EXISTENCE THEOREMS FOR CERTAIN CLASSES OF TWO-POINT BOUNDARY PROBLEMS BY VARIATIONAL METHODS

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Prefatory remarks. The principal results of this paper are existence theorems for solutions of two classes of vector differential systems; in each case the existence theorem is established by variational methods. In particular, the second system considered is a generalization of a scalar system, including as a special case the so-called Fermi-Thomas equation, studied by Sansone [8; pp. 445-450]. In spite of similarities occurring in the discussion of the two systems considered, the two problems are sufficiently distinct to warrant separate treatment. Accordingly, we shall divide the remaining sections of this paper into two parts, in which the numbering of sections and of displayed material will be independent; the bibliography, however, will apply to both parts.

Matrix notation will be used throughout and all matrices will have real elements; in particular, a vector $u = (u_j)$, $(j = 1, 2, \cdots, n)$, will be regarded as an $n \times 1$ matrix. If M is a matrix, M^* will denote the transpose of M, while for a vector $u = (u_j)$, $(j = 1, 2, \cdots, n)$, we define $|u| = (u_1^2 + \cdots + u_n^2)^{1/2}$. For $F(u)$ a scalar function of the vector u, the symbol $F_u(u)$ will denote the vector function $(F_{u_j}(u))$; if $G(u)$ is a vector function $(G_i(u))$, $(i = 1, 2, \cdots, m)$, of the vector u, then $G_u(u)$ will denote the $m \times n$ matrix $\|\partial G_i/\partial u_j\|$, $(i = 1, \cdots, m; j = 1, \cdots, n)$. If M and N are matrices, the notation $M \geq N$ is used to signify that M and N are real symmetric matrices of the same dimensions and $M - N$ is non-negative. As usual, the symbol $C^{(n)}$ represents the class of finite dimensional matrix functions which are continuous and have continuous derivatives of the first n orders on some given set.

PART I

1. Introduction. This part of the paper will be concerned with vector differential systems of the form

$$(1.1)\qquad \begin{aligned} &y'' = f(x, y, y')\,, \qquad\qquad a \leq x \leq b\,, \\ &y(a) = y_1, \quad y(b) = y_2\,, \end{aligned}$$

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where $f(x, y, z)$ is an n-dimensional real vector function of the real scalar x and the real n-dimensional vectors y and z. It will be shown that the system (1.1) has a solution, under the hypotheses H_1, H_2, H_3, H_5, of § 3 and H_4^* of § 4. For reasons of convenience, we shall work primarily with the system

$$(1.2) \qquad \begin{gathered} y'' = f(x, y, y') , \qquad a \leq x \leq b , \\ y(a) = 0 = y(b) , \end{gathered}$$

and show in § 4 how a system (1.1) may be reduced to such a system.

The existence proof will use variational methods applied to the functional

$$(1.3) \qquad I(y, z) \equiv \int_a^b (|y' - z|^2 + |z' - f(x, y, z)|^2) dx ,$$

with (y, z) in the class K of function pairs defined below. In § 2 there are listed some lemmas to be used later. In § 3 an existence theorem for a solution of (1.2) is established by showing, in effect, that $I(y, z)$ has a minimum for (y, z) in K, and that this minimum is zero. The relation between systems of the forms (1.1) and (1.2) is considered in § 4, while § 5 contains a comment on a modification of hypotheses. Finally, § 6 is devoted to an example of a class of problems to which the existence theorem proved here is applicable.

In what follows, A_2 will denote the class of vector functions $y(x)$ which are absolutely continuous and for which $|y'(x)|^2$ is integrable on $a \leq x \leq b$, while K is the class of vector function pairs (y, z) with $y(x)$ and $z(x)$ in A_2 and with $y(a) = 0 = y(b)$.

2. Some useful lemmas. For future reference we collect here certain auxiliary results.

LEMMA 2.1. *Suppose that the matrix $f_z(x, y, z)$ exists and is continuous for $a \leq x \leq b$, all y, and all z. If for each $\rho > 0$ the elements of f_z are bounded for $a \leq x \leq b$, $|y| < \rho$ and z arbitrary, then there are values $K_1 = K_1(\rho)$ and $K_2 = K_2(\rho)$ such that*

$$|f(x, y, z)| \leq K_1|z| + K_2 , \text{ for } a \leq x \leq b, \ |y| < \rho, \ z \text{ arbitrary.}$$

LEMMA 2.2. *If $\{w_m(x)\}$, $(m = 1, 2, \cdots)$, is a sequence of vector functions of class A_2 such that the two sequences $\left\{\int_a^b |w_m|^2 dx\right\}$ and $\left\{\int_a^b |w_m'|^2 dx\right\}$ are bounded, then the $w_m(x)$ are uniformly bounded on $a \leq x \leq b$, and there exists a $w(x)$ in A_2 and a subsequence $\{w_{m_j}(x)\}$ such that $w_{m_j}(x) \to w(x)$ uniformly and $w_{m_j}'(x) \to w'(x)$ weakly in the class of integrable square functions on $a \leq x \leq b$.*

This lemma is a ready consequence of well-known results for the Hilbert space of real-valued measurable functions whose squares are Lebesgue integrable on $a \leq x \leq b$, see, for example, [7; §§ 32, 99].

LEMMA 2.3. *If $y(x)$ is in A_2 and $y(a) = 0$, then*

$$\int_a^b | y' |^2 dx \geq \frac{\pi^2}{4(b-a)^2} \int_a^b | y |^2 dx .$$

This is a well-known condition on the smallest proper value of the differential system $y'' + \lambda y = 0$, $y(a) = 0 = y'(b)$. For an independent proof see [2; p. 184]; the present inequality follows from (7.7.1) of [2] by a simple change of variable.

We will also need some results related to non-oscillation of the scalar differential equation

$$(\psi_1(x)u'(x))' - \psi_2(x)u(x) = 0 , \quad a \leq x \leq b , \tag{2.1}$$

where ψ_1 is of class C' and ψ_2 continuous on $a \leq x \leq b$. The equation (2.1) is termed non-oscillatory on $a \leq x \leq b$ if for two arbitrary points x_1, x_2 satisfying $a \leq x_1 < x_2 \leq b$, any solution $u(x)$ of (2.1) vanishing at x_1 and at x_2 vanishes identically on $a \leq x \leq b$. It is well-known that if $\psi_1(x) > 0$ on $a \leq x \leq b$, then (2.1) is non-oscillatory on $a \leq x \leq b$ if and only if

$$J(u) \equiv \int_a^b (\psi_1(x)u'^2(x) + \psi_2(x)u^2(x))dx > 0 \tag{2.2}$$

holds for all non-identically vanishing $u(x)$ belonging to A_2 and satisfying $u(a) = 0 = u(b)$. For a proof of this statement see, for example, [5; Theorem 2.1], where a more general result is proved. Moreover, if (2.1) is non-oscillatory on $a \leq x \leq b$, the infimum of $J(u)$ for $u(x)$ in A_2 and satisfying $u(a) = 0 = u(b)$, $\int_a^b u^2 dx = 1$ is greater than zero, as can be seen from an indirect argument. Indeed, if the infimum were equal to zero, then there would be a sequence of functions u_j in A_2 with $u_j(a) = 0 = u_j(b)$, $\int_a^b u_j^2 dx = 1$, $j = 1, 2, \cdots$, and with $J(u_j) \to 0$. One readily verifies that the sequence $\left\{\int_a^b u_j'^2 dx\right\}$ would be bounded, so that, by Lemma 2.2, there would be a $u(x)$ in A_2 and a subsequence of $\{u_j\}$, denoted again by $\{u_j\}$, such that $u_j(x) \to u(x)$ uniformly on $a \leq x \leq b$, and $u_j'(x) \to u'(x)$ weakly on this interval. The identity

$$J(u_j) = J(u) + \int_a^b \psi_1(x)[2u'(u_j' - u') + (u_j' - u'^2)]dx + \int_a^b \psi_2(x)(u_j^2 - u^2)dx$$

would then imply that $0 = \lim J(u_j) \geq J(u)$, contrary to (2.2), since $u(a) = 0 = u(b)$ and $\int_a^b u^2 dx = 1$. With these comments one readily establishes the following result.

LEMMA 2.4. *If* (2.1) *is non-oscillatory on* $a \leq x \leq b$, *and* $\psi_1(x) > 0$ *on this interval, then there exists an* $\varepsilon > 0$ *such that if* $h(x)$ *is any function continuous and satisfying* $|h(x)| < \varepsilon$ *on* $a \leq x \leq b$, *then the equation* $(\psi_1(x)u')' - (\psi_2(x) + h(x))u = 0$ *is non-oscillatory on* $a \leq x \leq b$.

3. Existence theorem for a solution of (1.2). In the future sections we will make reference to the following hypotheses on the real-valued vector function $f(x, y, z)$:

H_1. $f(x, y, z)$ is continuous for (x, y, z) in $\Omega : a \leq x \leq b$, $|y| < \infty$, $|z| < \infty$.

H_2. *The matrices* f_y *and* f_z *exist and are continuous for* (x, y, z) *in* Ω.

H_3. *For any* $\rho > 0$, *there exists a* $K = K_\rho$ *such that* $|\partial f_i / \partial z_j| \leq K$ *for* $|y| < \rho$, $a \leq x \leq b$, $|z| < \infty$, $(i, j = 1, \cdots, n)$.

