SELF-INTERSECTION OF A SPHERE ON A COMPLEX QUADRIC

I. FÁRY

1. The real part S^n of a quadric V in complex, affine $(n+1)$-space is a sphere. The self-intersection of S^n *in* V is the same as the self-intersection of a "vanishing cycle," introduced by Lefschetz, and plays a certain role in [4], [5]. We will compute here this self-intersection number, using elementary tools.

Let us introduce some notations. P_{n+1} denotes the complex projective space of algebraic dimension $n+1$, hence of topological dimension

$$\dim P_{n+1} = 2n + 2 .$$

To each projective sub-space P_k of P_{n+1} a *positive* orientation can be given, thus it can be considered as a *cycle* p_{2k}. Then we agree that

$$\text{(1)} \qquad \textit{if } k + l = n + 1, \text{ then } (p_{2k}, p_{2l}) = 1 \text{ in } P_{n+1}$$

be true for the *intersection numbers* of cycles. This is the usual convention, the one in [1], for example; in [7] another convention is adopted.

Let $x_1, \cdots, x_{n+2}$ be a fixed system of projective coordinates in P_{n+1}. Then

$$\text{(2)} \qquad Q_n : x_1^2 + \cdots + x_{n+2}^2 = 0$$

is a *non-singular quadric*; $\dim Q_n = 2n$. The points of P_{n+1} whose last coordinate is non-zero form a complex affine space C_{n+1}, and

$$V = Q_n \cap C_{n+1} = [x : x \in Q_n, x_{n+2} \neq 0]$$

is a *non-singular affine quadric.* If $z \in C_{n+1}$, we denote by $z_1, \cdots, z_{n+2}$ those coordinates for which $z_{n+2} = i$ where $i^2 = -1$; thus $z_1, \cdots, z_{n+1}$ are affine coordinates in C_{n+1}. Then

$$V\colon\ z_1^2 + \cdots + z_{n+1}^2 = 1 \qquad (z \in C_{n+1})$$
$$S^n\colon\ z_1^2 + \cdots + z_{n+1}^2 = 1, z_1 \cdots, z_{n+1} \textit{ reals}$$

are the equations of an affine quadric and its real part respectively; *this real part S^n* is, of course, a sphere. We consider S^n with an arbitrarily chosen and fixed orientation as a cycle s. It is well known (see, for example, [2], p. 35, (g)) that

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(3) *the homology class s, of the cycle whose carrier is* S^n, *generates* $H_n(V; Z)$,

where Z denotes the ring of integers.

As dim $V = 2$ dim S^n, the self intersection number

$$(s, s) = (S^n S^n) , \qquad (\text{in } V) , \tag{4}$$

of s in V, is well defined; we may write (S^n, S^n) for this self intersection number, because (s, s) does not depend on the orientation of S^n, used in (3).

2. M. F. Atiyah communicated to me his computation of the intersection number (4) for $n = 2$, showing that the sign in [2], p. 35 (10) is not the right one.[1] The determination of the sign of (4) given below is a generalization to n dimensions of the construction of Atiyah. In [2] we used only the fact that (4) is not zero, if n is even, hence other results of that paper are not invalidated by the false sign in (10), p. 35. The mistaken sign is "classical." Wrong sign appears in [4], p. 93, Théorème sur les Γ_{d-1} de C_u, *I*, [5] on top of p. 16, [8], p. 102, (3), and [7], p. 104, Theorem 45 (although in [7] not the convention (1) is used, the alternation of the sign in question is independent of any convention). After the completion of the present paper [6] appeared, where the classical mistake in sign is corrected (see (11.3) on p. 161). The results of [1] are in agreement with the sign (5) below.

3. Using the notations and conventions introduced above, we will prove the following theorem.

THEOREM. *Let s be the homology class of the oriented sphere* S^n *in* $H_n(Q_n; Z)$ *where* $n = 2h$ *is even. Let us denote by* (s, s) *the self-intersection number of s computed with the convention* (1). *Then*

$$(s, s) = \begin{cases} -2 , & \text{if } h = \frac{n}{2} \text{ is odd} ; \\ +2 , & \text{if } h = \frac{n}{2} \text{ is even} ; \end{cases} \tag{5}$$

holds true.

[1] I take the opportunity to correct another mistake in [2], also noticed by Atiyah. In Proposition 2, p. 27, we have to suppose that the singularity in question is *conical*. In [2], Proposition 2 is stated without proof; Atiyah gave an example showing that the statement does not hold true, if the singularity is *not conical*, and gave a proof with the correct hypothesis. Proposition 2 is used in [2] only in connection with conical singularities; thus other results of [2] are not affected by the incomplete formulation of that Proposition.

4. We prepare the proof of this theorem; for the first part of the proof, see [1]. (See also [3], pp. 230–232.) In order to describe easily linear sub-spaces of Q_n, we introduce new projective coordinates in P_{n+1}:

$$\begin{matrix} u_j = x_{2j-1} + ix_{2j} \\ v_j = x_{2j-1} - ix_{2j} \end{matrix} \; j = 1, \cdots, h+1 \qquad (i^2 = -1) .$$

Let us notice that

$$u_j = v_j = 0 \textit{ if and only if } x_{2j-1} = x_{2j} = 0 . \tag{6}$$

The equation of Q_n is

$$u_1 v_1 + \cdots + u_{h+1} v_{h+1} = 0 ,$$

in the new coordinates.

