gamma}}{i\gamma}\Big)$$

where $\hat{g}(x)$ denotes the Fourier transform of $g(x)$.

LEMMA 3. *The expression* $\hat{K}_{N,\varepsilon}(x) - \dfrac{1}{i\gamma\varepsilon^{i\gamma}}$ *is in absolute value less than* $c(\gamma) = C\dfrac{(|\gamma|+1)^2}{|\gamma|}$ *where* C *is an absolute constant. As* $M \to \infty$ *the expression converges to a function* $\hat{K}_\varepsilon(x)$ *except for* $x = 0$. *Furthermore, as* $\varepsilon \to 0$, $\hat{K}_\varepsilon(x)$ *converges to a function* $\hat{K}(x)$ *except for* $x = 0$.

From its definition $\hat{K}_{N,\varepsilon}(x) - \dfrac{1}{i\gamma\varepsilon^{i\gamma}}$ is equal to

$$\int_\varepsilon^N \frac{e^{ixt}}{t^{1+i\gamma}}dt - \frac{1}{i\gamma\varepsilon^{i\gamma}} = |x|^{i\gamma}\int_{\varepsilon|x|}^{N|x|} \frac{e^{it \operatorname{sgn} x}}{t^{1+i\gamma}}dt - \frac{\varepsilon^{-i\gamma}}{i\gamma} .$$

Now

$$\int_a^b \frac{e^{it \operatorname{sgn} x}}{t^{1+i\gamma}}dt = \frac{e^{it \operatorname{sgn} x}}{i \operatorname{sgn} x t^{1+i\gamma}}\Big|_a^b + \frac{1+i\gamma}{i \operatorname{sgn} x}\int_a^b \frac{e^{it \operatorname{sgn} x}}{t^{2+i\gamma}}dt \tag{4.1}$$

and

$$\int_a^b \frac{e^{it \operatorname{sgn} x}}{t^{1+i\gamma}}dt = \frac{e^{it \operatorname{sgn} x}}{-i\gamma t^{i\gamma}}\Big|_a^b + \frac{\operatorname{sgn} x}{\gamma}\int_a^b \frac{e^{it \operatorname{sgn} x}}{t^{i\gamma}}\,dt . \tag{4.2}$$

If necessary, split the integral

$$\int_{\varepsilon|x|}^{N|x|} \frac{e^{it \operatorname{sgn} x}}{t^{1+i\gamma}}dt$$

into two parts, the first with limits less than or equal to one, and the second with limits greater than or equal to one. Then applying (4.2) to the first part and (4.1) to the second part, it is clear that the whole integral is in absolute value less than $C\frac{(|\gamma|+1)^2}{|\gamma|}$ for some absolute constant C.

Using (4.1), it is clear that

$$\lim_{N\to\infty}\hat{K}_{N,\varepsilon}(x)-\frac{1}{i\gamma\varepsilon^{i\gamma}}=|x|^{i\gamma}\left(\int_{\varepsilon|x|}^{1}\frac{e^{it\operatorname{sgn}x}}{t^{1+i\gamma}}dt+\lim_{N\to\infty}\frac{e^{it\operatorname{sgn}x}}{it^{1+i\gamma}\operatorname{sgn}x}\Big|_{1}^{N|x|}\right.$$
$$\left.+\lim_{N\to\infty}\frac{1+i\gamma}{i\operatorname{sgn}x}\int_{1}^{N|x|}\frac{e^{it\operatorname{sgn}x}}{t^{2+i\gamma}}\right)-\frac{1}{i\gamma\varepsilon^{i\gamma}}$$

and the two limits certainly exist. From this and (4.2) it follows that

$$\lim_{\varepsilon\to0}\hat{K}_{\varepsilon}(x)=\lim_{\varepsilon\to0}\left[|x|^{i\gamma}\left(\frac{e^{it\operatorname{sgn}x}}{-i\gamma t^{i\gamma}}\Big|_{\varepsilon|x|}^{1}+\int_{\varepsilon|x|}^{1}\frac{e^{it\operatorname{sgn}x}}{\gamma t^{i\gamma}\operatorname{sgn}x}dt-\frac{e^{i\operatorname{sgn}x}}{i\operatorname{sgn}x}\right.\right.$$
$$\left.\left.+\frac{1+i\gamma}{i\operatorname{sgn}x}\int_{1}^{\infty}\frac{e^{it\operatorname{sgn}x}}{t^{2+i\gamma}}dt\right)-\frac{1}{i\gamma\varepsilon^{i\gamma}}\right].$$

The limit of the integral clearly exists. The lower limit on the first integrated part and the last term combined give

$$\lim_{\varepsilon\to0}|x|^{i\gamma}\left(-\frac{|x|^{-i\gamma}e^{i\varepsilon x}}{-i\gamma\varepsilon^{i\gamma}}\right)-\frac{1}{i\gamma\varepsilon^{i\gamma}}=\lim_{\varepsilon\to0}\frac{\varepsilon^{-i\gamma}}{i\gamma}(e^{i\varepsilon x}-1)=0\,.$$

It follows that

$$\hat{K}(x)=\lim_{\varepsilon\to0}\hat{K}_{\varepsilon}(x)=|x|^{i\gamma}\left(\frac{ie^{i\operatorname{sgn}x}}{\gamma}+\frac{ie^{i\operatorname{sgn}x}}{\operatorname{sgn}x}+\int_{0}^{1}\frac{e^{it\operatorname{sgn}x}}{\gamma t^{i\gamma}\operatorname{sgn}x}dt\right.$$
$$\left.+\frac{1+i\gamma}{i\operatorname{sgn}x}\int_{1}^{\infty}\frac{e^{it\operatorname{sgn}x}}{t^{2+i\gamma}}dt\right) \tag{4.3}$$

Corollary 1. *If $f(x)$ belongs to L^2, then the transformation $\tilde{f}_\varepsilon(x)=\int_\varepsilon^\infty\frac{f(x-t)}{t^{1+i\gamma}}dt-\frac{f(x)}{i\gamma\varepsilon^{i\gamma}}$ satisfies $||\tilde{f}_\varepsilon(x)||_2\le c(\gamma)||f(x)||_2$. As $\varepsilon\to0$, $\tilde{f}_\varepsilon(x)$ converges in L^2 norm to a function $\tilde{f}(x)$ which also satisfies $||\tilde{f}(x)||_2\le c(\gamma)||f(x)||_2$*

The expression $\left(\hat{K}_{N,\varepsilon}(x)-\frac{1}{i\gamma\varepsilon^{i\gamma}}\right)\hat{f}(x)$ converges in L^2 norm to $\hat{K}_\varepsilon(x)\hat{f}(x)$ because the first part of the product converges boundedly. Consequently, taking Fourier transforms, $\tilde{f}_{N,\varepsilon}(x)$ converges in L^2 norm to $\tilde{f}_\varepsilon(x)$. Similarly, since $\hat{K}_\varepsilon(x)\ \hat{f}(x)$ converges in L^2 norm, the Fourier transform, $\tilde{f}_\varepsilon(x)$, converges in L^2 norm to a function $\tilde{f}(x)$. The statements concerning the norms follow immediately from the estimate in Lemma 3.