H_4. *For arbitrary* $\rho > 0$ *there exist scalar functions* $\psi_1(x) = \psi_1(x; \rho) \in C$, $\psi_2(x) = \psi_2(x; \rho) \in C'$ *with* $\psi_2(x) > 0$ *on* $a \leq x \leq b$, *and a constant* $N = N(\rho)$ *such that*:

(a) *the scalar differential equation* $(\psi_2(x)w')' - \psi_1(x)w = 0$ *is non-oscillatory on* $a \leq x \leq b$;

(b) *the integral inequality*

$$2\int_a^b y^* f(x, y, z)dx \geq \int_a^b [(\psi_2 - 1)|y'|^2 + \psi_1 |y|^2]dx - N$$

holds for all $y(x)$, $z(x)$ *in* A_2 *satisfying* $y(a) = 0 = y(b)$ *and*

$$\int_a^b |y' - z|^2 dx \leq \rho .$$

H_5. *For arbitrary* $y(x)$, $z(x)$ *in* A_2, *the vector differential system*

$$\begin{aligned} &w'' - f_z(x, y(x), z(x))w' - f_y(x, y(x), z(x))w = 0 , \quad a \leq x \leq b \\ &w(a) = 0 = w(b) \end{aligned} \tag{3.1}$$

has only the solution $w(x) \equiv 0$, $a \leq x \leq b$.

We now prove the following theorem.

THEOREM 3.1. *Under the hypotheses* H_1–H_5 *there exists a solution of the system* (1.2).

Let $\{y_m(x), z_m(x)\}$, $m = 1, 2, \cdots$, be a sequence of function pairs of class K such that $I(y_m, z_m) \to I_0$, where I_0 denotes the infimum of $I(y, z)$ on K. Since $\{I(y_m, z_m)\}$ is a convergent sequence, there exists a constant M_0 such that $I(y_m, z_m) \leq M_0$, $m = 1, 2, \cdots$. It will be shown first that the inequality

$$\int_a^b (|y_m'(x)|^2 + 2y_m^*(x)f(x, y_m(x), z_m(x)))dx \leq M , \quad m = 1, 2, \cdots \tag{3.2}$$

holds for

$$M = 2M_0/k\ , \quad \text{where} \quad k = \operatorname{Min}(1, \pi^2/[4(b-a)^2])\ . \tag{3.3}$$

Let $v_m(x) = \int_a^x f_m(s)ds + z_m(a)$, where $f_m(x) = f(x, y_m(x), z_m(x))$. Then $u_m(x) = z_m(x) - v_m(x)$ is in A_2, and $u_m(a) = 0$, so that by Lemma 2.3,

$$\int_a^b |z'_m - f_m|^2 dx = \int_a^b |u'_m|^2 dx \geq \frac{\pi^2}{4(b-a)^2} \int_a^b |u_m|^2 dx\ .$$

This inequality yields

$$M_0 \geq k\int_a^b (|y'_m - z_m|^2 + |z_m - v_m|^2)dx \geq (k/2)\int_a^b |y'_m - v_m|^2 dx\ , \tag{3.4}$$

where k is as in (3.3). Since

$$\int_a^b y'^*_m v_m dx = y^*_m v_m \Big|_a^b - \int_a^b y^*_m v'_m dx = -\int_a^b y^*_m f_m dx\ ,$$

relation (3.2), with M given by (3.3), results from (3.4) and the obvious inequality

$$\int_a^b |y'_m - v_m|^2 dx \geq \int_a^b (|y'_m|^2 - 2y^{*\prime}_m v_m)dx\ .$$

Since the sequence $\left\{\int_a^b |y'_m - z_m|^2 dx\right\}$ is bounded, we may use H_4 to write

$$2\int_a^b y^*_m f(x, y_m, z_m)dx \geq \int_a^b [(\psi_2 - 1)|y'_m|^2 + \psi_1 |y_m|^2]dx - N\ ,$$

where $\psi_2(x)$, $\psi_1(x)$, and N have the properties stated in H_4. This relation with (3.2) yields

$$\int_a^b (\psi_2 |y'_m|^2 + \psi_1 |y_m|^2)dx \leq M + N\ .$$

Since $(\psi_2(x)u')' - \psi_1(x)u = 0$ is non-oscillatory on $a \leq x \leq b$, Lemma 2.4 implies that there is an r with $0 < r < 1$ such that $(\psi_2 u')' - (1/r)\psi_1 u = 0$ is non-oscillatory on $a \leq x \leq b$. As $y_m(a) = 0 = y_m(b)$, we then have

$$\int_a^b (r\psi_2 |y'_m|^2 + \psi_1 |y_m|^2)dx \geq 0$$

and

$$\int_a^b (\psi_2 |y'_m|^2 + \psi_1 |y_m|^2)dx \geq (1-r)\int_a^b \psi_2 |y'_m|^2 dx \geq r_0 \int_a^b |y'_m|^2 dx\ ,$$

where $r_0 = (1-r)\operatorname{Min}_{a \leq x \leq b} \psi_2(x)$. Thus, the sequence $\left\{\int_a^b |y'_m|^2 dx\right\}$ is bounded, and since each $y_m(x)$ vanishes at a and b, the vector functions $y_m(x)$ are uniformly bounded on $a \leq x \leq b$. Moreover,

$$\begin{aligned}\int_a^b |z_m|^2 dx &\leq \int_a^b |y'_m + (z_m - y'_m)|^2 dx \\ &\leq 2\int_a^b |y'_m|^2 dx + 2\int_a^b |z_m - y'_m|^2 dx \\ &\leq 2\int_a^b |y'_m|^2 dx + 2M_0\end{aligned}$$

so that the sequence $\left\{\int_a^b |z_m|^2 dx\right\}$ is bounded. Finally, with $f_m(x)$ continuing to denote $f(x, y_m(x), z_m(x))$, we have

$$\begin{aligned}\int_a^b |z'_m|^2 dx &= \int_a^b |(z'_m - f_m) + f_m|^2 dx \\ &\leq 2\int_a^b |z'_m - f_m|^2 dx + 2\int_a^b |f_m|^2 dx \\ &\leq 2M_0 + 2\int_a^b |f_m|^2 dx .\end{aligned}$$

As the vector functions $y_m(x)$, $(m = 1, 2, \cdots)$, are bounded uniformly on $a \leq x \leq b$, in view of hypothesis H_3 and the result of Lemma 2.1, this latter inequality implies $\int_a^b |z'_m|^2 dx \leq K' + K''\int_a^b |z_m|^2 dx + 2M_0$, for suitable constants K', K''. Hence, the two sequence $\{y_m(x)\}$, $\{z_m(x)\}$ satisfy the hypotheses of Lemma 2.2, and we conclude that there exist subsequences, which will be denoted simply by $\{y_m(x)\}$ and $\{z_m(x)\}$, and a pair of functions $y(x)$, $z(x)$ in A_2, such that $y_m(x) \to y(x)$ and $z_m(x) \to z(x)$ uniformly on $a \leq x \leq b$, while $y'_m(x) \to y'(x)$ and $z'_m(x) \to z'(x)$ weakly on the same interval.

With $f_m(x)$ as above and $f(x) = f(x, y(x), z(x))$ we have

$$I(y_m, z_m) = I(y, z) + I_{1,m} + I_{2,m} ,$$

where

$$I_{1,m} = \int_a^b [|(z_m - z) - (y'_m - y')|^2 + |(f_m - f) - (z'_m - z')|^2] dx$$

and

$$\begin{aligned}I_{2,m} = 2\int_a^b \{&(y' - z)^*[(y'_m - y') - (z_m - z)] \\ &+ (z' - f)^*[(z'_m - z') - (f_m - f)]\} dx .\end{aligned}$$

Since $y_m(x) \to y(x)$, $z_m(x) \to z(x)$ uniformly, we also have $f_m(x) \to f(x)$ uniformly on $a \leq x \leq b$. This, and the fact that $y'_m \to y'$, $z'_m \to z'$ weakly on the same interval, implies that $I_{2,m} \to 0$ as $m \to \infty$. As $I_{1,m} \geq 0$, it follows that

$$I_0 = \lim_{m\to\infty} I(y_m, z_m) \geq I(y, z) .$$

On the other hand, the definition of I_0 requires $I_0 \leq I(y, z)$, so that $I_0 = I(y, z)$; that is, (y, z) renders $I(y, z)$ a minimum in the class of function pairs K.

It follows that for arbitrary $\eta(x)$, $\zeta(x)$ in A_2 with $\eta(a) = 0 = \eta(b)$, and θ a real parameter, the functional $I(y + \theta\eta, z + \theta\zeta)$ has a minimum at $\theta = 0$, and therefore $(d/d\theta)I(y + \theta\eta, z + \theta\zeta) = 0$ for $\theta = 0$; that is,

$$\int_a^b [(y^{*\prime} - z^*)(\eta' - \zeta) + (z^{*\prime} - f^*)(\zeta' - f_y\eta - f_z\zeta)]dx = 0 , \tag{3.5}$$

where the arguments of f, f_y, f_z are x, $y(x)$, $z(x)$.

In view of H_5, (see [4; pp. 213–214]), for an arbitrary continuous function $g(x)$, $a \leq x \leq b$, there exists a solution $(\eta(x), \zeta(x))$ of the differential system

$$\begin{aligned} &\eta' - \zeta = 0 , \\ &\zeta' - f_y(x, y(x), z(x))\eta - f_z(x, y(x), z(x))\zeta = g(x) , \quad a \leq x \leq b , \\ &\eta(a) = 0 = \eta(b) . \end{aligned}$$

Therefore, $\int_a^b [z^{*\prime} - f^*(x, y, z)]g(x)dx = 0$ for arbitrary $g(x)$ continuous on $a \leq x \leq b$, and consequently $z'(x) - f(x, y(x), z(x)) = 0$ a.e. on the same interval. Relation (3.5), with $\eta(x)$ chosen identically zero on $a \leq x \leq b$, then yields $\int_a^b \zeta^*(y' - z)dx = 0$ for arbitrary ζ in A_2, and hence $y'(x) = z(x)$ a.e. on $a \leq x \leq b$. From the relations $z(x) = z(a) + \int_a^x f(s, y(s), z(s))ds$, $y(x) = \int_a^x z(s)ds$, it then follows that $y(x)$ and $z(x)$ are of class C', and that $y'(x) = z(x)$, $z'(x) = f(x, y(x), z(x))$ for $a \leq x \leq b$, so that $y(x)$ is of class C'' and satisfies (1.2).