We consider the following linear sub-spaces of Q_n:

$$A: \; u_j = 0 , \; j = 1, \cdots, h, h+1 ; \tag{7}$$

$$B: \; u_j = 0 , \; j = 1, \cdots, h ; \; v_{h+1} = 0 ; \tag{8}$$

$$C: \; v_j = 0 , \; j = 1, \cdots, h+1 . \tag{9}$$

Let us remark that,

$$A \cap C = \phi , \; B \cap C \textit{ is just one point,} \tag{10}$$

by (6).

LEMMA 1. *Let X be one of the projective spaces A, B, C. If, in the system of equations defining X, we replace an even number of equations $u_j = 0$ by the corresponding $v_j = 0$, or vice versa, we define a new linear sub-space of Q_n belonging to the same continuous system as X. Similarly, without leaving the continuous system containing B, we may replace $u_h = 0$, $v_{h+1} = 0$ in* (8) *by $v_h = 0$ and $u_{h+1} = 0$.*

Proof. Let us suppose that we want to replace $v_1 = 0$, $v_2 = 0$ in (9) by $u_1 = 0$, $u_2 = 0$. Let us consider the linear space

$$\begin{matrix} \alpha v_2 + \beta u_1 = 0 , \\ -\alpha v_1 + \beta u_2 = 0 , \end{matrix} \quad v_3 = 0, \cdots, v_{h+1} = 0 ,$$

defined for every $(\alpha, \beta) \neq (0, 0)$. This projective space is clearly contained in Q_n. For $(1, 0)$ we have C and for $(0, 1)$ the desired replacement. The last statement of the lemma is proved similarly using the system

$$\begin{matrix} \alpha u_h + \beta u_{h+1} = 0 , \\ -\beta v_h + a v_{h+1} = 0 . \end{matrix}$$

Let us consider now A, B, C as cycles of Q_n, and let us denote by a, b, c their respective homology classes in $H_n(Q_n; Z)$.

LEMMA 2. *If h is odd, then $c = a$. If h is even, then $c = b$.*

Proof. If h is odd, the $h+1$ equations of (9) can be replaced by the equations $u_j = 0$, $j = 1, \cdots, h+1$. Hence A and C belong to the same continuous system. If h is even, we can replace the first h equations defining C by $u_j = 0$, $j = 1, \cdots, h$. Hence C and B belong to the same continuous system.

LEMMA 3. *As to the intersection numbers, we have*

$$\text{(11)} \qquad \textit{if } h \textit{ is odd, then } (a, a) = 0,\ (b, b) = 0,\ (a, b) = 1\,,$$

$$\text{(12)} \qquad \textit{if } h \textit{ is even, then } (a, a) = 1,\ (b, b) = 1,\ (a, b) = 0\,.$$

Proof. (1) Let h be *odd*. By Lemma 2 and the first equation of (10), we have $(a, a) = 0$. Similarly, the second equation of (10) and Lemma 2 prove $(a, b) = 1$. In order to prove $(b, b) = 0$, we consider the space

$$B' : \quad v_j = 0\,, \quad j = 1, \cdots, h\,, \quad u_{h+1} = 0\,.$$

We claim that B and B' are in the same continuous system. In order to prove this statement, we use Lemma 1 twice. First, we replace the last two equations of (8) by $v_h = 0$, and $u_{h+1} = 0$. Second, in the system obtained by the first step, we replace the first $h-1$ equations by $v_j = 0$. Now $B \cap B' = \phi$, and this proves $(b, b) = 0$.

(2) Let h be *even*. The proof of (12) is similar to the previous one. The last two equations of (12) are immediate from (10) and Lemma 2. Using Lemma 1, we can find presently *a* B'', such that $B \cap B''$ be just one point.

LEMMA 4. *Using the previous notations s, a, b, for homology classes,*

$$s = \pm(a - b)\,, \tag{13}$$

the sign depending on the chosen orientation of S^n.

Proof. Let us denote by I the hyperplane $x_{n+2} = 0$, Then, clearly,

$$A \cap I = B \cap I\,.$$

We denote by J this intersection ($J = A \cap B$). Let us consider a pencil of k-planes, $2k + \dim A = 2n + 2$, in general position. If N is a neighborhood of J in B, the k-planes of the pencil project N into a neighborhood M of J in A. Given now a Riemann metric of P_{n+1}, if N is a small enough neighborhood of J, the corresponding points of N, M determine unique geodesic segments. We consider now B as a cycle, whose simplexes are so small that those intersecting J are contained in N. Using the geodesic segments introduced above which start at points of the simplexes of B intersecting J, it is easy to construct a chain E of Q_n, such that

$$A - B + \partial E \tag{14}$$

be a sum of simplexes of $V = Q_n - I$. Hence, s being a generator of $H_n(V; Z)$, (14) will be homologous to a multiple of s. Thus $a - b = ms$ for some integer m. Now $(a - b, a) = m(s, a)$ is ± 1 by Lemma 3, hence $m = \pm 1$.

Proof of the Theorem. (1) Let us suppose that h is *odd*. We use (13) and (11): $(s, s) = (a - b, a - b) = (a, a) - (b, a) - (a, b) + (b, b) = -(b, a) - (a, b) = -2$.

(2) Let us suppose that h is *even*. This time we use (12): $(s, s) = (a, a) + (b, b) = +2$. Hence the proof of (5) is complete.

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UNIVERSITY OF CALIFORNIA, BERKELEY