STRONGLY CONTINUOUS MARKOV PROCESSES

S. R. FOGUEL

Introduction. This paper is a continuation of [3]. We deal here with Markov processes with continuous parameter, while in [3] the discrete parameter case was studied. The notion of a "Markov Process" (here and in [3]) is different from the standard one: A stationary probability measure is assumed to exist, but the Chapman-Kolmogoroff Equation is replaced by a weaker condition. The exact definitions are given in § 1.

All problems are discussed from a Hilbert space point of view and convergence will mean, always, either strong of weak convergence.

1. Notation and background. We shall repeat here, for completeness, the notation of [3] and some of the results.

Let (Ω, Σ, μ) be a given measure space where $\mu(\Omega) = 1$, and $\mu \geqq 0$. The measure will be called the probability measure. The space of real square integrable functions is denoted by L_2.

Let $X_t(\omega)$ be a family of measurable real functions where $0 \leqq t < \infty$ and $\omega \in \Omega$. This will be called the Markov process and we assume:

If A is a Borel set on the real line and $t_1 < t_2 < t_3$ then the conditional probability that $X_{t_3} \in A$ given X_{t_1} and X_{t_2} is equal to the conditional probability that $X_{t3} \in A$ given X_{t_2}.

Also we assume that the process is stationary. Namely:

$$\mu(X_{t_1+s} \in A_1 \cap X_{t_2+s} \in A_2) = \mu(X_{t_1} \in A_1 \cap X_{t_2} \in A_2)$$

for all t_1, t_2, s positive real numbers and A_1A_2 Borel sets.

For any set $\sigma \subset \Omega$, χ_σ denotes the characteristic function of this set. Let B_t be the closed subspace of L_2 generated by the functions $\chi_{X_t \in A}$. The self adjoint projection on B_t is denoted by E_t. Finally, let T_t be the transformation from B_0 to B_t defined by

$$T_t\chi_{X_0 \in A} = \chi_{X_t \in A}$$

where we used additivity to extend it to whole of B_0. In [3] the following equations are proved:

1.1 $$E_{t_1}E_{t_2}E_{t_3} = E_{t_1}E_{t_3} \qquad \text{if } t_1 < t_2 < t_3 .$$

1.2 a. $$\| T_tx \| = \| x \| , \qquad \text{for } x \in B_0 .$$

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b. $$T_tB_0 = B_t\ .$$

c. $$(T_{t_1+s}x,\ T_{t_2+s}y) = (T_{t_1}x,\ T_{t_2}y)\ , \qquad \text{for } x \in B_0\ y \in B_0\ .$$

See Theorem 2.1 and Lemma 2.4.

Let P_t be the operator on B_0 defined by $P_t = E_0T_t$.

THEOREM 1.1. *The operators P_t form a semi group of contractions on B_0. The adjoint semi group is given by $P_t^* = T_t^{-1}E_t$.*

Proof. It is clear that $\| P_t \| \leqq 1$. Let x and y be vectors of B_0 and choose $z \in B_0$ so that $T_sz = E_sy$. Thus $z = T_s^{-1}E_sy$. Then

$$\begin{aligned}(P_sP_tx, y) &= (E_0T_sE_0T_tx, y) = (T_sE_0T_tx, y)\\ &= (T_sE_0T_tx, E_sy) = (E_0T_tx, z) = (T_tx, z)\ .\end{aligned}$$

Where we used Equation 1.2c. On the other hand

$$\begin{aligned}(P_{s+t}x, y) &= (E_0T_{s+t}x, y) = (E_0E_sT_{s+t}x, y) = (E_sT_{s+t}x, y)\\ &= (T_{s+t}x, E_sy) = (T_{s+t}x, T_sz) = (T_tx, z)\ .\end{aligned}$$

Here we used Equations 1.1 and 1.2c. Now

$$(P_sx, y) = (T_sx, y) = (T_sx, E_sy) = (x, z) = (x,\ T_s^{-1}E_sy)\ .$$

The fact that P_t is a semi group is our version of the Chapman-Kolmogoroff Equation.

In most of this paper it will be assumed that the semi group P_t is strongly continuous. We shall say, in this case that the Markov process is strongly continuous.