For later proofs there is a more convenient form for $\hat{K}(x)$. Adding the identity

$$0 = -\frac{|x|^{i\gamma}}{i\gamma} + \frac{|x|^{i\gamma} e^{i \operatorname{sgn} x}}{i\gamma} - \frac{|x|^{i\gamma} \operatorname{sgn} x}{\gamma} \int_0^1 e^{it \operatorname{sgn} x} dt$$

to (4.3) gives

$$\begin{aligned} \hat{K}(x) = \frac{-|x|^{i\gamma}}{i\gamma} + |x|^{i\gamma}\Big[\operatorname{sgn} x \int_0^1 \frac{t^{-i\gamma} - 1}{\gamma} e^{it \operatorname{sgn} x} dt \\ + i e^{i \operatorname{sgn} x} \operatorname{sgn} x + \frac{1 + i\gamma}{i \operatorname{sgn} x} \int_1^\infty \frac{e^{it \operatorname{sgn} x}}{t^{2+i\gamma}} dt \Big] . \end{aligned} \tag{4.4}$$

Now for $|\gamma| \leq 1$ the expression in brackets is uniformly bounded. This is obvious for the last two terms. Furthermore,

$$\left| \frac{t^{-i\gamma} - 1}{\gamma} \right| = \left| \frac{\log t \int_0^\gamma t^{-iu} du}{\gamma} \right| \leq |\log t|$$

so that the first integral is also uniformly bounded. This leads to the following.

COROLLARY 2. *The transform* $\tilde{F}_\gamma(x) = \lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{|t| > \varepsilon} \frac{f(x-t) \operatorname{sgn} t}{|t|^{1+i\gamma}} dt$, $|\gamma| \leq 1$, *satisfies* $\|\tilde{F}_\gamma(x)\|_2 \leq A \|f(x)\|_2$ *where* A *is independent of* γ *and* f. *As* $\gamma \to 0$, $\tilde{F}_\gamma(x)$ *converges in* L^2 *to the ordinary Hilbert transform of* $f(x)$.

$\tilde{F}_\gamma(x)$ may be written in the form

$$\lim_{\varepsilon \to 0} \frac{1}{\pi} \int_\varepsilon^\infty \frac{f(x-t)}{t^{1+i\gamma}} dt - \frac{f(x)}{i\gamma \varepsilon^{i\gamma}} - \int_{-\infty}^{-\varepsilon} \frac{f(x-t)\, dt}{(-t)^{1+i\gamma}} + \frac{f(x)}{i\gamma \varepsilon^{i\gamma}} .$$

Now observing that

$$\int_{-N}^{-\varepsilon} \frac{e^{ixt}}{(-t)^{1+i\gamma}} dt = \int_\varepsilon^N \frac{e^{i(-x)t}}{t^{1+i\gamma}} dt,$$

it is clear that

$$\hat{\tilde{F}}_\gamma(x) = \frac{1}{\pi} \lim_{\varepsilon \to 0} \lim_{N \to \infty} \Big(\hat{K}_{N,\varepsilon}(x) - \hat{K}_{N,\varepsilon}(-x)\Big) \hat{f}(x) = \frac{1}{\pi}\Big(\hat{K}(x) - \hat{K}(-x)\Big)\hat{f}(x) .$$

From (4.4) it is clear that $\hat{K}(x) - \hat{K}(-x)$ is bounded uniformly in γ since the unbounded terms cancel. Letting $\gamma \to 0$ in (4.4) then gives $\lim_{\gamma \to 0} (\hat{K}(x) - \hat{K}(-x))$

$$= -2i \operatorname{sgn} x \int_0^1 \log t \cos t \, dt + 2i \operatorname{sgn} x \cos 1 + \frac{2}{i \operatorname{sgn} x} \int_1^\infty \frac{\cos t}{t^2} dt$$

$$= 2i \operatorname{sgn} x\left(\lim_{N\to\infty}\int_0^N \frac{\sin t}{t}dt\right) = \pi i \operatorname{sgn} x \ .$$

Therefore,

$$\lim_{\gamma\to 0} \hat{\hat{F}}_\gamma(x) = i\hat{f}(x) \operatorname{sgn} x \ .$$

The Fourier transform of the Hilbert transform of $f(x)$ may be written as

$$\frac{1}{\pi}\lim_{\varepsilon\to 0}\lim_{N\to 0}\left(\int_\varepsilon^N \frac{e^{ixt} - e^{-ixt}}{t}dt\right)\hat{f}(x) = i\hat{f}(x) \operatorname{sgn} x \ .$$

Thus, the two transforms are the same.

5. The N dimensional case. Most of the important results for the n dimensional case can be obtained from one dimensional results quite simply by the method of rotation which is treated in §8. Rotation methods, however, fail in certain cases, and for these a direct approach must be used. This will be similar to the one dimensional methods and is actually just a generalization of them.

In n dimensions the transforms will be of the form

$$\hat{f}(x) = \int_{E^n} \frac{f(x-t)\Omega(t)}{|t|^{n+i\gamma}}dt \text{ where } \Omega(t) = \Omega\left(\frac{t}{|t|}\right)$$

is a function only of angle and is integrable on the unit sphere, Σ. The part of the integral for which $0 \le |t| \le 1$ is obtained by using the same summation methods as before. The same reasoning shows that the existence of

$$\lim_{\varepsilon\to 0}\int_{|t|\ge\varepsilon} \frac{f(x-t)\Omega(t)}{|t|^{n+i\gamma}}dt - \frac{f(x)}{i\gamma\varepsilon^{i\gamma}}\int_\Sigma \Omega(t)\,d\sigma \tag{5.1}$$

where $d\sigma$ is the element of "area" of the unit sphere, implies the existence of the original definition. The convergence in norm implies the convergence in norm of the original definition.

In n dimensions define

$$K_{N,\varepsilon}(t) = \begin{cases} \dfrac{\Omega(t)}{|t|^{n+i\gamma}} & \varepsilon \le |t| \le N \\ 0 & \text{elsewhere} \ . \end{cases}$$

LEMMA 4. *The expression* $\hat{K}_{N,\varepsilon}(x) - \dfrac{1}{i\gamma\varepsilon^{i\gamma}}\int_\Sigma \Omega(t)\,d\sigma$ *is in absolute vvlue less than* $c(\gamma) = C\dfrac{(|\gamma|+1)^2}{|\gamma|}\int_\Sigma |\Omega(t)|\,d\sigma$ *where* C *is an absolute* *constant. As $N \to \infty$ the expression converges to a function $\hat{K}_\varepsilon(x)$ except for $x = 0$. Furthermore, as $\varepsilon \to 0$, $\hat{K}_\varepsilon(x)$ converges to a function $\hat{K}(x)$ except for $x = 0$.*

Let θ be the angle between x and t. Then using polar coordinates

$$\begin{aligned}(5.2)\qquad &\hat{K}_{N,\varepsilon}(x) - \frac{1}{i\gamma\varepsilon^{i\gamma}}\int_\Sigma \Omega(t)\,d\sigma \\ &= \int_\Sigma \Omega(t)\,d\sigma\Big(\int_\varepsilon^N \frac{e^{ir|x|\cos\theta}}{r^{1+i\gamma}}dr - \frac{1}{i\gamma\varepsilon^{i\gamma}}\Big).\end{aligned}$$

The inner expression is the same at the one dimensional Fourier transform except that x has been replaced by $|x|\cos\theta$. Hence by Lemma 3 it is in absolute value less than $C\dfrac{(|\gamma|+1)^2}{|\gamma|}$. The convergence as $N \to \infty$, and $\varepsilon \to 0$ follow from this. Applying Holder's inequality then shows these conclusions hold for the whole expression.

COROLLARY 3. *If $f(x)$ belongs to L^2, the transform*

$$\hat{f}_\varepsilon(x) = \int_{|t|\geq\varepsilon} \frac{f(x-t)\Omega(t)}{|t|^{n+i\gamma}}\,dt - \frac{f(x)}{i\gamma\varepsilon^{i\gamma}}\int_\Sigma \Omega(t)\,d\sigma$$

satisfies $\|\hat{f}_\varepsilon(x)\|_2 \leq c(\gamma)\|f(x)\|_2$. As $\varepsilon \to 0$, $\hat{f}_\varepsilon(x)$ converges in L^2 norm to a function $\hat{f}(x)$ which also satisfies $\|\hat{f}(x)\|_2 \leq c(\gamma)\|f(x)\|_2$.