4. Existence of a solution of (1.1). For the system

$$\begin{aligned} y'' &= f(x, y, y') , \qquad a \leq x \leq b , \\ y(a) &= y_1 , \qquad y(b) = y_2 , \end{aligned} \tag{1.1}$$

let $F(x, y, z) \equiv f(x, y + \lambda(x), z + \lambda'(x)) - \lambda''(x)$, where $\lambda(x)$ is any vector function of class C'' on $a \leq x \leq b$ satisfying $\lambda(a) = y_1$, $\lambda(b) = y_2$. Then (1.1) is equivalent, with $u = y - \lambda$, to

$$\begin{aligned} u'' &= F(x, u, u') , \qquad a \leq x \leq b , \\ u(a) &= 0 = u(b) . \end{aligned} \tag{4.1}$$

This leads us to formulate the following hypothesis.

H_4^*. *There exists $\lambda(x)$ of class C'' on $a \leq x \leq b$ with $\lambda(a) = y_1$, $\lambda(b) = y_2$, and such that for arbitrary $\rho > 0$ there exist scalar functions*

$\psi_1(x) = \psi_1(x; \rho)$, continuous on $a \leq x \leq b$, $\psi_2(x) = \psi_2(x; \rho)$ of class C' on $a \leq x \leq b$ with $\psi_2(x) > 0$, and a constant $N = N(\rho)$ such that:

(a) *the scalar differential system $(\psi_2(x)w')' - \psi_1(x)w = 0$ is non-oscillatory on $a \leq x \leq b$;*

(b) *the integral inequality*

$$2\int_a^b y^* f(x, y + \lambda, z + \lambda')dx \geq \int_a^b [(\psi_2 - 1)|y'|^2 + \psi_1 |y|^2]dx - N$$

holds for all $y(x)$, $z(x)$ in A_2 satisfying $y(a) = 0 = y(b)$ and

$$\int_a^b |y' - z|^2 dx < \rho .$$

THEOREM 4.1. *Under hypotheses H_1, H_2, H_3, H_4^*, H_5, the system* (1.1) *has a solution.*

Let $F(x, y, z) = f(x, y + \lambda(x), z + \lambda'(x)) - \lambda''(x)$, where $\lambda(x)$ is the function described in H_4^*. Clearly, $F(x, y, z)$ satisfies H_1, H_2, H_3. Since f satisfies H_4^*, we have

$$2\int_a^b y^* f(x, y + \lambda, z + \lambda')dx \geq \int_a^b [(\psi_2 - 1)|y'|^2 + \psi_1 |y|^2]dx - N$$

for arbitrary $y(x)$, $z(x)$ satisfying $y(a) = 0 = y(b)$ and $\int_a^b |y' - z|^2 dx \leq \rho$. Hence, for such $y(x)$, $z(x)$ we have

$$\begin{aligned} 2\int_a^b y^* F(x, y, z)dx &\geq \int_a^b [(\psi_2 - 1)|y'|^2 + \psi_1 |y|^2]dx - N - 2\int_a^b y^* \lambda''(x)dx \\ &\geq \int_a^b [(\psi_2 - 1)|y'|^2 + (\psi_1 - \varepsilon)|y|^2]dx \\ &\quad - \left(N + \frac{1}{\varepsilon}\int_a^b |\lambda''|^2 dx\right), \end{aligned}$$

for any $\varepsilon > 0$. But by Lemma 2.4, ε can be chosen so small that $(\psi_2 w')' - (\psi_1 - \varepsilon)w = 0$ is still non-oscillatory on $a \leq x \leq b$. Thus, $F(x, y, z)$ satisfies H_4. Finally, one easily verifies that if $f(x, y, z)$ satisfies H_5 then $F(x, y, z)$ satisfies H_5. Thus, whenever $f(x, y, z)$ satisfies the hypotheses of Theorem 4.1, the corresponding function $F(x, y, z)$ of (4.1) satisfies the hypotheses of Theorem 3.1, so that the result of Theorem 4.1 is a direct corollary of Theorem 3.1.

5. A comment on altered hypotheses. We note here that hypothesis H_4 is implied by the more restrictive but simpler hypotheses H_4' and H_4''.

H_4'. *There exists a constant C such that*

$$|y^*(f(x, y, z_2) - f(x, y, z_1))| \leq C|y||z_2 - z_1| , \textit{ for } (x, y, z_1), (x, y, z_2) \textit{ in } \Omega.$$

H_4''. *There exist scalar functions $\psi_1(x)$, continuous on $a \leq x \leq b$,* *and* $\psi_2(x) > 0$ *of class* C' *on* $a \le x \le b$, *and a constant* N *such that:*

(a) *the scalar differential system* $(\psi_2(x)w')' - \psi_1(x)w = 0$ *is non-oscillatory on* $a \le x \le b$;

(b) *the integral inequality*

$$2\int_a^b y^* f(x, y, y')dx \ge \int_a^b [(\psi_2 - 1)|\, y'\,|^2 + \psi_1 |\, y\,|^2]dx - N$$

holds for all $y(x)$ *in* A_2 *satisfying* $y(a) = 0 = y(b)$.

To see that H_4 is implied by H_4' and H_4'' (assuming, of course, H_1, H_2), for $y(x)$, $z(x)$ in A_2 and $\varepsilon > 0$ we write,

$$\begin{aligned} 2\int_a^b y^* f(x, y, z)dx &= 2\int_a^b y^* f(x, y, y')dx + 2\int_a^b y^*[f(x, y, z) - f(x, y, y')]dx \\ &\ge 2\int_a^b y^* f(x, y, y')dx - 2C\int_a^b |\, y\,|\,|\, y' - z\,|\, dx \\ &\ge 2\int_a^b y^* f(x, y, y')dx - C\varepsilon\int_a^b |\, y\,|^2 dx - (C/\varepsilon)\int_a^b |\, y' - z\,|^2 dx \\ &\ge \int_a^b [(\psi_2 - 1)|\, y'\,|^2 + (\psi_1 - C\varepsilon)|\, y\,|^2]dx - [(C\rho)/\varepsilon + N] \end{aligned}$$

for all $y(x)$, $z(x)$ in A_2 with $y(a) = 0 = y(b)$ and $\int_a^b |\, y' - z\,|^2 dx < \rho$. Since ε can be chosen so small that $(\psi_2 w')' - (\psi_1 - \varepsilon C)w = 0$ is still non-oscillatory on $a \le x \le b$, we see that H_4' and H_4'' imply H_4.

It is to be noted that if the elements of $f_z(x, y, z)$ are bounded for (x, y, z) in Ω, then $f(x, y, z)$ satisfies both H_3 and H_4'.

6. An example. Let $f(x, y, z) = g(x, y)(1 + z^2)^{1/2}$, where z is a scalar and $g(x, y)$ is a scalar function of the scalars x and y satisfying the conditions:

(6.1)
(a) $g(x, y)$ *and* $g_y(x, y)$ *are continuous for* $a \le x \le b$, $-\infty < y < \infty$;
(b) $g_y(x, y) \ge 0$ *for* $a \le x \le b$, $-\infty < y < \infty$;
(c) *there exists a constant* $A > 0$ *such that if* $|\, y\,| \ge A$ *then*

$$yg(x, y) \ge 0\,, \qquad a \le x \le b\,.$$

One may verify that $f(x, y, z)$ satisfies hypotheses H_1, H_2, H_3, H_4^*, and H_5.

PART II

1. Introduction. Sansone [8; pp. 445–450] has proved the existence and uniqueness of a solution of the scalar differential system

$$\begin{aligned} &y'' = \psi(x)\phi(x, y)\,, \qquad 0 < x < \infty\,, \\ (1.1) \qquad &y(0) = y_0\,, \qquad y(+\infty) = 0\,, \\ &y \in C' \quad \text{on} \quad 0 \le x < \infty\,, \end{aligned}$$

under assumptions which are related to hypotheses H_1–H_6 (see §§ 2, 7) of this paper. The product $\psi(x)\phi(x, y)$ appears in (1.1) to facilitate stating the hypotheses in such a way as to include the Fermi-Thomas system (see [8; p. 445]),

$$\begin{gathered} y'' = x^{-1/2}y^{3/2} , \\ y(0) = 1 , \qquad \lim_{x\to\infty} y(x) = 0 . \end{gathered} \tag{1.2}$$

In this paper we consider solutions of a vector differential system, for which we prove an existence and uniqueness theorem which includes the results of Sansone.