THEOREM 2.1. *The Markov process is strongly continuous if and only if*

$$\lim_{t\to 0} \mu(X_0 \in A \cap X_t \in A) = \mu(X_0 \in A)\ .$$

Proof. Note that

$$\begin{aligned}\mu(X_0 \in A) &= \| \chi_{X_0\in A} \|^2\\ \mu(X_0 \in A \cap X_t \in A) &= (T_t\chi_{X_0\in A}, \chi_{X_0\in A}) = (P_t\chi_{X_0\in A}, \chi_{X_0\in A})\ .\end{aligned}$$

Thus

$$\mu(X_0 \in A) - \mu(X_0 \in A \cap X_t \in A) = (\chi_{X_0\in A} - P_t\chi_{X_0\in A}, \chi_{X_0\in A})$$

and this converges to zero if P_t converges to the identity operator strongly. On the other hand

$$\begin{aligned}\| P_t\chi_{X_0\in A} - \chi_{X_0\in A} \|^2 &= \| P_t\chi_{X_0\in A} \|^2 + \| \chi_{X_0\in A} \|^2 - 2(P_t\chi_{X_0\in A}, \chi_{X_0\in A}) \\ &\leqq 2(\| \chi_{X_0\in A} \|^2 - (P_t\chi_{X_0\in A}, \chi_{X_0\in A})) \\ &= 2(\mu(X_0 \in A) - \mu(X_0 \in A \cap X_t \in A)) .\end{aligned}$$

Thus the condition of the Theorem implies that $P_t x$ converges to x for a set of functions, x, that span B_0 and because $\| P_t \| \leqq 1$ this must hold for every x in B_0.

2. Limit of transition probabilities as $t \to \infty$. This section is an extension of § 3 of [3]. Throughout this section we assume:

CONDITION D. *There exist a finite a measure φ, on the real line, and an $\varepsilon > 0$ such that if A is a Borel set and $\varphi(A) < \varepsilon$ then*

$$E_0\chi_{X_t\in A} \neq \chi_{X_t\in A} .$$

This condition was given in]3] and is similar to Doeblin's condition as given in [1] page 192. Another form of the condition is: if $\varphi(A)<\varepsilon$ then

$$\| T_t\chi_{X_0\in A} \|^2 = \| \chi_{X_0\in A} \|^2 > \| P_t\chi_{X_0\in A} \|^2 .$$

In this form it is seen immediately that t can be replaced by any larger number. Thus one can choose t to be of the form $n\delta$ for any fixed $\delta > 0$. (n a positive integer). For a fixed $\delta > 0$ $X_{n\delta}$ form a discreet Markov process for which a Doeblin condition holds. Let H_δ be the space of all functions in B_0 such that

$$x \in \bigcap_{n=0}^{\infty} B_{n\delta},\ T_{k\delta}x \in \bigcap_{n=0}^{\infty} B_{n\delta} \qquad k = 1, 2, \cdots .$$

In [3] Theorem 3.7 it was proved that if x is orthogonal to H_δ then $T_{k\delta}\, x$ tends weakly to zero as k tends to infinity (k integer).

THEOREM 1.2. *$x \in H_\delta$ if and only if $T_t x = x$ for some $t > 0$. Thus H_δ is the same for all δ and will be denoted by H. The space H is generated by a finite number of disjoint characteristic functions and is invariant under T_t for all $t > 0$.*

Proof. It is enough to prove first statement for the rest follows from Theorem 3.8 and Corollary 2 of Theorem 3.11 of [3].

In Corollary 2 of Theorem 3.11 of [3] it was shown that if $x \in H_\delta$ then $T_{k\delta}x = x$ for some x. Thus it is enough to show that if $T_t x = x$ for some $t > 0$, then $x \in H_\delta$. Now if $T_t x = x$ then

$$(T_{t+a}x, T_a x) = (T_t x, x) = \| x \|^2 = \| T_a x \|^2$$

Thus

$$T_{t+a}x = T_a x\ .$$

In particlar

$$x = T_t x = T_{2t}x = \cdots\ .$$

Thus

$$x \in \bigcap_{k=0}^{\infty} B_{tk}\ .$$

But by Theorem 2.2 of [3]

$$\bigcap_{k=0}^{\infty} B_{tk} = \bigcap_{n=0}^{\infty} B_{\delta n}\ .$$

Now

$$T_{m\delta}x = T_{m\delta+t}x = T_{m\delta+2t}x = \cdots$$

or

$$T_{m\delta}x \in \bigcap_{k=0}^{\infty} B_{m\delta+kt} = \bigcap_{n=m}^{\infty} B_{n\delta}\ .$$