The existence almost everywhere of $\hat{f}_\varepsilon(x)$ follows from the reasoning of [3] p. 292. The result then follows from Lemma 4 in the same way that Corollary 1 followed from Lemma 3.

COROLLARY 4. *If $\int_\Sigma \Omega(t)d\sigma = 0$ and $\Omega(t)$ belongs to $L\log^+ L$ on Σ, then the transform $\tilde{F}_\gamma(x) = \lim_{\varepsilon\to 0}\int_{|t|\geq\varepsilon} \frac{f(x-t)\Omega(t)}{|t|^{n+i\gamma}}\,dt$ for $|\gamma| \leq 1$ satisfies $\|\tilde{K}_\gamma(x)\|_2 \leq A\|f(x)\|_2$ where A is independent of γ and f. As $\gamma \to 0$, $F(x)$ converges in L^2 to the ordinary Calderon and Zygmund singular integral $\lim_{\varepsilon\to 0}\int_{|t|\geq\varepsilon} \frac{f(x-t)\Omega(t)}{|t|^n}\,dt$.*

Using the one dimensional formula (4.4) in (5.2) shows that

$$(5.3)\qquad \hat{K}(x) = \int_\Sigma \Omega(t)\Big(\frac{-|x\cos\theta|^{i\gamma}}{i\gamma}\Big)d\sigma + \int_\Sigma \Omega(t)H(|x|\cos\theta, \gamma)\,d\sigma$$

where $H(|x|\cos\theta, \gamma)$ is uniformly bounded in both arguments. The first term may be written as

$$-\frac{|x|^{i\gamma}}{i\gamma}\int_\Sigma \Omega(t)\Big(\frac{|\cos\theta|^{i\gamma}-1}{\gamma}\Big)\,d\sigma$$

since $\int_\Sigma \Omega(t)d\sigma = 0$. Now

$$\left| \frac{|\cos\theta|^{i\gamma} - 1}{\gamma} \right| = \left| \frac{\log|\cos\theta| \int_0^\gamma i |\cos\theta|^{iu} du}{\gamma} \right| \leq \log \frac{1}{|\cos\theta|},$$

and since $\Omega(t)$ belongs to $L \log^+ L$ on Σ, an application of Young's inequality[1] shows that the first part of (5.3) is also uniformly bounded. Convergence follows from the pointwise convergence of the expressions in the integral signs and the bounded convergence theorem. The first part converges to 0 and the second part as in Corollary 2 converges to $\int_\Sigma \pi i \operatorname{sgn}(\cos\theta)\Omega(t)\,d\sigma$. That the Fourier transform in the case of the ordinary singular integral converges to the same value follows by expressing the transform in polar coordinates and again applying the reasoning of Corollary 2.

6. **Convergence in norm.** Let $\beta^f(y) = \sup|S|$ where all sets S such that $\int_S f(x)\,dx \geq |S|y$ are considered. Further, given a function $\Omega(x)$ of the type considered in the last section, let $\omega(r)$ be its modulus of continuity; that is $\omega(r) = \sup|\Omega(x) - \Omega(y)|$ where x and y both lie on the unit sphere and $|x - y| \leq r$.

LEMMA 5. *Let $f(x)$ be non negative and belong to $L^p, 1 \leq p \leq 2$, in E^n. Let $\Omega(t) = \Omega\left(\frac{t}{|t|}\right)$ be such that its modulus of continuity satisfies $\int_0^1 \frac{\omega(r)}{r} dr < \infty$. Let E_y be the set where*

$$\tilde{f}_\varepsilon(x) = \int_{|t|\geq\varepsilon} \frac{\Omega(t)}{|t|^{n+i\gamma}} f(x-t)\,dt - \frac{f(x)}{i\gamma\varepsilon^{i\gamma}} \int_\Sigma \Omega(t)\,d\sigma \tag{6.1}$$

exceeds y in absolute value. Then $|E_y| \leq \frac{c(\gamma)}{y^2}\int_{E^n}[f(x)]_y^2 dx + c(\gamma)\beta^f(y)$, where $[f(x)]_y = \min(f(x), y)$ and $c(\gamma) = \frac{C(|\gamma| + 1)^2}{|\gamma|}$ where C depends only on Ω.

Note. The primary use of this lemma will be for the one dimensional case where the continuity condition is automatically satisfied and the constant C is an absolute constant.

This lemma is the same as Lemma 2, Chapter I of [1] except that the transform

$$\int_{|t|\geq 1/\lambda} \frac{\Omega(t)}{|t|^n} f(x-t)\,dt$$

has been replaced by (6.1) and λ by $1/\varepsilon$. The proof is almost identical, and therefore will not be repeated. The few minor differences will be

[1] See [7] Vol. I, p. 116.

mentioned.

When $f(x)$ is split into the two parts $g(x)$ and $h(x)$, the proof for the one in L^2, $h(x)$, is a consequence of Corollary 3. The proof that

$$k(x) = \int_{|t|\geq\varepsilon} \frac{\Omega(t)}{|t|^{n+i\gamma}} g(x-t)\,dt$$

satisfies

$$\int_{\bar{D}'_y} |k(x)|\,dx \leq c\int_{D_y} |g(x)|\,dx$$

is the same except where the expression for the difference of the kernels is obtained. The principal difference there is that the expression

$$\frac{1}{|t|^{n+i\gamma}} - \frac{1}{|t_k|^{n+i\gamma}}$$

arises instead of

$$\left|\frac{1}{|t|^n} - \frac{1}{|t_k|^n}\right| .$$

However, using the fact that

$$\frac{1}{a^{n+i\gamma}} - \frac{1}{b^{n+i\gamma}} = -\int_a^b \frac{n+i\gamma}{u^{n+1+i\gamma}}\,du ,$$

the same inequality can be obtained. Now

$$\tilde{g}_\varepsilon(x) = k(x) - \frac{g(x)}{i\gamma\varepsilon^{i\gamma}}\int_\Sigma \Omega(t)\,d\sigma$$

so that

$$\int_{\bar{D}'_y} |\tilde{g}_\varepsilon(x)|\,dx = \int_{\bar{D}'_y} |k(x)|\,dx \leq c\int_{D_y} |g(x)|\,dx .$$

From this point the proofs are again identical. Following the details closely also shows that the constants are of the desired form.

From this result Theorems 1 through 7 of Chapter I of [1] follow immediately, either with the same proofs or with minor modifications. In some cases where only norms are concerned it is more convenient to carry through the proof for

$$\int_{|t|\geq\varepsilon} \frac{\Omega(t)}{|t|^{n+i\gamma}} f(x-t)\,dt$$

and then to add in the other term for which the theorems are obviously true. Lemma 5 is also obviously valid for just this term of the transform. Thus, for example, the following are true.

THEOREM 1. *Let $f(x)$ belong to $L^p, 1 < p < \infty$, in E^n. Then with the continuity condition on Ω of Lemma 5, the function $\hat{f}_\varepsilon(x)$ of (6.1) also belongs to L^p. Furthermore, $\|\hat{f}_\varepsilon(x)\|_p \le c(\gamma, p)\|f(x)\|_p$ where $c(\gamma, p) = C\dfrac{(|\gamma| + 1)^2p^2}{|\gamma|(p-1)}$ and C depends only on Ω.*

The form of $c(\gamma, p)$ can be obtained by using the reasoning of the remark on page 99 of [1], following the constants through the proof, and using the fact that for

$$p > 1, \left(\frac{(|\gamma| + 1)^2}{|\gamma|}\right)^{\frac{1}{p}} \le \frac{(|\gamma| + 1)^2}{|\gamma|} .$$

THEOREM 2. *Let $f(x)$ be a function such that*

$$\int_{E^n} |f(x)|(1 + \log^+|f(x)|)\, dx < \infty .$$

Then with the continuity condition of Ω of Lemma 5 $\tilde{f}_\varepsilon(x)$ is integrable over any set S of finite measure and

$$\int_S |\tilde{f}_\varepsilon(x)|dx \le c(\gamma)\int_{E^n}|f(x)|dx$$
$$+ c(\gamma)\int_{E^n} f(x) \log^+ \left(|S|^{1+\frac{1}{n}}f(x)\right) dx + c(\gamma)S^{-\frac{1}{n}}$$

where $c(\gamma) = C\dfrac{(|\gamma| + 1)^2}{|\gamma|}$ and C depends only on Ω.