The proof given in [8] may be considered in two parts. In the first part the author proves, in effect, that under his hypotheses the system

$$\begin{aligned} & y'' = \psi(x)\phi(x, y) , \qquad 0 < x < \infty , \\ & y(0) = y_0 , \quad y(x) \textit{ bounded on } 0 \le x < \infty, \\ & y \in C' \quad \textit{on} \quad 0 \le x < \infty , \end{aligned} \tag{1.3}$$

has a unique solution. Essential to Sansone's proof of this result is the fact that his hypotheses guarantee a local uniqueness property for solutions of

$$y'' = \psi(x)\phi(x, y) ; \tag{1.4}$$

that is, under his hypotheses, (1.4) has for $0 \le x_0 < \infty$ exactly one solution satisfying $y(x_0) = y_0$, $y'(x_0) = y'_0$. The hypotheses of the present paper, however, are not strong enough to imply such local uniqueness, as will be shown by an example in § 2. In the second phase of his proof, Sansone appeals to hypotheses which are designed to guarantee that the bounded solutions of (1.3) actually satisfy (1.1). In this paper we make a similar step, but again our hypothesis is weaker than the corresponding ones in [8], as will be made clear in § 7.

Sections 2–5 of this paper present an existence and uniqueness theorem for a solution of the vector generalization of Sansone's system mentioned above. This proof is primarily by variational methods, and the solutions are shown to be characterized by an extremal property. In § 6 there is given a different characterization of these solutions, while § 7 contains several theorems relating to the asymptotic behavior of solutions. Finally, § 8 is devoted to properties of solutions of (2.1) as functions of initial values.

2. Formulation of the problem. Let $g(x, y)$ be a real-valued scalar function of the scalar x and the n-dimensional vector $y = (y_j)$. We will denote by $g_y(x, y)$ the vector $(g_{y_j}(x, y))$, and consider the problem of solving the vector differential system

$$(2.1)\qquad \begin{aligned} &y''(x) = g_y(x, y(x)) \,, \qquad 0 < x < \infty \,, \\ &y(0) = y_0 \,, \qquad y(x) \textit{ bounded on } 0 \leq x < \infty \,, \end{aligned}$$

where $y(x) \in C'$ on $0 \leq x < \infty$ and $y(x) \in C''$ on $0 < x < \infty$. We will suppose that $g(x, y)$ has the form $g(x, y) = \psi(x)G(x, y)$, where $\psi(x)$ and $G(x, y)$ are real-valued functions which satisfy the following hypotheses:

H_1. *$G(x, y)$ is continuous in (x, y) on Ω: $0 \leq x < \infty$, $|y| < \infty$, and $G(x, 0) \equiv 0$ for $0 \leq x < \infty$.*

H_2. *$G_y(x, y)$ exists and is continuous in (x, y) on Ω.*

H_3. *$y^*G_y(x, y) \geq 0$ for (x, y) on Ω.*

H_4. *$\eta^*[G_y(x, y + \eta) - G_y(x, y)] \geq 0$ for (x, y), (x, η) on Ω.*

H_5. *$\psi(x)$ is continuous and positive for $x > 0$ and integrable on any finite closed interval $0 \leq x \leq A$.*

It is to be noted that $g(x, y)$ may satisfy H_1–H_5 without the equation $y'' = g_y(x, y)$ having the local uniqueness property mentioned in § 1. Indeed, if we take

$$g(x, y) = \begin{cases} 8y^{3/2} \,, & y \geq 0 \,, \\ 0 \,, & y \leq 0 \,, \end{cases}$$

so that

$$g_y(x, y) = \begin{cases} 12y^{1/2} \,, & y \geq 0 \,, \\ 0 \,, & y \leq 0 \,, \end{cases}$$

it is easily verified that $g(x, y)$ satisfies H_1–H_5, with $\psi(x) \equiv 1$. However, the function $y_1(x) = (x - x_0)^4$, $x_0 > 0$, satisfies the equation $y''(x) = g_y(x, y(x))$, as does the function $y_2(x) \equiv 0$. Since we have $y_1(x_0) = y_2(x_0)$, $y_1'(x_0) = y_2'(x_0)$, it follows that the local uniqueness property does not obtain here.

A few consequences of the above hypotheses are worthy of comment. First, observe that H_2 and H_3 imply $G_y(x, 0) \equiv 0$ for $0 \leq x < \infty$. Also, since $G(x, 0) \equiv 0$ by H_1, and

$$G(x, y) = \int_0^1 \left[\frac{d}{ds} G(x, sy) ds \right] = \int_0^1 y^*G_y(x, sy)ds = \int_0^1 s^{-1}(sy^*G_y(x, sy))ds \,,$$

we have by H_3 that $G(x, y) \geq 0$ on Ω. Moreover, if $y(x)$ is continuous on $0 \leq x \leq A$, $A > 0$, and $y''(x)$ exists and satisfies $y''(x) = g_y(x, y(x))$ for $0 \leq x \leq A$, then $y \in C'$ on $0 \leq x \leq A$. To see this we write

$$y'(x) = y'(A) - \int_x^A y''(t)dt = y'(A) - \int_x^A \psi(t)G_y(t, y(t))dt \,, \quad 0 < x \leq A \,.$$

Hence, $\lim_{x \to 0} y'(x)$ exists. This, with the fact that $y(x)$ is continuous for $0 \leq x \leq A$, implies $y'(0)$ exists and that $\lim_{x \to 0} y'(x) = y'(0)$.

Next we note that if $G(x, y)$ satisfies H_1 and H_2, then H_4 is equivalent to the statement that $G(x, y)$ is convex in y; that is,

$$G(x, y_2) - G(x, y_1) - (y_2 - y_1)^* G_y(x, y_1) \geq 0$$

for arbitrary (x, y_1), (x, y_2) in Ω. Finally, we note that the condition $G(x, 0) \equiv 0$ of H_1 is no essential restriction, since if $G(x, y)$ satisfies H_1–H_5 with the exception of this condition, then the function $G_1(x, y) \equiv G(x, y) - G(x, 0)$ satisfies H_1–H_5 and presents the same differential system (2.1).

3. Some properties of solutions. In addition to the system (2.1), we will consider also the system

$$\begin{aligned} y''(x) &= g_y(x, y(x)) , \qquad 0 \leq a \leq x \leq b , \\ y(a) &= y_a , \qquad y(b) = y_b , \end{aligned} \tag{3.1}$$

where y is of class C'' on $a \leq x \leq b$, with the obvious modification in case $a = 0$. For these systems we prove the following theorem.

THEOREM 3.1. *Under hypotheses H_1–H_5, the systems* (2.1) *and* (3.1) *have at most one solution.*

We will give the proof for (2.1); the proof for (3.1) is similar. If $y_1(x)$ and $y_2(x)$ are two solutions of (2.1), let $\eta(x) = y_1(x) - y_2(x)$. By H_4 and H_5 we have for $0 < x < \infty$,

$$0 \leq \eta^*[g_y(x, y_2 + \eta) - g_y(x, y_2)] = \eta^*[g_y(x, y_1) - g_y(x, y_2)] = \eta^*\eta'' ,$$

and hence,

$$\int_0^x \eta^*(t)\eta''(t)dt \geq 0 , \qquad 0 < x < \infty .$$

Consequently, upon integration by parts, we get

$$\eta^*(x)\eta'(x) \geq \int_0^x |\eta'|^2 dt \geq 0 .$$

Since $(|\eta|^2)' = 2\eta^*\eta'$ and $(|\eta^2|)'' = 2|\eta'|^2 + 2\eta^*\eta''$, it then follows that

$$(|\eta(x)|^2)' \geq 0 , \quad \text{and} \quad (|\eta(x)|^2)'' \geq 0 , \quad 0 \leq x < \infty .$$

Consequently, either $\eta(x) \equiv 0$, $0 \leq x < \infty$, or else $|\eta(x)| \to \infty$ as $x \to \infty$. Since the latter is impossible, (2.1) has at most one solution.

The following result will be of use later.

LEMMA 3.1. *If $g(x, y)$ satisfies H_1–H_5, and $y(x)$ is a solution of $y''(x) = g_y(x, y(x))$ on $0 < x < \infty$ with $\int_0^\infty |y'|^2 dx < \infty$, then $y(x)$ is bounded on $0 \leq x < \infty$.*

If $y(x)$ satisfies $y'' = g_y(x, y)$, then, since $(|y|^2)'' = 2|y'|^2 + 2y^*y'' = 2|y'|^2 + 2y^*g_y(x, y) \geq 0$, we know that either there is an x_1 such that

$|y| \equiv 0$ for $x \geq x_1$, or else there is an x_2 such that $|y| \neq 0$ for $x \geq x_2$. In the latter case we have

$$|y||y|' = y^*y' , \qquad x \geq x_2 ,$$

and

$$|y|^3|y|'' = |y|^2(y^*y'') + (|y|^2|y'|^2 - (y^*y')^2) \geq 0 , \qquad x \geq x_2 ,$$

since $y^*y'' = y^*g_y(x, y) \geq 0$. Hence, either $|y|' \leq 0$ for $x \geq x_2$, in which case $\lim_{x\to\infty}|y(x)| \leq |y(x_2)|$, or there is an $\alpha > 0$ and an $x_3 \geq x_2$ such that $|y|' \geq \alpha > 0$ for $x \geq x_3$. In this latter case, for $x \geq x_3$ we have $|y||y'| \geq y^*y' = |y||y|' \geq \alpha|y|$, so that $|y'| \geq \alpha > 0$ and consequently $\int_0^\infty |y'|^2dx = \infty$. Since this is the only case in which $y(x)$ would be unbounded, we conclude that if $y(x)$ is unbounded then $\int_0^\infty |y'|^2dx = \infty$.

4. A preliminary existence theorem. In what follows $I(y; a, b)$ will denote the functional

$$I(y; a, b) = \int_a^b [|y'|^2 + 2g(x, y)]dx , \qquad y(x) \text{ in } K(a, b) ,$$

where $K(a, b)$ is the class of absolutely continuous vector functions $y(x)$ with $|y'(x)|^2$ integrable on $a \leq x \leq b$, and satisfying $y(a) = y_a$, $y(b) = y_b$. We prove the following result.