Again by Theorem 2.2 of [3]. Thus it suffices to show that $T_{m\delta}x \in B_0$ for then $T_{m\delta}x \in \bigcap_{n=0}^{\infty} B_{n\delta}$ by the same Theorem. Now

$$\begin{aligned}\sup_{z\in B_0,\|z\|=1}(T_{m\delta}x, z) &= \sup_{z^1\in B_{kt},\|z^1\|=1}(T_{m\delta+kt}x, z^1)\\ &= \sup_{z^1\in B_{kt},\|z^1\|=1}(T_{m\delta}x, z^1) = \| T_{m\delta}x \|\end{aligned}$$

for

$$T_{m\delta}x \in \bigcap_{n=m}^{\infty} B_{n\delta} \subset B_{kt} \quad \text{if} \quad kt > m\delta\ .$$

Thus

$$T_{m\delta}x \in B_0 \quad \text{and} \quad x \in H_\delta\ .$$

Notice that on H $P_t = T_t$, and P_t is a unitary operator.

In the rest of the paper we shall assume that the process $\{X_t\}$, *is strongly continuous.*

LEMMA 2.2. *On the space* H T_t *is the identity operator for all* t.

Proof. Let χ be one of the atoms generating H. Thus χ is a characteristic function that is not the sum of two characteristic functions in H. Let t be so small that $(T_t\chi, \chi) \neq 0$. Now $T_t\chi$ is also a characteristic function in H and $\| T_t\chi \| = \| \chi \|$. Thus $T_t\chi = \chi$ because χ is an atom. Also for every $n T_{nt}\chi = P_{nt}\chi = (P_t)^n\chi = \chi$, hence $T_t\chi = P_t\chi = \chi$ for all t.

THEOREM 3.2. *Let $x \in B_0$ and let y be the projection of x on H, then*

$$\text{weak}\lim_{t\to\infty} P_t x = \text{weak}\lim_{t\to\infty} T_t x = y \, .$$

Proof. By the previous lemma it suffices to show that if x is orthogonal to H then $T_t x$ tends weakly to zero. Let $z \in B_0$, $\| z \| = 1$ be a given vector and let $\varepsilon > 0$. Choose δ_0 so that $\| T_\delta x - x \| \leqq \varepsilon/2$ if $\delta \leqq \delta_0$. By Theorem 3.7 of [3] if n is large enough then

$$| (T_{n\delta_0}x, z) | \leqq \varepsilon/2 \, .$$

Thus

$$\begin{aligned} | (T_t x, z) | &= | ((T_t - T_{n\delta_0})x, z) + (T_{n\delta_0}x, z) | \\ &\leqq \varepsilon/2 + \| (T_t - T_{n\delta_0})x \| \, . \end{aligned}$$

Now

$$\begin{aligned} \| (T_t - T_{n\delta_0})x \|^2 &= 2 \| x \|^2 - 2(T_t x, T_{n\delta_0}x) \\ &= 2 \| x \|^2 - 2(T_{t-n\delta_0}x, x) = \| T_{t-n\delta_0}x - x \|^2 \end{aligned}$$

by Equation 1.2.c. If n is so chosen that

$$t - n\delta_0 < \delta_0 \quad \text{then} \quad \| (T_t - T_{n\delta_0})x \| \leqq \varepsilon/2 \, .$$

3. **Differentiability.** In this section we do not assume Condition D. The process $\{X_t\}$ is assumed to be strongly continuous. It is known that in this case the function $P_t x$ is differentiable at the origin for x in a dense subset of B_0. The derivative, Q, of P_t is an unbounded closed operator. Let $D(Q)$ be the domain of Q. The simplest case is when Q is bounded. A necessary and sufficient condition for this is that the semi group P_t is continuous in the uniform topology. (See 2 Theorem VIII. 2)

THEOREM 1.3. *The operator Q is everywhere defined if and only if the expression*

$$1 - \frac{\mu(X_0 \in A \cap X_t \in A)}{\mu(X_0 \in A)}$$

tends to zero uniformly, for all Borel sets A.