THEOREM 3. *Let f be integrable in E^n and Ω satisfy the continuity condition of Lemma 5. Then if S is a set of finite measure,*

$$\int_S |\tilde{f}_\varepsilon(x)|^{1-\alpha} dx \le \frac{c}{\alpha}|S|^\alpha\left(\int_{E^n}|f(x)|dx\right)^{1-\alpha}$$

where c is a constant independent of α, S, ε and f.

THEOREM 4. *Let $\mu(x)$ be a mass-distribution, that is a completely additive function of Borel set in E^n, and suppose that the total variation V of μ in E^n is finite. Let $\mu'(x)$ denote the derivative of $\mu(x)$ which exists almost everywhere. Then if Ω satisfies the continuity condition of Lemma 5 and if*

$$\tilde{f}_\varepsilon(x) = \int_{|t|>\varepsilon} \frac{\Omega(t)}{|t|^{n+i\gamma}}\, d\mu(x-t) - \frac{\mu'(x)}{i\gamma\varepsilon^{i\gamma}}\int_\Sigma \Omega(t)\, d\sigma ,$$

over every set S of finite measure

$$\int_S |\tilde{f}_\varepsilon(x)|^{1-\alpha} dx \leq \frac{c}{\alpha} |S|^\alpha V^{1-\alpha} .$$

THEOREM 5. *Let $f(x)$ belong to $L^p, 1 < p < \infty$, and let Ω satisfy the continuity condition of Lemma 5. Then $\tilde{f}_\varepsilon(x)$ converges in the mean of order p as $\varepsilon \to 0$ to a function $\tilde{f}(x)$.*

From this last theorem it follows by use of Lemma 2 that the original summation definition of $\tilde{f}(x)$ also converges in L^p norm if f is in L^p and $1 < p < \infty$.

7. Pointwise convergence.

THEOREM 6. *If $f(x)$ belongs to $L^p, 1 < p < \infty$, then $\tilde{f}_\varepsilon(x)$ converges almost everywhere to a function $\tilde{f}(x)$ as $\varepsilon \to 0$. Moreover, the function $\sup_\varepsilon |\tilde{f}_\varepsilon(x)|$ belongs to L^p and $\|\sup_\varepsilon |\tilde{f}_\varepsilon(x)|\|_p \leq c\|f(x)\|_p$, c being a constant which depends on p, γ, and Ω only.*

The proof is similar to that of Theorem 1, Chapter II of [1]. Define

$$K_\varepsilon(x) = \begin{cases} \dfrac{\Omega(x)}{|x|^{n+i\gamma}} & |x| \geq \varepsilon \\ 0 & |x| < \varepsilon . \end{cases}$$

Let $H(x)$ be non negative, zero outside the unit sphere, have continuous first derivatives, and have $\int_{E^n} H(x)dx = 1$. Denote by $\tilde{f}(x)$ the limit in norm of $\tilde{f}_\varepsilon(x)$ and define

$$\hat{f}_\varepsilon(x) = \frac{1}{\varepsilon^n}\int_{E^n} H\Big(\frac{x-t}{\varepsilon}\Big)\tilde{f}(t)\, dt .$$

By the lemmas in Chapter II of [1], $\hat{f}_\varepsilon(x)$ converges almost everywhere to $\tilde{f}(x)$ and $\|\sup_\varepsilon \hat{f}_\varepsilon(x)\|_p \leq c\|\tilde{f}(x)\|_p \leq c\|f(x)\|_p$. As in [1] every constant not depending on f will be denoted by c simply.

Using the fact that $\tilde{f}_\varepsilon(x)$ converges in norm to $\tilde{f}(x)$,

$$\hat{f}_\varepsilon(x) = \lim_{\lambda\to 0} \int_{E^n} \frac{1}{\varepsilon^n} H\Big(\frac{x-t}{\varepsilon}\Big)\Big[\int_{E^n} f(t-v)K_\lambda(v)\, dv - \frac{f(t)}{i\gamma\lambda^{i\gamma}}\int_\Sigma \Omega(w)dw\Big] dt .$$

This may be considered as the difference of two integrals and written

$$\hat{f}_\varepsilon(x) = \lim_{\lambda\to 0}\Big[\int_{E^n}\int_{E^n} \frac{1}{\varepsilon^n} H\Big(\frac{x-t}{\varepsilon}\Big) f(t-v)K_\lambda(v)\, dt\, dv - \int_{E^n} \frac{1}{\varepsilon^n} H\Big(\frac{x-t}{\varepsilon}\Big)\frac{f(t)}{i\gamma\lambda^{i\gamma}}\, dt \int_\Sigma \Omega\Big] .$$

Making the substitutions $t=x-u+v$ in the first integral and $t=x-u$ in the second gives

$$\hat{f}_\varepsilon(x) = \lim_{\lambda\to 0}\int_{E^n} f(x-u)\left[\int_{E^n}\frac{K_\lambda(v)}{\varepsilon^n}H\left(\frac{u-v}{\varepsilon}\right)dv - \frac{1}{\varepsilon^n}H\left(\frac{u}{\varepsilon}\right)\frac{\lambda^{-i\gamma}}{i\gamma}\int_\Sigma \Omega\right]du$$

$$= \lim_{\lambda\to 0}\int_{E^n} f(x-u)\left[\int_{|v|\geq\varepsilon}\frac{K_\lambda(v)}{\varepsilon^n}H\left(\frac{u-v}{\varepsilon}\right)dv\right.$$

$$\left. + \int_{|v|<\varepsilon}\frac{K_\lambda(v)}{\varepsilon^n}\left[H\left(\frac{u-v}{\varepsilon}\right) - H\left(\frac{u}{\varepsilon}\right)\right]dv - \frac{H\left(\frac{u}{\varepsilon}\right)\int_\Sigma \Omega}{i\gamma\varepsilon^{n+i\gamma}}\right]du\,.$$

Since $H(x)$ is differentiable, the limit may be taken inside the integral signs to give

$$\hat{f}_\varepsilon(x) = \int_{E^n} f(x-u)\left[\int_{|v|\geq\varepsilon}\frac{K(v)}{\varepsilon^n}H\left(\frac{u-v}{\varepsilon}\right)dv\right.$$

$$\left. + \int_{|v|<\varepsilon}\frac{K(v)}{\varepsilon^n}\left[H\left(\frac{u-v}{\varepsilon}\right) - H\left(\frac{u}{\varepsilon}\right)\right]dv - \frac{H\left(\frac{u}{\varepsilon}\right)\int_\Sigma \Omega}{i\gamma\varepsilon^{n+i\gamma}}\right]du$$

where $K(v) = K_0(v)$.