THEOREM 4.1. *If $g(x, y)$ satisfies hypotheses H_1–H_5, then for any a, b satisfying $0 \leq a < b$, the system* (3.1) *has a unique solution. Moreover, this solution is a unique minimizing function for $I(y; a, b)$ in the class $K(a, b)$.*

By H_5 and the fact that $g(x, y) \geq 0$, we see that $I(y; a, b) \geq 0$ for y in $K(a, b)$. Let $\bar{I}(a, b)$ denote the infimum of $I(y; a, b)$ for y in $K(a, b)$, and let $\{y_m(x)\}$ be a sequence of elements of $K(a, b)$ such the $I(y_m; a, b) \to \bar{I}(a, b)$. As $g(x, y_m(x)) \geq 0$ on $a < x \leq b$, we have

$$\int_a^b |y_m'|^2dx = I(y_m; a, b) - 2\int_a^b g(x, y_m)dx \leq I(y_m; a, b) ,$$

so that there exists an N such that $\int_a^b |y_m'|^2dx < N$ for $m = 1, 2, \cdots$. Moreover, for $a \leq x \leq b$,

$$|y_m(x) - y_a|^2 = \left|\int_a^x y_m'(t)dt\right|^2 \leq (x - a)\int_0^x |y_m'|^2dt \leq (b - a)\int_a^b |y_m'|^2dt ,$$

so that $|y_m(x) - y_a| \leq [(b - a)N]^{1/2}$, and hence $|y_m(x)| \leq |y_a| + [(b - a)N]^{1/2}$. Consequently, we may use Lemma 2.2 of Part I to conclude that there is a subsequence, which we will denote again by $\{y_m(x)\}$, and a function

$y(x)$ in $K(a, b)$, such that $y_m(x) \to y(x)$ uniformly on $a \leq x \leq b$, and $y'_m(x) \to y'(x)$ weakly on this interval.

From the identity

$$I(y_m; a, b) - I(y; a, b) = \int_a^b [|y'_m - y'|^2 + 2(g(x, y_m) - g(x, y)) + 2(y'_m - y')^* y'] dx ,$$

and the fact that $y_m(x) \to y(x)$ uniformly on $a \leq x \leq b$ while $y'_m(x) \to y'(x)$ weakly on this interval, one obtains the lower semi-continuity relation

$$\bar{I}(a, b) = \lim_{m \to \infty} I(y_m; a, b) \geq I(y; a, b) .$$

Since the definition of $\bar{I}(a, b)$ requires that $\bar{I}(a, b) \leq I(y; a, b)$, we see that $\bar{I}(a, b) = I(y; a, b)$; that is, $y(x)$ minimizes $I(y; a, b)$ in the class $K(a, b)$.

It follows that if $\eta(x)$ is absolutely continuous with $\eta(a) = 0 = \eta(b)$ and $|\eta'(x)|^2$ is integrable on $a \leq x \leq b$, and θ is a real parameter, then $I(y + \theta\eta; a, b)$ has a minimum at $\theta = 0$. From this it follows that $(d/d\theta)I(y + \theta\eta; a, b) = 0$ at $\theta = 0$; that is,

$$\int_a^b [\eta'^* y' + \eta^* g_y(x, y)] dx = 0 .$$

In particular, this last equality holds for arbitrary η of class C'' on a $\leq x \leq b$ with $\eta(a) = 0 = \eta(b) = \eta'(a) = \eta'(b)$, and for such an η integration by parts leads to

$$\int_a^b \eta''^* \Big[y(x) - \int_a^x ds \int_a^s g_y(t, y(t)) dt \Big] dx = 0 . \tag{4.1}$$

By the fundamental lemma of the calculus of variations, there exist constant vectors ξ_1 and ξ_2 such that

$$y(x) = \int_a^x ds \int_a^s g_y(t, y(t))\, dt + \xi_1 x + \xi_2 , \qquad a \leq x \leq b .$$

Therefore, $y''(x)$ exists and satisfies

$$y''(x) = g_y(x, y(x)) , \qquad a \leq x \leq b ,$$

with the understanding that if $a = 0$, then $y''(x)$ may fail to exist at $x = 0$. Since $y(a) = y_a$, $y(b) = y_b$, it follows that $y(x)$ satisfies (3.1). The uniqueness of this solution follows from Theorem 3.1. Moreover, since the above argument shows that any function of class $K(a, b)$ that minimizes $I(y; a, b)$ is a solution of (3.1), it follows that the above determined $y(x)$ is the unique minimizing function for $I(y; a, b)$ in $K(a, b)$.

5. An existence theorem for a solution of (2.1). In what follows, K will denote the class of absolutely continuous vector functions $y(x)$ with $|y'|^2$ integrable on $0 \leq x < \infty$ and satisfying $y(0) = y_0$, $I(y; 0, \infty) < \infty$, where

$$I(y; 0, \infty) = \int_0^\infty [|y'|^2 + 2g(x, y)]dx .$$

We now prove the following result. .

THEOREM 5.1. *Under hypotheses H_1-H_5 the system* (2.1) *has a unique solution; moreover, this solution is a unique minimizing function for $I(y; 0, \infty)$ in the class K.*

Let $\{y_m(x)\}$, $m = 1, 2, \cdots$, be a sequence of functions in K such that $I(y_m; 0, \infty) \to \bar{I}$, where $\bar{I}$ denotes the infimum of $I(y; 0, \infty)$ for y in K. Then the non-negativeness of $g(x, y)$ implies that the sequence $\left\{\int_0^\infty |y'_m|^2 dx\right\}$ is bounded, and since $y_m(0) = y_0$ for every m, as in the proof of Theorem 4.1, the $y_m(x)$ are uniformly bounded on each finite interval. Hence, by Lemma 2.2 of Part I, there is a subsequence, say $\{y_m(x)\}$ again, and an absolutely continuous function $y(x)$, such that on each finite interval $y_m(x) \to y(x)$ uniformly, and $y'_m(x) \to y'(x)$ weakly. Now for any $A > 0$ we have $I(y_m; 0, \infty) \geq I(y_m; 0, A)$; moreover, as in § 4 we have

$$I(y_m; 0, A) - I(y; 0, A) \geq 2\int_0^A [(g(x, y_m) - g(x, y)) + (y'_m - y')^* y']dx$$

and consequently $\liminf_{m\to\infty} I(y_m; 0, A) \geq I(y; 0, A)$. Hence

$$\bar{I} = \lim_{m\to\infty} I(y_m; 0, \infty) \geq I(y; 0, A) , \qquad A > 0 ,$$

and finally,

$$\bar{I} \geq I(y; 0, \infty) = \lim_{A\to\infty} I(y; 0, A) .$$

In particular, this latter relation implies that $y(x)$ is in K, and in view of the definition of $\bar{I}$ we have $I(y; 0, \infty) \geq \bar{I}$, so that $I(y; 0, \infty) = \bar{I}$. That is, $y(x)$ minimizes $I(y; 0, \infty)$ in the class K.

Now on any finite interval $0 \leq x \leq A$, the thus determined $y(x)$ must coincide with the unique vector function which minimizes $I(y; 0, A)$ in the class $K(0, A)$ of curves joining $(0, y_0)$ and $(A, y(A))$, for otherwise one could piece together a curve which would give $I(y; 0, \infty)$ a smaller value than does $y(x)$. By Theorem 4.1 it then follows that $y(x)$ satisfies $y''(x) = g_y(x, y(x))$ on $0 < x \leq A$, where A is arbitrary, and consequently $y''(x) = g_y(x, y(x))$ on $0 < x < \infty$. Since $I(y; 0, \infty)$ is finite, Lemma 3.1 implies that $y(x)$ is bounded on $0 \leq x < \infty$ and therefore is a solution of (2.1). The uniqueness of this solution follows from Theorem 3.1, Inasmuch as we have actually shown that any $y(x)$ that minimizes $I(y; 0, \infty)$ in K is a solution of (2.1), the uniqueness of $y(x)$ as a minimizing function follows from its uniqueness as a solution of (2.1).

6. A further characterization of solutions of (2.1).

THEOREM 6.1. *Suppose that hypotheses H_1–H_5 are satisfied, and $y_\infty(x)$ is the unique solution of* (2.1) *guaranteed by Theorem* 5.1. *If, for a given vector, ξ, $y = y_N(x, \xi)$, $0 \leq x \leq N$, is the solution of*

$$y'' = g_y(x, y(x)) \,, \tag{6.1a}$$

$$y(0) = y_0 \,, \qquad y(N) = \xi \,, \qquad N = 1, 2, \cdots, \tag{6.1b}$$

then $y_N(x, \xi) \to y_\infty(x)$ and $y'_N(x, \xi) \to y'_\infty(x)$ uniformly on each subinterval $0 \leq x \leq A$.