Proof. If $\| I - P_t \| \to 0$ then

$$1 - \frac{\mu(X_0 \in A \cap X_t \in A)}{\mu(X_0 \in A)} = \frac{(\chi_{X_0 \in A} - P_t \chi_{X_0 \in A}, \chi_{X_0 \in A})}{\| \chi_{X_0 \in A} \|^2} \leqq \| I - P_t \| .$$

Thus the condition is necessary. Conversely let

$$x = \sum a_i \chi_i \quad \text{where} \quad \sum a_i^2 \| \chi_i \|^2 = 1 \quad \text{and} \quad \chi_i = \chi_{X_0 \in A_i}, A_i \cap A_j = \phi .$$

Then

$$\begin{aligned} 1 - (P_t x, x) &= \sum_{ij} a_i a_j ((\chi_i, \chi_j) - (P_t \chi_i, \chi_j)) \\ &\leqq \Big(\sum_{i,j} a_i^2 | (\chi_i, \chi_j) - (P_t \chi_i, \chi_j) | \Big)^{1/2} \Big(\sum_{ij} a_j^2 | (\chi_i, \chi_j) - (P_t \chi_i, \chi_j) | \Big)^{1/2} . \end{aligned}$$

By Schwarz's inequality. Let us consider each term separately.

$$\sum_{i,j} a_i^2 | (\chi_i, \chi_j) - (P_t \chi_i, \chi_j) | = \sum_i a_i^2 \sum_j | (\chi_i, \chi_j) - (P_t \chi_i, \chi_j) | .$$

For a fixed i we have

$$\begin{aligned} \sum_j | (\chi_i, \chi_j) - (P_t \chi_i, \chi_j) | &= \sum_{j \neq i} (P_t \chi_i, \chi_j) + \| \chi_i \|^2 - (P_t \chi_i, \chi_i) \\ &= \sum_j (P_t \chi_i, \chi_j) - (P_t \chi_i, \chi_i) + \| \chi_i \|^2 - (P_t \chi_i, \chi_i) \\ &= (P_t \chi_i, 1) - (P_t \chi_i, \chi_i) + \| \chi_i \|^2 - (P_t \chi_i, \chi_i) \end{aligned}$$

where 1 is the identity function. Now

$$(P_t \chi_i, 1) = (T_t \chi_i, 1) = (T_t \chi_i, T_t 1) = (\chi_i, 1) = \| \chi_i \|^2 .$$

Thus the sum over j is equal to

$$2 \| \chi_i \|^2 \Big(1 - \frac{(P_t \chi_i, \chi_i)}{\| \chi_i \|^2} \Big)$$

and

$$\sum_{i,j} a_i^2 | (\chi_i, \chi_j) - (P_t \chi_i, \chi_j) | \leqq 2 \sup_i \Big(1 - \frac{(P_t \chi_i, \chi_i)}{\| \chi_i \|^2} \Big) .$$

$$\sum a_i^2 \| \chi_i \|^2 = 2 \sup \Big(1 - \frac{(P_t \chi_i, \chi_i)}{\| \chi_i \|^2} \Big) .$$

For the second term we get

$$\sum a_j^2 | (\chi_i, \chi_j) - (P_t \chi_i, \chi_j) | = \sum_j a_j^2 \sum_i | (\chi_i, \chi_j) - (P_t \chi_i, \chi_j) |$$

and

$$\begin{aligned}\sum_i |(\chi_i, \chi_j) - (P_t\chi_i, \chi_j)| &= \|\chi_j\|^2 - (P_t\chi_j, \chi_j) + \sum_{i\neq j}(P_t\chi_i, \chi_j)\\ &= \|\chi_j\|^2 - (P_t\chi_j, \chi_j) + \sum_i (P_t\chi_i, \chi_j) - (P_t\chi_j, \chi_j)\\ &= \|\chi_j\|^2 - (P_t\chi_j, \chi_j) + (P_t1, \chi_j) - (P_t\chi_j, \chi_j)\\ &= 2(\|\chi_j\|^2 - (P_t\chi_j, \chi_j)) .\end{aligned}$$

And the second term has the same bound. Thus

$$1 - (P_tx, x) \leqq 2\sup\left(1 - \frac{(P_t\chi_i, \chi_i)}{\|\chi_i\|^2}\right) .$$

Now

$$\begin{aligned}\|P_tx - x\|^2 &= \|P_tx\|^2 + \|x\|^2 - 2(P_tx, x)\\ &\leqq 2((I - P_t)x, x) \leqq 4\sup_i\left(1 - \frac{(P_t\chi_i, \chi_i)}{\|\chi_i\|^2}\right) .\end{aligned}$$

By assumption this tends to zero uniformly. Hence $\|P_tx - x\|$ tends to zero uniformly, for x in a dense subset of B_0, and hence everywhere because $\|P_t\| \leqq 1$.