Now it is also true that

$$\tilde{f}_\varepsilon(x) = \int_{E^n} f(x-u)\left[\int_{E^n}\frac{K_\varepsilon(u)}{\varepsilon^n}H\left(\frac{u-v}{\varepsilon}\right)dv\right]du - \frac{f(x)}{i\gamma\varepsilon^{i\gamma}}\int_\Sigma \Omega$$

since the integral $\int_{E^n} H(x)dx = 1$. For $|u| \geq 3\varepsilon$ it is clear that

$$\left|\int_{E^n}\frac{K_\varepsilon(v)}{\varepsilon^n}H\left(\frac{u-v}{\varepsilon}\right)dv - \int_{E^n}\frac{K_\varepsilon(u)}{\varepsilon^n}H\left(\frac{u-v}{\varepsilon}\right)dv\right|$$

$$= \left|\int_{E^n}\varepsilon^{-n}H\left(\frac{u-v}{\varepsilon}\right)(K_\varepsilon(v) - K_\varepsilon(u))\,dv\right|$$

$$\leq \int_{E^n}\varepsilon^{-n}H\left(\frac{u-v}{\varepsilon}\right)\frac{c\omega\left(\frac{c\varepsilon}{|u|}\right)}{|u|^n}\,dv = \frac{c\omega\left(\frac{c\varepsilon}{|u|}\right)}{|u|^n}\,.$$

As before ω is the modulus of continuity of Ω and c is independent of ε. The last inequality for $|K_\varepsilon(v) - K_\varepsilon(u)|$ is the one used in the proof of Lemma 5; it is valid here because $|u-v| < \varepsilon$ when the integrand is not zero.

For $|u| \leq 3\varepsilon$ it is clear that both

$$\left|\int_{E^n}\frac{K_\varepsilon(v)}{\varepsilon^n}H\left(\frac{u-v}{\varepsilon}\right)dv\right| \quad\text{and}\quad \left|\int_{E^n}\frac{K_\varepsilon(u)}{\varepsilon^n}H\left(\frac{u-v}{\varepsilon}\right)dv\right|$$

are less than or equal to

$$\int_{|v|\leq 4\varepsilon} \frac{c}{\varepsilon^{2n}} H\left(\frac{u-v}{\varepsilon}\right) dv \leq \frac{c}{\varepsilon^n} \chi_{(0,5)}\left(\frac{|u|}{\varepsilon}\right).$$

Here $\chi_{(0,5)}$ is the characteristic function of the interval (0, 5).

Similarly

$$\left|\int_{|v|<\varepsilon} \frac{K(v)}{\varepsilon^n}\left[H\left(\frac{u-v}{\varepsilon}\right) - H\left(\frac{u}{\varepsilon}\right)\right] dv\right| \leq c\left|\int_{|v|<\varepsilon} \frac{K(v)}{\varepsilon^n} \frac{|v|}{\varepsilon} \chi_{(0,2)}\left(\frac{|u|}{\varepsilon}\right) dv\right|$$
$$\leq \frac{c}{\varepsilon^n} \chi_{(0,2)}\left(\frac{|u|}{\varepsilon}\right).$$

Combining all of these results

$$|\hat{f}_\varepsilon(x) - \tilde{f}_\varepsilon(x)| \leq \int_{E^n} |f(x-u)| \left[\frac{c\omega\left(\frac{c\varepsilon}{|u|}\right)}{|u|^n} \chi_{(3,\infty)}\left(\frac{|u|}{\varepsilon}\right)\right.$$
$$\left. + \frac{c}{\varepsilon^n} \chi_{(0,5)}\left(\frac{|u|}{\varepsilon}\right) + \frac{c}{\varepsilon^n} \chi_{(0,2)}\left(\frac{|u|}{\varepsilon}\right)\right] du$$
$$+ \left|\frac{\varepsilon^{-i\gamma}}{i\gamma}\left[f(x) - \frac{1}{\varepsilon^n}\int_{E^n} f(x-u) H\left(\frac{u}{\varepsilon}\right) du\right]\int_\Sigma \Omega\right|.$$

From this the lemmas of the second chapter of [1] give

$$\|\sup_\varepsilon |\hat{f}_\varepsilon(x) - \tilde{f}_\varepsilon(x)|\|_p \leq c\|f(x)\|_p .$$

Then since $\lim_{\varepsilon\to 0} \hat{f}_\varepsilon(x) = \tilde{f}(x)$ almost everywhere and $\|\sup_\varepsilon |\hat{f}_\varepsilon(x)|\|_p \leq c\|f(x)\|_p$ and $\tilde{f}_\varepsilon \to \tilde{f}$ in mean of order p, the theorem follows.

THEOREM 7. *Let $\mu(x)$ be a mass distribution, that is, a completely additive function of Borel set in E^n and suppose that the total variation V of $\mu(x)$ in E^n is finite. Then the expression*

$$\tilde{f}_\varepsilon(x) = \int_{|t|>\varepsilon} \frac{\Omega(t)}{|t|^{n+i\gamma}} d\mu(x-t) - \frac{\mu'(x)}{i\gamma\varepsilon^{i\gamma}}\int_\Sigma \Omega ,$$

where $\mu'(x)$ is the derivative of $\mu(x)$ where this exists, has a limit $\tilde{f}$ almost everywhere as ε tends to zero, and over every set S of finite measure $\int_S |\tilde{f}(x)|^{1-\alpha} dx \leq \frac{c}{\alpha}|S|^\alpha V^{1-\alpha}$.

This corresponds to Theorem 2, Chapter II of [2]. The proof is the same except that Theorem 6 is used to obtain the convergence of the integral involving $g(x)$.

8. Other theorems. With this basis all the basic theorems in [2] and [3] can easily be shown to have their analogues for transforms of the type considered here. The periodic and discrete cases can be done simply; in the discrete case the subtracted term even disappears.

The rotation method as presented in [3] can also be applied in this case but the proof is much simpler. The method that applies only to odd kernels in the ordinary case applies to all kernels in this case. To illustrate this the following important theorem is given.

THEOREM 8. *Let $f(x)$ belong to $L^p, 1 < p < \infty$, in E^n. Let $\Omega(t) = \Omega\left(\frac{t}{|t|}\right)$ be merely integrable on the unit sphere Σ. Then if*

$$\tilde{f}_\varepsilon(x) = \int_{|t|>\varepsilon} \frac{\Omega(t)}{|t|^{n+i\gamma}} f(x-t)\,dt - \frac{f(x)}{i\gamma\varepsilon^{i\gamma}} \int_\Sigma \Omega(t)\,d\sigma\,,$$

it satisfies

$$||\tilde{f}_\varepsilon(x)||_p \leq \frac{C(|\gamma|+1)^2 p^2}{|\gamma|(1-p)} ||b(x)||_p$$

where C depends only on Ω. As $\varepsilon \to 0$, $\tilde{f}_\varepsilon(x)$ converges in L^p norm to a function $\tilde{f}(x)$. Furthermore, $||\sup_\varepsilon |\tilde{f}_\varepsilon(x)|\,||_p \leq c||f(x)||_p$ where c is independent of f, and $\tilde{f}_\varepsilon(x)$ converges almost everywhere to $\tilde{f}(x)$ as $\varepsilon \to 0$.

That $\tilde{f}_\varepsilon(x)$ exists almost everywhere is shown on page 292 of [3].