We will suppose in what follows that the definition of $y_N(x, \xi)$ has been extended so that $y_N(x, \xi) = \xi$ for $x \geq N$. The inequality $(|\, y_N(x, \xi)\,|^2)'' \geq 0$ and the end conditions (6.1b) then imply that

$$|\, y_N(x, \xi)\,| \leq \operatorname{Max}(|\, y_0\,|, |\, \xi\,|) \,, \quad 0 \leq x < \infty \,, \quad N = 1, 2, \cdots. \tag{6.2}$$

Moreover, the identity

$$y'_N(x, \xi) = \frac{1}{A}\Big[y_N(A, \xi) - y_0 - \int_0^A ds \int_x^s g_y(t, y_N(t, \xi))dt \Big] \,, \tag{6.3}$$
$$0 \leq x \leq A \,, \quad N > A \,,$$

shows that the sequence $\{|\, y'_N(x, \xi)\,|\}$ is uniformly bounded on $0 \leq x \leq A$. Consequently, the sequence $\{y_N(x, \xi)\}$ is uniformly bounded and equicontinuous on any finite interval, so that we may select a subsequence $\{y_{N_j}(x, \xi)\}$ which converges uniformly on any finite interval to a continuous function $y(x)$. From (4.1) it follows that if $\eta(x)$ is of class C'' on $0 \leq x < \infty$, and $\eta(0) = 0 = \eta'(0) = \eta'(A)$, $\eta(x) \equiv 0$ for $x \geq A$, then

$$\int_0^A \eta''^* \Big[y_{N_j}(x, \xi) - \int_0^x ds \int_0^s g_y(t, y_{N_j}(t, \xi))dt \Big] dx = 0 \,, \quad N > A \,.$$

Since $y_{N_j}(x, \xi) \to y(x)$ uniformly on $0 \leq x \leq A$, we then have

$$\int_0^A \eta''^* \Big[y(x) - \int_0^x ds \int_0^s g_y(t, y(t))dt \Big] dx = 0 \,.$$

As before, application of the fundamental lemma of the calculus of variation yields the result that $y''(x)$ exists and $y'' = g_y(x, y)$ for $0 < x \leq A$. Since A is arbitrary, it follows that $y''(x) = g_y(x, y(x))$ on $0 < x < \infty$. Moreover, $y(0) = y_0$, while the relation (6.2) shows that $y(x)$ is bounded on $0 \leq x < \infty$, so that in view of Theorem 5.1 we have $y(x) = y_\infty(x)$.

Now for $0 \leq x_0 < \infty$, let η be any accumulation point of the bounded sequence $\{y_N(x_0, \xi)\}$, and let a subsequence $\{y_{N_i}(x_0, \xi)\}$ be chosen such that $y_{N_i}(x_0, \xi) \to \eta$. Then, as before, the sequence $\{y_{N_i}(x, \xi)\}$ is uniformly bounded and equicontinuous on any finite interval, so that we may select a subsequence which approaches $y_\infty(x)$ on $0 \leq x < \infty$. Consequently, the sequence $\{Y_N(x_0, \xi)\}$ has only one accumulation point, namely $\eta = y_\infty(x_0)$, from which it follows that $y_N(x, \xi) \to y_\infty(x)$ for $0 \leq x < \infty$.

With $\zeta_N(x) = y_N(x, \xi) - y_\infty(x)$, as in the proof of Theorem 3.1 we have that $(|\zeta_N(x)|^2)' \geq 0$, $0 \leq x \leq N$. This implies that for any $A > 0$ and $N > A$ we have $|\zeta_N(x)| \leq |\zeta_N(A)|$ on $0 \leq x \leq A$, and thus $y_N(x, \xi) \to y_\infty(x)$ uniformly on $0 \leq x \leq A$.

The fact that $y'_N(x, \xi) \to y'_\infty(x)$ uniformly on $0 \leq x \leq A$ now follows from (6.3), and the corresponding identity obtained by replacing $y_N(x, \xi)$ by $y_\infty(x)$.

7. Asymptotic behavior of solutions of (2.1). At this point we introduce the following hypotheses:

H_6. *For each $c > 0$ there is an $x_c \geq 0$ and a $\Psi(x, c) \geq 0$ with $x\Psi(x, c)$ integrable on every finite subinterval of $x_c \leq x < \infty$, $\int_{x_c}^\infty x\Psi(x, c)dx = \infty$, and such that for $x \geq x_c$, $|y| \geq c$ we have $y^* g_y(x, y) \geq \Psi(x, c)$.*

H_7. *If $y(x)$ is in C' and $|y(x)| \geq c > 0$ for $0 \leq x < \infty$, then $I(y(x); 0, \infty) = \infty$.*

We have the following result.

THEOREM 7.1. *If in addition to H_1–H_5, either H_6 or H_7 is also satisfied, then any solution of* (2.1) *approaches zero as $x \to \infty$.*

If $y(x)$ is a solution of (2.1), then $(|y|^2)'' = 2y^*y'' + 2|y'|^2 \geq 0$. Since $y(x)$ is bounded on $0 \leq x < \infty$, it follows that $(|y|^2)' \leq 0$, so that either $|y(x)|$ is bounded away from zero or else $y(x) \to 0$. If H_7 is satisfied then, in view of the fact that $I(y(x); 0, \infty)$ is finite for $y(x)$ a solution of (2.1), it follows that $|y(x)|$ cannot be bounded from zero; that is, $y(x) \to 0$.

Suppose now that H_6 is satisfied. As was noted in the preceding paragraph, $(|y|^2)'$ is non-decreasing and non-positive, so that $\lim_{x\to\infty}(|y|^2)'$ exists. This limit is zero, since $|y|^2$ is non-negative, and hence $\lim_{x\to\infty} y^*y' = 0$. This fact leads to the following relations,

$$-2y^*(x)y'(x) = \int_x^\infty (|y|^2)''dt = 2\int_x^\infty (y^*y'' + |y'|^2)dt ,$$

$$-2y^*(x)y'(x) = 2\int_x^\infty (y^*(t)g_y(t, y(t)) + |y'(t)|^2)dt ,$$

$$-y^*(x)y'(x) \geq \int_x^\infty y^*(t)g_y(t, y(t))dt .$$

Integration now yields

$$\frac{1}{2}|y(x)|^2 - \frac{1}{2}|y(A)|^2 \geq \int_x^A ds \int_s^\infty y^* g_y dt \ ,$$

and hence

$$\frac{1}{2}|y(x)|^2 \geq \int_x^A ds \int_s^\infty y^* g_y dt \ .$$

Finally, upon integration by parts we obtain

$$\frac{1}{2}|y(x)|^2 \geq A\int_A^\infty y^* g_y dt - x\int_x^\infty y^* g_y dt + \int_x^A s y^*(s) g_y(s, y(s)) ds \ .$$

If there is a $c > 0$ such that $|y(x)| \geq c$ on $x_c \leq x < \infty$, then by H_6 it follows that for all x and A satisfying $x_c \leq x < A < \infty$

$$\frac{1}{2}|y(x)|^2 \geq \int_x^A s\Psi(s, c) ds - x\int_x^\infty y^* g_y(t, y(t)) dt \ .$$

But this implies that $\int_x^\infty s\Psi(s, c)ds < \infty$, contrary to assumption. Thus, there is no $c > 0$ such that $|y(x)| \geq c$ on an interval of the form $x_c \leq x < \infty$, and since $|y(x)|$ is non-increasing it follows that $|y(x)| \to 0$ as $x \to \infty$.

In connection with the comments in § 1 of this paper, it is to be noted that the hypotheses used in [8] to establish the analogue of our Theorem 7.1 correspond to the assumption that the $\Psi(x)$ of H_6 satisfies $\int_0^\infty \Psi(x)dx = \infty$, instead of the weaker requirement made here.

For the next two theorems we will make use of the following hypothesis.

H_8. *There exists a function* $\phi(x)$ *such that*

$$|g_y(x, y_1) - g_y(x, y_2)| \leq \phi(x)|y_2 - y_1| \ ,$$
$$\text{for } 0 \leq x < \infty, \ |y_1| < \infty \ , \ |y_2| < \infty \ ,$$

where $\phi(x) \geq 0$, $x\phi(x)$ *is integrable on any finite subinterval of* $0 \leq x < \infty$, *and* $\int_0^\infty x\phi(x)dx < \infty$.

We prove the following theorem:

THEOREM 7.2. *If* $g(x, y)$ *satisfies* H_1, H_2, H_5, H_8, *and* $g_y(x, 0) \equiv 0$, *and if* α *is any constant vector, then there is a unique solution* $y(x)$ *of* $y'' = g_y(x, y)$ *for which* $y(x) \to \alpha$ *as* $x \to \infty$.

Let $G(x) = \int_x^\infty t\phi(t)dt$. Imitating Hille [3; p. 238], we consider the following successive approximations corresponding to a given vector α,

$$y_0(x) \equiv \alpha \ , \qquad 0 \leq x < \infty \ .$$
$$y_k(x) = \alpha + \int_x^\infty (t - x) g_y(t, y_{k-1}(t)) dt \ , \qquad 0 \leq x < \infty \ .$$

We will show by induction that for $0 \leq x < \infty$,

$$(7.1)\quad \begin{aligned} &\text{(a)}\quad y_k(x) \text{ is defined;} \\ &\text{(b)}\quad |y_k(x) - y_{k-1}(x)| \leq |\alpha| \frac{[G(x)]^k}{k!} \leq \frac{|\alpha| [G(0)]^k}{k!}, \qquad k = 1, 2, \cdots. \end{aligned}$$

We have $|y_1(x) - y_0(x)| = \left| \int_0^\infty (t - x) g_y(t, \alpha) dt \right|$. The integral here exists since on $x \leq t < \infty$ we have $|t - x| |g_y(t, \alpha)| \leq t |g_y(t, \alpha)| \leq t\phi(t) |\alpha|$, which is integrable on $x \leq t < \infty$. Moreover,

$$|y_1(x) - y_0(x)| \leq |\alpha| \int_x^\infty t\phi(t) dt = |\alpha| G(x),$$

so that (7.1) is satisfied for $k = 1$.