REMARKS. It is enough to assume the condition of the Theorem for a family of Borel sets, A, such that the functions χ_A generate B_0. It follows, from the fact that Q is bounded, that

$$1 - \frac{\mu(X_0 \in A \cap X_t \in A)}{\mu(X_0 \in A)} \leqq (\text{const})t .$$

Theorem 1.3 is well known for processes with countable state space. A brief discussion of this case is given in [1] page 265.

The function P_tx is differentiable for many x's exen if Q is unbounded. In order to study this we will need:

LEMMA 2.3. *Let R_t be strongly continuous semi group of operators, defined on a reflexive space X. If $x \in X$ then R_tx is differentiable if the expression $(1/t)\|R_tx - x\|$ is bounded for all t.*

This is included in Theorem 10.7.2 of [4]

Let $y \in L_2$ and Ω_1 be a subset of Ω such that $\chi_{\Omega_1 \in B_0}$. Then

$$\|E_0y\|^2 = \|\chi_{\Omega_1}\cdot E_0y\|^2 + \|\chi_{\Omega_2}\cdot E_0y\|^2$$

where $\Omega_2 = \Omega - \Omega_1$. Now $\chi_{\Omega_1}\cdot E_0y$ is the projection of y on the subspace generated by characteristic function, in B_0, of subsets of Ω_1. Thus

$$\|\chi_{\Omega_1}\cdot E_0y\| = \sup\{\textstyle\sum(y, \chi_i)a_i \mid \chi_i = \chi_{X_0 \in A_i} \in B_0 \text{ and } A_i \text{ are disjoint Borel sets, such that } X_0 \in A_i \subset \Omega_1, \text{ and } \sum a_i^2\|\chi_i\|^2 = 1\} .$$

But

$$|\sum(y,\chi_i)a_i| \leqq \sum\frac{|(y,\chi_i)|}{\|\chi_i\|}|a_i|\,\|\chi_i\| \leqq \left(\sum\frac{(y,\chi_i)^2}{\|\chi_i\|^2}\right)^{1/2}.$$

Hence

$$\|\chi_{\Omega_1}\cdot E_0 y\|^2 = \sup\left\{\sum\frac{(y,\chi_i)^2}{\|\chi_i\|^2}\,\middle|\,\chi_i = \chi_{X_0\in A_i}\in B_0,\right.$$

$$\left. A_i \text{ disjoint Borel sets and } X_0 \in A_i \subset \Omega_1\right\}$$

A similar expresion holds for $\|\chi_{\Omega_2}\cdot E_0 y\|^2$.

THEOREM 3.3. *Let A be a Borel set. The function $P_t\chi_{X_0\in A}$ is differentiable at zero if and only if the two expressions below, are bounded:*

1. $$\frac{1}{t^2}\sup\left\{\sum\frac{\mu(X_t\in A\cap X_0\in A_i)^2}{\mu(X_0\in A_i)}\,\middle|\,A_i \text{ disjoint Borel sets and } A_i\cap A=\phi\right\}.$$

2. $$\frac{1}{t^2}\sup\left\{\sum\frac{(\mu(X_t\in A\cap X_0\in A_i)-\mu(X_0\in A_i))^2}{\mu(X_0\in A_i)}\,\middle|\,A_i \text{ disjoint Borel sets and } A_i\subset A\right\}.$$

Proof. By Lemma 2.3 and the above discussion it is enough to show that

$$\frac{1}{t^2}\sup\left\{\sum\frac{(P_t\chi_{X_0\in A}-\chi_{X_0\in A},\chi_{X_0\in Ai})^2}{\|\chi_{X_0\in Ai}\|^2}\,\middle|\,A_i \text{ disjoint and } A_i\cap A=\phi\right\}$$

and

$$\frac{1}{t^2}\sup\left\{\sum\frac{(P_t\chi_{X_0\in A}-\chi_{X_0\in A},\chi_{X_0\in A_i})}{\|\chi_{X_0\in A_i}\|^2}\,\middle|\,A_i \text{ disjoint and } A_i\subset A\right\}$$

are both bounded. But these expressions are equal to 1 and 2 respectively.