Let the norm symbol $||\ \ ||_p$ apply to the variable x. To write the integrals in polar coordinates let $t = rt'$, t' on the unit sphere. Then

$$\begin{aligned} ||\sup_\varepsilon |\tilde{f}_\varepsilon(x)|\,||_p &= \left|\left|\sup_\varepsilon \left|\int_{|t|\geq\varepsilon} \frac{\Omega(t)}{|t|^{n+i\gamma}} f(x-t)dt - \frac{f(x)}{i\gamma\varepsilon^{i\gamma}}\int_\Sigma \Omega\right|\right|\right|_p \\ &= \left|\left|\sup_\varepsilon \left|\int_\Sigma \Omega(t')\,d\sigma\left(\int_\varepsilon^\infty \frac{f(x-rt')}{r^{1+i\gamma}}\,dr - \frac{f(x)}{i\gamma\varepsilon^{i\gamma}}\right)\right|\right|\right|_p \\ &\leq \left|\left|\int_\Sigma |\Omega(t')|\,d\sigma\left(\sup_\varepsilon \left|\int_\varepsilon^\infty \frac{f(x-rt')}{r^{1+i\gamma}}\,dr - \frac{f(x)}{i\gamma\varepsilon^{i\gamma}}\right|\right)\right|\right|_p . \end{aligned}$$

Using Minkowski's integral inequality this is less than or equal to

$$\int_\Sigma |\Omega(t')|\,d\sigma \left|\left|\sup_\varepsilon \left|\int_\varepsilon^\infty \frac{f(x-rt')}{r^{1+i\gamma}}\,dr - \frac{f(x)}{i\gamma\varepsilon^{i\gamma}}\right|\right|\right|_p .$$

Using the one dimensional version of theorem 6 on the inner integral by first integrating x parallel to t' and then over the space of such lines gives

$$||\sup_\varepsilon |\tilde{f}_\varepsilon(x)|\,||_p \leq \left(\int_\Sigma |\Omega(t')|\,d\sigma\right) c||f(x)||_p \leq c||f(x)||_p .$$

The inequality for $||\tilde{f}_\varepsilon(x)||_p$ follows using the same method and the one dimensional version of Theorem 1. The rest of the proof is the same as that of Theorem 3 of [3] once convergence for continuously differentiable f vanishing outside a bounded set is shown. Writing $\tilde{f}_\varepsilon(x)$ as

$$\tilde{f}_\varepsilon(x) = \int_{\varepsilon \le |t| \le 1} \frac{f(x-t) - f(x)}{|t|^{n+i\gamma}} \Omega(t)\, dt + \int_{|t| \ge 1} \frac{f(x-t)\Omega(t)}{|t|^{n+i\gamma}}\, dt - \frac{f(x)}{i\gamma} \int_\Sigma \Omega$$

shows clearly that it converges pointwise in this case.

9. Transforms of fractional integral type.[2]

DEFINITION. $T_z(f) = c_z \int_{E^n} s(t) \left(\frac{\theta(t)}{|t|^n} \right)^z f(x-t)\,dt$ for $0 \le R(z) < 1$,

$$T_z(f) = c_z \lim_{\varepsilon \to 0} \int_{|t| \ge \varepsilon} s(t) \left(\frac{\theta(t)}{|t|^n} \right)^z f(x-t)\,dt - \frac{f(x) \int_\Sigma s(t)(\theta(t))^z\, d\sigma}{(nz - n)\varepsilon^{nz-n}}$$

for $R(z) = 1$ and $z \ne 1$, and $T_1(f) = -\frac{1}{n} f(x) \int_\Sigma s(t)\theta(t)\,d\sigma$, where $c_z = \frac{z-1}{(z-2)^2}$, 0^0 is taken as 0, $\theta(t) = \theta\left(\frac{t}{|t|}\right) \ge 0$ is integrable on the unit sphere Σ, $s(t) = s\left(\frac{t}{|t|}\right)$ has absolute value one, and $R(z)$ denotes the real part of z.

To obtain the principal theorem of this section a theorem of Stein [4] p. 483 will be used. For this purpose it will be necessary to show that the operators T_z as defined above satisfy the conditions of this theorem. Using the terminology of [4], the following lemma may be proved.

LEMMA 6. *Consider the set T_z as a family of operators from functions in E^n that are zero off the sphere $|x| \le D$ to functions in E^n. The set T_z is then an analytic family of operators of admissible growth in the strip $0 \le R(z) \le 1$. For a simple function φ in the given set, the inequalities $\|T_{1+iy}\varphi\|_p \le \frac{Cp^2}{p-1}\|\varphi\|_p$ for $1 < p < \infty$, and $\|T_{iy}\varphi\|_\infty \le \|\varphi\|_1$ hold where C depends only on $\theta(t)$ and not on D.*

Throughout the proof φ and ψ will be simple non negative functions and M the maximum of φ. Since any simple function can be written as the difference of two such functions, it will be sufficient to prove the assertions for these. The lemma will be proved in parts as indicated.

a. Simple functions in the given set are transformed into measurable functions for $0 \le R(x) \le 1$. For $R(z) = 1$ this follows from the preced-

[2] The method of this section was suggested by A. P. Calderon.

ing sections. To consider the case $0 \leq R(z) < 1$, let $r = |t|$ and $t' = t/|t|$. Then changing to polar coordinates

$$T_z(\varphi) = c_z \int_\Sigma s(t')(\theta(t'))^z d\sigma \int_0^\infty \frac{f(x - rt')}{r^{1-n+nz}} dr \,. \tag{9.1}$$

Using this,

$$|T_z(\varphi)| \leq M\Big(\int_\Sigma (1 + \theta(t')) d\sigma\Big)\Big(\int_A^B \frac{dr}{r^{1-n+nR(z)}}\Big)$$

where A is the greater of $|x| - D$ and 0 and $B = |x| + D$. Both integrals are obviously finite so that $T_z\varphi$ exists. The measurability follows from the Fubini theorem.

b. If $R(z) = 1$ and $1 \geq \varepsilon > 1/3n$, then $|T_{z-\varepsilon}(\varphi)|$ is bounded by a constant that is independent of z and ε. For $\varepsilon = 1$

$$|T_{z-1}(\varphi)| \leq \int_{E^n} |\varphi(x - t)| dt$$

and is obviously bounded. For $1 > \varepsilon > 1/3n$

$$\begin{aligned} |T_{z-\varepsilon}(\varphi)| &\leq |c_{z-\varepsilon}| \int_{E^n} \Big(\frac{\theta(t)}{|t|^n}\Big)^{1-\varepsilon} \varphi(x - t) dt \\ &\leq \Big[\int \Big(\frac{\theta(t)}{|t|^n}\Big)^{\frac{3n-1}{3n}} \varphi(x - t) dt\Big]^{\frac{3n-3n\varepsilon}{3n-1}} \Big[\int \varphi(x - t) dt\Big]^{\frac{3n\varepsilon-1}{3n-1}} \end{aligned}$$

by use of Holder's inequality. The second integral is certainly bounded. Writing the first integral in polar coordinates shows that it is in absolute value less than

$$\int_\Sigma (1 + \theta(t')) d\sigma \int_0^{2D} \frac{M dr}{r^{2/3}}$$

so that it too is bounded. Since the exponents are between 0 and 1 the whole expression is bounded.

c. If $R(z) = 1$ and $1/3n \geq \varepsilon > |I(z)|$, where $I(z)$ denotes the imaginary part of z, then $|T_{z-\varepsilon}(\varphi)|$ is bounded by a constant that is independent of z and ε. Using polar coordinates,

$$\begin{aligned} |T_{z-\varepsilon}(\varphi)| &\leq |c_{z-\varepsilon}| \int_\Sigma [\theta(t')]^{1-\varepsilon} d\sigma \int_0^\infty \frac{f(x - rt')}{r^{1-n\varepsilon}} dr \\ &\leq 2\varepsilon \int_\Sigma (1 + \theta(t')) d\sigma \int_0^{2D} \frac{M}{r^{1-n\varepsilon}} dr \\ &\leq 2\varepsilon \int_\Sigma (1 + \theta(t')) d\sigma \frac{M}{n\varepsilon} (2D)^{n\varepsilon} \\ &\leq \frac{4DM}{n} \int_\Sigma (1 + \theta(t')) d\sigma \,. \end{aligned}$$

d. If $R(z) = 1$, $\varepsilon < |I(z)|$ and $\varepsilon < 1/2n$, then the integral

$$(9.2)\qquad c_{z-\varepsilon}\int_{|t|\geq 1} s(t)\left[\left(\frac{\theta(t)}{|t|^n}\right)^{z-\varepsilon} - \frac{(\theta(t))^{z-\varepsilon}}{|t|^{nz}}\right]\varphi(x-t)\,dt$$

is uniformly bounded. For $z \neq 1$ it converges to 0 as ε approaches 0. The integral of (9.2) is clearly dominated by $\int_{|t|\geq 1}\frac{2(1+\theta(t))}{|t|^{n-\frac{1}{2}}}\varphi(x-t)\,dt$ which is finite. Since $c_{z-\varepsilon}$ is bounded, the expression (9.2) is bounded; convergence follows from the dominated convergence theorem.

e. If $R(z) = 1$, $\varepsilon < |I(z)|$ and $\varepsilon < 1/2n$, then the integral

$$(9.3)\qquad c_{z-\varepsilon}\int_0^1 n\varepsilon\alpha^{n\varepsilon-1}d\alpha\int_{|t|\geq\alpha}\frac{s(t)(\theta(t))^{z-\varepsilon}\varphi(x-t)}{|t|^{nz}}\,dt\,.$$

has uniformly bounded L^2 norm. For $z \neq 1$ it converges in L^2 to $T_z(\varphi)$ as ε approaches 0.