Suppose (7.1) is true for $k = 1, 2, \cdots, N - 1$. Then $y_N(x)$ is defined, since $|g_y(t, y_{N-1}(t)| \leq \phi(t) |y_{N-1}(t)|$, where $y_{N-1}(t)$ is bounded on $0 \leq t < \infty$. Moreover,

$$\begin{aligned} |y_N(x) - y_{N-1}(x)| &= \left| \int_x^\infty (t - x)(g_y(t, y_{N-1}(t)) - g_y(t, y_{N-2}(t))) dt \right| \\ &\leq \int_k^\infty t\phi(t) |y_{N-1}(t) - y_{N-2}(t)| dt \\ &\leq \frac{|\alpha|}{(N-1)!} \int_x^\infty t\phi(t) G^{N-1}(t) dt. \end{aligned}$$

Since $G^{N-1}(t)$ is bounded, all the integrals above exist. Hence,

$$|y_N(x) - y_{N-1}(x)| \leq \frac{-|\alpha|}{(N-1)!} \int_x^\infty [G(t)]^{N-1} G'(t) dt = \frac{|\alpha| [G(x)]^N}{N!}.$$

Now $y_N(x) - \alpha = (y_1 - y_0) + (y_2 - y_1) + \cdots + (y_N - y_{N-1})$, and the series $\sum_{k=1}^\infty |y_k(x) - y_{k-1}(x)|$ converges uniformly on $0 \leq x < \infty$ by (7.1b). Hence $y(x) = \lim_{N\to\infty} y_N(x)$ exists; moreover $y(x)$ is continuous on $0 \leq x < \infty$ and satisfies $|y(x)| \leq |\alpha| \exp\{G(x)\}$. Therefore $|y(x)|$ is bounded on $0 \leq x < \infty$, and H_8 with the uniform convergence of $\{y_N(x)\}$ on $0 \leq x < \infty$ implies $y(x) = \alpha + \int_x^\infty (t - x) g_y(t, y(t)) dt$, so that

$$y''(x) = g_y(x, y(x)), \qquad 0 < x < \infty,$$
$$\lim_{x\to\infty} y(x) = \alpha.$$

If $Y(x)$ satisfies $Y''(x) = g_y(x, Y(x))$ on $0 < x < \infty$ and $Y(x) \to \beta$ as $x \to \infty$, then the integral $\int_x^\infty (t - x) g_y(t, Y(t)) dt$, $0 \leq x < \infty$, exists and $\eta(x) \equiv Y(x) - \beta - \int_x^\infty (t - x) g_y(t, Y(t)) dt$ is such that $\eta''(x) = 0$, $0 < x < \infty$, and $\eta(x) \to 0$ as $x \to \infty$. Hence, $\eta(x) \equiv 0$ and $Y(x) = \beta + \int_x^\infty (t - x) g_y(t, Y(t)) dt$. With $y(x)$ as above we then have

$$| y(x) - Y(x) | = \left| \alpha - \beta + \int_x^\infty (t - x)[g_y(t, y(t)) - g_y(t, Y(t))]dt \right| ,$$
$$\leq | \alpha - \beta | + \int_x^\infty t\phi(t) \, | y(t) - Y(t) | \, dt ,$$

so that by a simple modification of Gronwall's lemma, (see [1; p. 35]), it follows that

$$| y(x) - Y(x) | \leq | \alpha - \beta | \exp \{G(x)\} .$$

If $\beta = \alpha$ then $Y(x) \equiv y(x)$, which proves the uniqueness of solutions of $y'' = g_y(x, y)$ with given limit as $x \rightarrow \infty$. Moreover, $| y(x) - Y(x) | \leq | \alpha - \beta | \exp \{G(0)\}$, so that we have the following corollary.

COROLLARY 7.1. *The solution $y(x)$ described in Theorem 7.2 is a continuous function of $\alpha = y(\infty)$.*

We now prove the following theorem on the order of growth of solutions.

THEOREM 7.3. *If $g(x, y)$ satisfies H_1, H_2, H_5, H_8, and $g_y(x, 0) \equiv 0$, and if $y(x)$ satisfies $y'' = g_y(x, y)$ on $0 < x < \infty$, then $\eta = lim_{x \rightarrow \infty} y'(x)$ exists and is finite, and $y(x) = x[\eta + o(1)]$.*

Note the H_8 implies $| g_y(x, y) | \leq \phi(x) \, | y |$, which is all that is needed here. If $y(x)$ satisfies $y'' = g_y(x, y)$, then following Bellman [1; p. 114] we write

$$y(x) = y(0) + xy'(0) + \int_0^x (x - t) g_y(t, y(t)) dt .$$

Hence, for $x \geq 1$,

$$| y(x) | \leq x(| y(0) | + | y'(0) |) + x \int_0^x \phi(t) \, | y(t) | \, dt$$

or

$$\frac{| y(x) |}{x} \leq (| y(0) | + | y'(0) |) + \int_0^x t\phi(t) \frac{| y(t) |}{t} dt .$$

Therefore, by Gronwall's lemma, (see [1; p. 35]),

$$\frac{| y(x) |}{x} \leq (| y(0) | + | y'(0) |) \exp \left\{ \int_0^x t\phi(t) dt \right\} ,$$

and hence there is a constant M such that

$$| y(x) | \leq Mx , \qquad x \geq 1 .$$

Now for $x \geq 1$ we have

$$\int_1^x |g_y(t, y(t))|\,dt \leq \int_1^x \phi(t)\,|y(t)|\,dt \leq M\int_1^x t\phi(t)dt\ ,$$

so that $\int_0^\infty |g_y(t, y(t))|\,dt$ exists. Since

$$y'(x) = y'(0) + \int_0^x g_y(t, y(t))dt\ ,$$

we have that $y'(x) \to \eta$ as $x \to \infty$, where

$$\eta = y'(0) + \int_0^\infty g_y(t, y(t))dt\ .$$

The final equality in the theorem is a ready consequence of this finite limit of $y'(x)$.

8. Behavior of solutions of (2.1) with respect to initial values. We continue to suppose here that H_1–H_5 are satisfied, but not necessarily any other hypotheses. Let $y_1(x)$, $y_2(x)$ be two bounded solutions of $y'' = g_y(x, y)$ on $0 \leq x < \infty$, and set $\eta(x) = y_1(x) - y_2(x)$. Then by H_4, we have $\eta^*\eta'' \geq 0$, so that $(|\eta|^2)'' = 2|\eta'|^2 + 2\eta^*\eta'' \geq 0$. Since $\eta(x)$ is bounded, we must have $|\eta(x)|$ non-increasing; in particular, $|\eta(x)| \leq |\eta(0)|$ on $0 \leq x < \infty$. Suppose now we denote by $y(x;\alpha)$ the unique bounded solution of $y'' = g_y(x, y)$ which satisfies $y(0;\alpha) = \alpha$. Then $y(x;\alpha)$ is continuous in x and α jointly, as may be seen from the inequality

$$\begin{aligned} |y(\bar{x};\bar{\alpha}) - y(x,\alpha)| &\leq |y(\bar{x};\bar{\alpha}) - y(\bar{x};\alpha)| + |y(\bar{x};\alpha) - y(x;\alpha)|\ , \\ &\leq |\bar{\alpha} - \alpha| + |y(\bar{x};\alpha) - y(x;\alpha)|\ . \end{aligned}$$

Moreover, for any $A > 0$,

$$y'(x;\alpha) = \frac{1}{A}\Big[y(A;\alpha) - y(0;\alpha) - \int_0^A ds\int_x^s g_y(t, y(t;\alpha))dt\Big]\ , \tag{8.1}$$

so that $y'(x;\alpha)$ is also continuous in x and α.

We turn now to the question of differentiability of solutions with respect to initial values. The derivation of our results will involve the use of a lemma, the proof of which is based on certain theorems due to W. T. Reid. In [6], Reid has discussed a class of non-oscillatory linear matrix differential equations which includes as a special case the matrix equation

$$U'' = P(x)U\ , \qquad a \leq x < \infty \tag{8.2}$$

where $P(x)$ is a non-negative definite symmetric matrix with continuous real-valued elements. As shown in Theorem 6.1 of [6], if $U(x)$ is a solution of (8.2) which is non-singular on a subinterval $b \leq x < \infty$, and the necessarily constant matrix $U^*(x)U'(x) - U^{*\prime}(x)U(x)$ is the zero matrix, then

$$M(b;U)=\lim_{x\to\infty}\left(\int_b^x U^{-1}(t)U^{*-1}(t)dt\right)^{-1}$$

exists and is finite. Moreover, such a $U(x)$ is a *principal solution* of (8.2) in the sense of Reid [6] if and only if $M(b;U)=0$. In addition, a principal solution $U(x)$ is characterized by $U(x)=U_{b,\infty}(x)C$, where C is a non-singular constant matrix and $U_{b,\infty}=\lim_{t\to\infty}U_{b,t}(x)$, where $U_{b,t}(x)$, $t>b$, is the unique solution of (8.2) satisfying $U_{b,t}(b)=E$, $U_{b,t}(t)=0$.

It follows as a special case of Theorem 5.1 of this paper that the vector system

$$\begin{aligned} u''&=A(x)u\,, \qquad 0\le x<\infty \\ u(0)&=u_0\,, \quad |u(x)| \text{ bounded on } 0\le x<\infty\,, \end{aligned} \tag{8.3}$$

where $A(x)$ is a real symmetric non-negative matrix of functions continuous on $0\le x<\infty$, has a unique solution. Moreover, Theorem 6.1 shows that the solution $u(x)$ of (8.3) is the limit, as $N\to\infty$, of a function $u_N(x)$ satisfying $u_N''=A(x)u_N$, $u_N(0)=u_0$, $\mu_N(N)=0$, $N=1,2,\cdots$. In view of the similar characterization of this solution and of the principal solutions described above, it follows that the column vectors of $U(x)$, where $U(x)$ is a principal solution of $U''=A(x)U$, are particular bounded solutions of $u''=A(x)u$. This fact will be used in the proof of the following lemma.