REMARK. If A is an atom for B_0 then the second expression is

$$\frac{1}{t^2}\left(\frac{\chi(X_t\in A\cap X_0\in A)-\mu(X_0\in A)}{\mu(X_0\in A)}\right)^2\mu(X_0\in A)$$

$$=\left(\frac{1}{t}\left(1-\frac{\mu(X_t\in A\cap X_0\in A)}{\mu(X_0\in A)}\right)\right)^2\mu(X_0\in A).$$

A more precise information is available in the following special case.

Theorem 4.3. *Let $x \in B_0$. Then $x \in D(Q)$ and $(Qx, x) = 0$ if and only if $(1/t^2)(\| x \|^2 - (P_t x, x))$ is bounded. In this case $Q^* x$ exists and is equal to $-Qx$.*

Proof. If $y \in B_0$ then

$$\begin{aligned} \| y - P_t y \|^2 &= \| y \|^2 + \| P_t y \|^2 - 2(P_t y, y) \\ &\leqq 2(\| y \|^2 - (T_t y, y)) = \| y - T_t y \|^2 \end{aligned}$$

thus

a. $$\frac{\| T_t y - y \|}{\sqrt{t}} = \sqrt{2 \frac{(y - P_t y, y)}{t}} \geqq \frac{\| P_t y - y \|}{\sqrt{t}} .$$

Also if y and z are any two vectors in B_0 then

b. $$\begin{aligned} \left(\frac{1}{t}(P_t - 1)z, y\right) &= \frac{1}{t}(T_t z - z, y) = \frac{1}{t}(T_t z, y - T_t y) \\ &= \frac{1}{t}(T_t z - z, y - T_t y) + \frac{1}{t}(z, y - P_t y) \end{aligned}$$

where we used Equation 1.2.c for the third equality.

Let x be such that $(1/t^2)(\| x \|^2 - (P_t x, x))$ is bounded. Then from (a) we get

$$\| \frac{1}{t^2}(P_t x - x) \|^2 \leqq 2 \frac{(x - P_t x, x)}{t^2}$$

and is bounded by assumption. Thus we know from Lemma 2.3 that $x \in D(Q)$. Moreover

$$(Qx, x) = - \lim t \frac{(x - P_t, x)}{t^2} = 0 .$$

Conversely let $x \in D(Q)$ and $(Qx, x) = 0$. If $y \in D(Q)$ then it follows from (b) that

$$\begin{aligned} (Qx, y) &= \lim_{t \to 0} \frac{1}{t}((P_t - 1)x, y) \\ &= \lim_{t \to 0} \frac{1}{t}(T_t x - x, y - T_t y) + \frac{1}{t}(x, y - P_t y) \end{aligned}$$

the second term tends to $-(x, Qy)$ while the first is bounded by

$$\left|\frac{1}{t}(T_t x - x, y - T_t y)\right| \leqq \frac{\| T_t x - x \|}{\sqrt{t}} \frac{\| y - T_t y \|}{\sqrt{t}}$$
$$= \left(2\frac{(x - P_t x, x)}{t} \cdot 2\frac{(y - P_t y, y)}{t}\right)^{1/2}$$

as $t \to 0$ this tends to

$$(4(Qx, x)(Qy, y))^{1/2} = 0 .$$

Thus

$$(Qx, y) = -(x, Qy)$$

or

$$x \in D(Q^*) \quad \text{and} \quad Q^* x = -Qx .$$

Now

$$\begin{aligned}(x - P_t x, x) &= \int_0^t (QP_u x, x) du \leqq t \max_{u \leqq t} |(QP_u x, x)| \\ &= t \max_{u \leqq t} |(P_u x, Qx)| = t \max_{u \leqq t} |(P_u x - x, Qx)| \\ &\leqq \text{const. } t^2\end{aligned}$$

because $\| P_u x - x \| \leqq \text{const.}\, u$.

REMARK. If x is a characteristic function then it is easy to see that $Qx = 0$ if $(Qx, x) = 0$.

The referee called my attention to the fact that this theorem generalizes to arbitrary semi groups of contraction operators, when T_t is replaced by the group of unitary operators which project down to P_t as in s_z Nagy theorem (See Riesz Nagy appendix to the third edition). Some simple changes have to be done to take care of the complex case.

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