As before, let the norm symbol $\|\ \|_2$ apply to the variable x. Then changing to polar coordinates the L^2 norm of (9.3) is

$$\left\|c_{z-\varepsilon}\int_\Sigma s(t')(\theta(t'))^{z-\varepsilon}\,d\sigma\int_0^1 n\varepsilon\alpha^{n\varepsilon-1}d\alpha\int_\alpha^\infty\frac{\varphi(x-rt')dr}{r^{nz-n+1}}\right\|_2\,.$$

Then applying Minkowski's integral inequality twice shows that this is less than or equal to

$$|c_{z-\varepsilon}|\int_\Sigma(1+\theta(t'))\,d\sigma\int_0^1 n\varepsilon\alpha^{n\varepsilon-1}d\alpha\left\|\int_\alpha^\infty\frac{\varphi(x-rt')dr}{r^{nz-n+1}}\right\|_2\,.$$

Using Corollary 1 and performing the integration of x first over lines parallel to t' and then over the space of such lines shows that the whole expression is bounded by

$$\frac{2|I(z)|}{1+|I(z)|^2}\int_\Sigma(1+\theta(t'))\,d\sigma\frac{C(1+|I(z)|^2)}{|I(z)|}\|\varphi\|_2 = 2C\|\varphi\|_2\int_\Sigma(1+\theta(t'))\,d\sigma\,.$$

To prove the convergence consider the expression

$$(9.4)\qquad c_{z-\varepsilon}\int_\Sigma s(t')(\theta(t'))^z d\sigma\int_0^1 n\varepsilon\alpha^{n\varepsilon-1}d\sigma\int_\alpha^\infty\frac{\varphi(x-rt')}{r^{nz-n+1}}\,dr\,.$$

This converges in L^2 norm to $T_z(\varphi)$ by Corollary 3 and Lemma 2 since its limit is the Abel summation definition of $T_z(\varphi)$ written in polar coordinates. The reasoning used above to show that (9.3) had bounded L^2 norm can be applied to the difference of (9.3) and (9.4). This shows that the L^2 norm of the difference is less than or equal to

$$2C\int_\Sigma|(\theta(t'))^{z-\varepsilon} - (\theta(t))^z|\,d\sigma\|\varphi\|_2\,,$$

and this converges to 0 as ε approaches 0. Consequently, (9.3) converges to $T_z(\varphi)$.

f. $F(z) = \int \psi T_z(\varphi)\, dx$ is analytic in $0 \leq R(z) < 1$. For $1 - R(z) \geq |I(z)|$ or $1 - R(z) \geq 1/3n$ this follows immediately from the majorizing expressions for $T_z(\varphi)$ in parts b and c. Since $T_z(\varphi)$ is a uniformly convergent integral of an analytic function in these cases, $T_z(\varphi)$ and hence $F(z)$ are analytic. For $1 - R(z) < |I(z)|$ and $1 - R(z) < 1/3n$ observe that $T_z(\varphi)$ is the sum of (9.2) and (9.3). By the same reasoning as in the other case, the integral of the product of ψ with either (9.2) or (9.3) is analytic. Therefore, the sum of these parts, $F(z)$, is analytic.

g. $F(z) = \int \psi T_z(\varphi)\, dx$ is continuous on $R(z) = 1$. By its definition $T_z(\varphi)$ is the product of c_z and the transformation of the previous sections where $s(t)(\theta(t))^z$ has replaced $\Omega(t)$ and $(nz - n)/i$ has replaced γ. Using Fourier transforms then gives $\hat{T}_z(\varphi) = c_z \hat{K}_z \hat{\varphi}$ where $\hat{K}_z$ is the function $\hat{K}$ of Lemma 4 with $\gamma = (nz-n)/i$, provided that $z \neq 1$. Using the expression (4.4) in (5.2) gives an expression for $c_z\hat{K}_z$. Its form shows that $c_z\hat{K}_z$ is uniformly bounded in x and z. Furthermore, for $0 < a \leq x \leq b < \infty$, it is also clear that $c_z\hat{K}_z$ is continuous in z, uniformly in x. Both statements remain valid if $-\frac{1}{n}\int_\Sigma s(t)\theta(t)\, d\sigma$ is used for $c_1\hat{K}_1$. Using this, it is also clear that $\hat{T}_1(\varphi) = c_1\hat{K}_1\hat{\varphi}$.

Now let z be a complex number with $R(z) = 1$, and let $\varepsilon > 0$ be arbitrary. Choose real numbers a and b so that if S consists of points in E^n whose distance from the origin lies between a and b, and S' is the complement of S in E^n, then

$$\left(\int_{S'} (\varphi(x))^2 dx\right)^{\frac{1}{2}} \leq \frac{\varepsilon}{4\|\psi\|_2 \sup\limits_{x,z} |c_z\hat{K}_z|} .$$

Let w be another complex number with $R(w) = 1$. Then

$$\begin{aligned} |F(z) - F(w)| &\leq \|\psi\|_2 \|T_z\varphi - T_w\varphi\|_2 \leq \|\psi\|_2 \|\hat{T}_z\hat{\varphi} - \hat{T}_w\hat{\varphi}\|_2 \\ &\leq \|\psi\|_2 \left(\int_{S'} \hat{\varphi}^2 dx\right)^{\frac{1}{2}} \sup_{x\in S'} |c_z\hat{K}_z - c_w\hat{K}_w| \\ &\quad + \|\psi\|_2 \left(\int_{S} \hat{\varphi}^2 dx\right)^{\frac{1}{2}} \sup_{x\in S} |c_z\hat{K}_z - c_w\hat{K}_w| . \end{aligned}$$

The first part is less than $\varepsilon/2$ and the second part approaches 0 as w approaches z. This shows the desired continuity.

h. $F(z)$ is continuous and bounded on $0 \leq R(z) \leq 1$. From parts b through e it is clear that $F(z)$ is uniformly bounded in $0 \leq R(z) < 1$ and $\lim\limits_{\varepsilon\to 0} F(z - \varepsilon) = F(z)$ for $R(z) = 1$ and $z \neq 1$. These facts, together with the analyticity and continuity on $R(z) = 1$, give the desired continuity and boundedness.

i. $\|T_{iy}\varphi\|_\infty \leq \|\varphi\|_1$ and $\|T_{1+iy}\varphi\|_p \leq A\dfrac{p^2}{p-1}\|\varphi\|_p, 1 < p < \infty$,

where y is real and A depends only on $\theta(t)$ and not on D or φ. The first is trivial. The second follows from Theorem 8 since $(1 + |y|^2)|c_{1+iy}|/|y|$ is bounded.

This completes the proof of the lemma.