LEMMA 8.1. *Suppose $A(x;h)$ is an $n\times n$ non-negative definite symmetric matrix, continuous jointly in the scalar x and the vector h for $0\le x<\infty$ and h in some open set H. Let $W_h(x)$ be the unique principal solution of*

$$W_h''(x)=A(x;h)W_h(x)\,,$$

satisfying

$$W_h(0)=E\,.$$

Then, if h_0 is in H we have $\lim_{h\to h_0}W_h(x)=W_{h_0}(x)$, uniformly for x on any interval $0\le x\le X$.

We consider the solutions $U=U_h(x)$ of the system

$$\begin{aligned} U''&=A(x;h)U \\ U(0)&=E\,, \qquad U'(0)=E \end{aligned} \tag{8.4}$$

or, equivalently,

$$\begin{aligned} U'&=V \\ V'&=A(x;h)U \\ U(0)&=E\,, \qquad V(0)=E\,. \end{aligned} \tag{8.5}$$

The latter is of the form (2.4′) of [6] with $A = 0$, $B = E$, $C = A(x; h)$. The solution of (8.4) is non-singular on $0 \leq x < \infty$, since if ξ is a constant vector such that $u = U(x)\xi$ satisfies $u(x_0) = 0$ with $x_0 > 0$, then

$$\begin{aligned} 0 &= \int_0^{x_0} u^*(u'' - Au)dx \ , \\ &= u^*u' \Big|_0^{x_0} - \int_0^{x_0} (|\, u' \,|^2 + u^*Au)dx \ , \\ &= -|\, \xi \,|^2 - \int_0^{x_0} (|\, u' \,|^2 + u^*Au)dx \ , \end{aligned}$$

so that $\xi = 0$.

Continuing to use the notation of [6; § 3], we compute the value of the constant matrix $\{U, U\} = U^*(x)V(x) - V^*(x)U(x)$ to be $U^*(0)V(0) - V^*(0)U(0) = 0$, and we find that $T = E$. By Theorem 3.1 of [6] we know that any solution $Y(x)$ of $Y'' = A(x)Y$ with $Y(0) = E$ has the form

$$Y(x) = U(x)\Big[E + \Big(\int_0^x U^{-1}(t)U^{*-1}(t)dt\Big)K_0\Big] \ ,$$

for some constant matrix K_0.

Now by Theorems 5.1 and 6.1 of [6] we have $W_h(x) = \lim_{N\to\infty} Y_{0N}(x)$, where $Y''_{0N} = A(x, h)Y_{0N}$, $Y_{0N}(0) = E$, $Y_{0N}(N) = 0$. But in view of the boundary conditions satisfied by $Y_{0N}(x)$ we have

$$Y_{0N}(x) = U(x)\Big[E + \Big(\int_0^x U^{-1}(t)U^{*-1}(t)dt\Big)K_0\Big] \ ,$$

with

$$K_0 = -\Big(\int_0^N U^{-1}(t)U^{*-1}(t)dt\Big)^{-1} \ .$$

Hence,

$$Y_{0N}(x) = U(x)\Big[E - \Big(\int_0^x U^{-1}U^{*-1}dt\Big)\Big(\int_0^N U^{-1}U^{*-1}dt\Big)^{-1}\Big] \ ,$$

and finally,

$$W_h(x) = U_h(x)\Big[E - \Big(\int_0^x U_h^{-1}U_h^{*-1}dt\Big)M(0; U_h)\Big] \ . \tag{8.6}$$

We now need an estimate of $U_h^{-1}(x)U_h^{*-1}(x)$ for large x. To this end put $Z_h(x) = (1 + x)^{-1}U_h(x)$. In view of (8.4), one readily verifies that

$$((1 + x)^2 Z_h')' - (1 + x)^2 A(x; h)Z_h = 0 \ , \quad Z_h(0) = E \ , \quad Z_h'(0) = 0 \ .$$

From this fact it follows that

$$0 = \int_0^x Z_h^*[((1+t)^2 Z_h')' - (1+t)^2 A(t;h) Z_h]dt$$
$$= (1+t)^2 Z_h^* Z_h' \Big|_0^x - \int_0^x (1+t)^2 [Z_h^{*\prime} Z_h' + Z_h^* A(t;h) Z_h]dt ,$$

and therefore

$$(1+x)^2 Z_h^*(x) Z_h'(x) = \int_0^x (1+t)^2 [Z_h^{*\prime} Z_h' + Z_h^* A Z_h]dt .$$

Consequently, $(Z_h^* Z_h)' = Z_h^* Z_h' + Z_h^{*\prime} Z_h = 2Z_h^* Z_h' \geq 0$ on $0 \leq x < \infty$, and $Z_h^*(x) Z_h(x) \geq Z_h^*(0) Z_h(0) = E$ for $x \geq 0$; that is, $U_h^*(x) U_h(x) \geq (1+x)^2 E$ and hence $U_h^{-1}(x) U_h^{*-1}(x) \leq (1+x)^{-2} E$ on $0 \leq x < \infty$ for h in H.

Since as $h \to h_0$ we have $U_h(x) \to U_{h_0}(x)$ uniformly on each finite interval $0 \leq x \leq X$, it follows that

$$\lim_{h \to h_0} M(0; U_h) = M(0; U_{h_0}) .$$

This result, with (8.6), proves the lemma.

We can now prove the following theorem:

THEOREM 8.1. *If $g_{yy}(x, y) = \| g_{y_i y_j} \|$ exists and is continuous for (x, y) in Ω: $0 \leq x < \infty$, $|y| < \infty$, and if $g(x, y)$ satisfies H_1–H_5, then with $y(x; \alpha)$ as in the beginning of this section, we have that $\partial y(x; \alpha)/\partial \alpha_j$ and $\partial y'(x; \alpha)/\partial \alpha_j$ exist and are continuous in x, α for $0 \leq x < \infty$, $|\alpha| < \infty$, $j = 1, 2, \cdots, n$.*

Note that if the hypotheses of this theorem are satisfied, then $g_{yy}(x, y) \geq 0$ for (x, y) in Ω. We denote by $e^{(j)}$ the unit vector having all components zero but the jth, and we let $\Delta\alpha = e^{(j)} h$, $\Delta y = y(x; \alpha + \Delta\alpha) - y(x; \alpha)$, where h is a real scalar. Then

$$(\Delta y)'' = g_y(x, y(x; \alpha + \Delta\alpha)) - g_y(x, y(x; \alpha))$$
$$= \left(\int_0^1 g_{yy}(x, y(x; \alpha) + \theta \Delta y) d\theta\right) \Delta y ,$$

so that

$$\left(\frac{\Delta y}{h}\right)'' = \left(\int_0^1 g_{yy}(x, y(x; \alpha) + \theta \Delta y) d\theta\right)\left(\frac{\Delta y}{h}\right), \quad h \neq 0 .$$

In Lemma 8.1 we identity $A(x; h)$ as $\int_0^1 g_{yy}(x, y(x; \alpha) + \theta \Delta y) d\theta$, where α is fixed, and we identify h_0 as zero. We note that

$$\left(\frac{\Delta y}{h}\right)_{x=0} = e^{(j)} , \quad \text{and} \quad \left|\frac{\Delta y(x)}{h}\right| = \frac{|\Delta y(x)|}{|h|} \leq \frac{|\Delta y(0)|}{|h|} \leq 1 ,$$

$0 \leq x < \infty$. Hence, $(1/h)\Delta y$ is the unique bounded solution of $z'' = A(x; h) z$ satisfying $z(0) = e^{(j)}$. As explained above, the unique principal solution of

$$(8.7') \qquad Z_h'' = A(x\,;h)Z_h\,,$$

satisfying

$$(8.7'') \qquad Z_h(0) = E\,,$$

is the same as the bounded solution of (8.7'), (8.7'') guaranteed by Theorem 6.1 of this paper, of which $(1/h)\Delta y$ is the jth column vector, for $h \neq 0$. Lemma 8.1 then implies that $\lim_{h\to 0}(1/h)\Delta y(x)$ exists and is equal to the jth column vector of $Z_0(x)$; that is, for all α, the vector function $y_{\alpha_j}(x\,;\alpha) = (\partial/\partial\alpha_j)y(x\,;\alpha)$ exists and satisfies

$$(8.8) \qquad (y_{\alpha_j}(x\,;\alpha))'' = g_{yy}(x, y(x\,;\alpha))y_{\alpha_j}(x\,;\alpha)\,; \qquad 0 \leq x < \infty\,.$$

Since $|y_{\alpha_j}(x\,;\alpha)| \leq 1$, we may use Lemma 8.1 with $h = \alpha$ in conjunction with the inequality

$$|y_{\alpha_j}(\bar{x}, \bar{\alpha}) - y_{\alpha_j}(x, \alpha)| \leq |y_{\alpha_j}(\bar{x}\,;\bar{\alpha}) - y_{\alpha_j}(\bar{x}\,;\alpha)| + |y_{\alpha_j}(\bar{x}\,;\alpha) - y_{\alpha_j}(x\,;\alpha)|$$

to show that $y_{\alpha_j}(x\,;\alpha)$ is continuous in x and α. Differentiation of the right hand member of (8.1) with respect to α_j shows the existence of $(\partial/\partial\alpha_j)y'(x\,;\alpha)$ and its continuity with respect to x and α.

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