THEOREM 9. *Let p, q, and λ be positive numbers such that $1 < p < q < \infty$ and $\dfrac{1}{q} = \dfrac{1}{p} - \lambda$. Let f be in L^p on E^n and $\Omega(t) = \Omega\left(\dfrac{t}{|t|}\right)$ be in L^s, $s = \dfrac{1}{1-\lambda}$, on the unit sphere. Then the integral*

$$D_\lambda(f) = \int_{E^n} \frac{\Omega(t)}{|t|^{n(1-\lambda)}} f(x-t)\,dt$$

exists for almost all x and

$$\|D_\lambda(f)\|_q \leq C\frac{1}{\lambda}\left(\frac{pq}{p-1}\right)^{1-\lambda}\|f\|_p$$

where C depends only on Ω.

Applying the theorem of Stein [4] p. 483 to the T_z with $p_1 = 1$, $q_1 = \infty$, $p_2 = q_2 = q(1-\lambda)$, $z = 1 - \lambda$ gives for simple φ,

$$\|T_z\varphi\|_q \leq A\left(\frac{p_2^2}{p_2 - 1}\right)^{R(z)}\|\varphi\|_p \,.$$

Now let $\theta(t) = |\Omega(t)|^{\frac{1}{1-\lambda}}$, and $s(t) = \operatorname{sgn}\Omega(t)$. Then dividing the above inequality by $c_{1-\lambda}$ gives

$$\begin{aligned}\|D_\lambda(\varphi)\|_q &\leq A\left(\frac{q^2(1-\lambda)^2}{q(1-\lambda)-1}\right)^{1-\lambda}\frac{1+\lambda^2}{\lambda}\|\varphi\|_p \\ &= A\left(\frac{pq(1-\lambda)^2}{p-1}\right)^{1-\lambda}\frac{1+\lambda^2}{\lambda}\|\varphi\|_p \\ &\leqq \frac{2A}{\lambda}\left(\frac{pq}{p-1}\right)^{1-\lambda}\|\varphi\|_p\end{aligned}$$

Now if $\Omega \geq 0$ all the integrands are positive. Given an arbitrary positive function f in L^p, take a sequence of simple functions φ_n that vanish off bounded sets and converge in L^p norm to f. Then taking the limit in the inequality above gives

$$|D_\lambda(f)\|_q \leq \frac{2A}{\lambda}\left(\frac{pq}{p-1}\right)^{1-\lambda}\|f\|_p \,.$$

From this $D_\lambda(f)$ exists almost everywhere. In the case where f and Ω are not positive the integrand of $D_\lambda(f)$ is majorized by a positive function that does satisfy the desired inequality. This completes the proof.

It is known that the usual fractional integration theorem and, as a result, Theorem 9 fail for the cases $p = 1$ and $q = \infty$. Zygmund [8] p. 605-6 proved substitute results for the usual fractional integral case, and these results can be extended to the present case. The proof of Theorem 10 is an adaptation of the corresponding proof in [8].

THEOREM 10. *Let $p = 1/\lambda$ be a positive number greater than 1. Let f be in L^p on E^n, vanish off a bounded set R and $||f||_p \leq 1$. Let $\Omega(t) = \Omega\left(\frac{t}{|t|}\right)$ be in L^s, $s = \frac{1}{1-\lambda}$ on the unit sphere. Then the expression $D_\lambda(f)$ exists for almost all x. Furthermore, if $\Phi(x) = e^{x^s} - x^s - 1$, there exist constants a and A, independent of f and R, such that*

$$\int_{E^n} \Phi(a|D_\lambda(f)|) \leq A|R| .$$

Using Theorem 9

$$\int_{E^n} \Phi(a|D_\lambda(f)|) \leq \sum_2^\infty \frac{a^{ns}}{n!}\int_R |D_\lambda(f)|^{ns}$$
$$\leq \sum_2^\infty \frac{1}{n!}\left(\frac{n^2}{(n-1)(1-\lambda)^2}\right)^n \left(\frac{aC}{\lambda}\right)^{ns} ||f||_{p_n}^{ns}$$

where $p_n = \frac{n}{1 - \lambda + \lambda n}$. Now using the fact that $\left(\frac{1}{|R|}\int_R |f|^p\right)^{\frac{1}{p}}$ increases with p shows that the preceding sum is less than or equal to $\sum_2^\infty \frac{(a^s Dn)^n}{n!}|R|||f||_p^{ns}$ where D is a constant independent of n, f, and R. Then using the fact that $||f||_p \leq 1$ and Stirling's formula shows that for $a^s = 1/(2eD)$ the series converges to a constant A.

THEOREM 11. *Let $q = 1/(1-\lambda)$ be a positive number, $1 < q < \infty$. Let $\Psi(x) = (1+x)[\log(1+x)]^{1-\lambda}$ and f be a function in E^n such that $\int_{E^n} \Psi(|f|)$ is finite. Let $\Omega(t) = \Omega\left(\frac{t}{|t|}\right)$ be in L^s, $s = 1/(1-\lambda)$ on the unit sphere. Then the expression $D_\lambda(f)$ exists for almost all x, and over any set R of finite measure*

$$\left(\int_R |D_\lambda(f)|^q\right)^{\frac{1}{q}} \leq A\left(|R| + \int_R \Psi(|f|)\right)$$

where A is independent of f and R.

By differentiating it is clear that $\Psi(x)$ is greater than the function conjugate to $\Phi(x) = e^{x^s} - x^s - 1$ in the sense of Young.[3] Consequently, for real positive numbers b and d, $bd \leq \Phi(b) + \Psi(d)$ by Young's inequality. Now consider a function g in L^p, $p = 1/\lambda$, vanishing outside R and with $\|g\|_p \leq 1$. Then using Theorem 10

$$\left|\int_{E^n} D_\lambda(g)f\right| \leq \frac{1}{a}\int_{E^n} a|D_\lambda(g)||f| \leq \frac{1}{a}\left(\int_{E^n}\Phi(a|D_\lambda(g)|)\right) + \int_{E^n}\Psi(|f|)$$
$$\leq \frac{1}{a}\left(A|R| + \int_{E^n}\Psi(|f|)\right).$$

However, by interchanging the order of integration

$$\left|\int_{E^n} D_\lambda(g)f\right| = \left|\int_R gD_\lambda(f)\right|.$$

Since g is an arbitrary function in L^p on R, the least upper bound for this integral is $\left(\int_R |D_\lambda(f)|^q\right)^{1/q}$ by the converse of Holder's inequality. Therefore $\left(\int_R |D_\lambda(f)|^q\right)^{1/q} \leq \frac{A|R|}{a} + \frac{1}{a}\int_{E^n}\Psi(|f|)$.

References

1. A. P. Calderon and A. Zygmund, *On the existence of certain singular integrals*, Acta Mathematica, **88** (1952), 85–139.
2. ———, *Singular integrals and periodic functions*, Studia Mathematica, **14** (1954), 249–271.
3. ———, *On singular integrals*, Amer. J. Math. **88** (1956), 289–309.
4. E. M. Stein, *Interpolation of linear operators*, Trans. Amer. Math. Soc. **83** (1956), 482–492.
5. E. M. Stein and G. Weiss, *Fractional integrals on n dimensional space*, J. Math. and Mech. **7** (1958), 503–514.
6. G. O. Thorin, *Convexity theorems*, Meddelanden fran Lunds Universitets Matematiska Seminarium, **9** (1948).
7. A. Zygmund, *Trigonometrical Series*, Second Ed., Vol. I, p. 1–383, Vol. II, p. 1–343, Cambidge University Press, 1959.
8. ———, *Some points in the theory of trigonometric and power series*, Trans. Amer. Math. Soc. **36** (1934), 586–617.

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[3] See [7] Vol. I, p. 16