SIMPLE PATHS ON CONVEX POLYHEDRA

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1. Introduction. In problems of linear programming, one sometimes wants to find all vertices of a given convex polyhedron. An algorithm for finding all such vertices will often define a path which passes from vertex to vertex along the edges of the polyhedron in question [1], and thus it is natural to ask, as Balinski does in [2], whether or not one can always find a path along the edges of a convex polyhedron which visits each vertex once and only once. This question has been answered in the negative independently by Grünbaum and Motzkin [5] and the author [3]. The purpose of the present paper is to present a modification of the results of [3], and answer certain questions which were asked by Grünbaum and Motzkin.

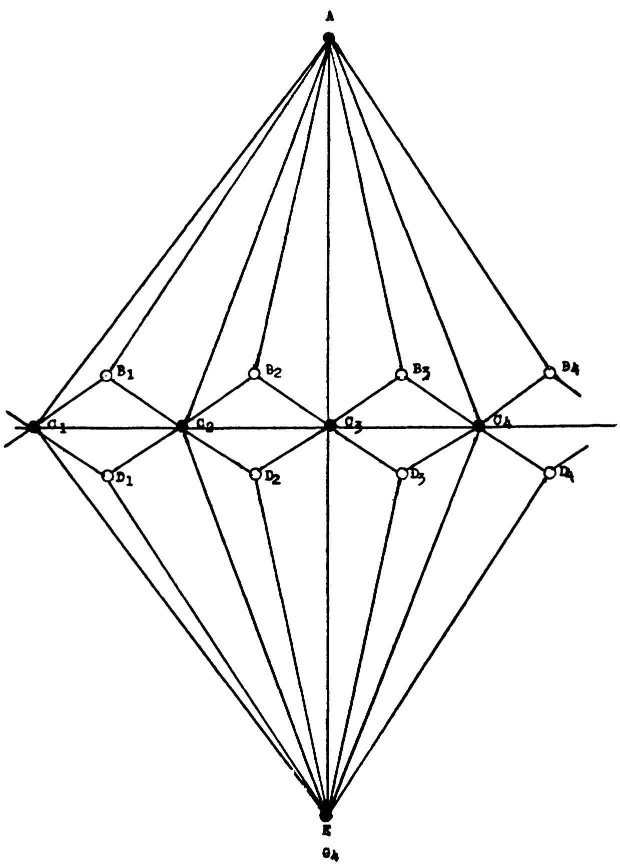

Figure 1.

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2. Path numbers and path lengths. For any graph G with $n(G)$ nodes we let $m(G)$ denote the number of disjoint simple paths required to cover all vertices of G, and let $p(G)$ denote the maximum number of nodes contained in a simple path on G. We call $m(G)$ the "path number" of G and $p(G)$ the "path length" of G. If G can be represented as the edges and vertices of a convex polyhedron in three-dimensional space, we say that G is "3-polyhedral". Now let

$$p(n) = \min\{p(G)\colon\ G \text{ is 3-polyhedral and } n(G) = n\}$$
$$m(n) = \max\{m(G)\colon G \text{ is 3-polyhedral and } n(G) = n\}\ .$$

We will show, by means of a class of examples, that $m(n) \geqq (n-10)/3$ and $p(n) \leqq (2n+13)/3$ for all n.

3. The graphs G_k. Let the graph $G_k (k \geqq 3)$ have $3k+2$ vertices, which we will denote by a, b_i, c_i, d_i, and e(i ranging from 1 to k). Let the edges of G_k be (a, b_i), (a, c_i), (e, d_i), (e, c_i), (c_i, c_{i+1}), (c_i, b_i), (c_i, d_i), (d_i, c_{i+1}), and (b_i, c_{i+1}). Thus a and e are of valence $2k$, the c_i are of valence 8, and the b_i and d_i are of valence 3. See Figure 1 for a drawing of G_4. G_k can be represented as a triangulation of the plane, and it is easy to show by induction [4] that if $n(G) \geqq 4$ and G can be

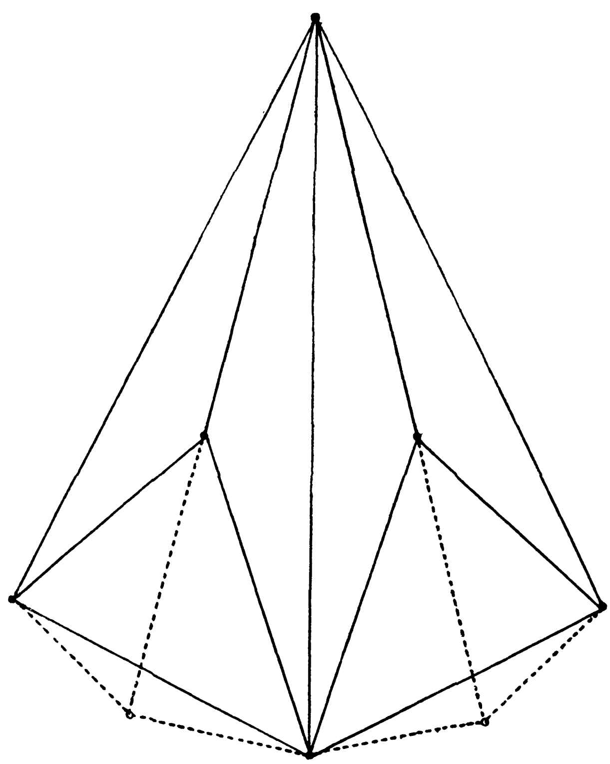

Figure 2.

represented as a triangulation of the plane, then G can be represented as the edges and vertices of a convex polyhedron in 3-space. Alternatively, one could apply the "Fundamentalsatz der Konvexen Typen" of E. Steinitz [6]. But in the case of G_k it is really unnecessary to use any such general results, for G_k is clearly the graph of the polyhedron obtained by appropriately truncating a bipyramid whose base is a regular $2k$-gon (Figure 2 illustrates how the top half of a bipyramid should be truncated in obtaining G_4).

If we color a, c_i, and e black and let b_i and d_i be white (where i ranges from 1 to k), then G_k consists of $n + 2$ black nodes and $2n$ white ones. Since each white node has only black neighbors, each simple path in G_k must contain at most one more white node than black. Thus at least $2k - (k + 2) = k - 2$ disjoint simple paths are required to visit every node of G_k. The following set of paths shows that the path-number of G_k is, in fact, exactly $k - 2$:

$$b_1 \to c_1 \to d_1 \to e \to d_2 \to c_2 \to b_2 \to a \to b_3 \to c_3 \to d_3$$
$$b_i \to c_i \to d_i \quad (i = 4, \cdots, k) .$$

Similarly, since no simple path can contain more than $k + 2$ black vertices, it follows that no simple path can contain more than

$$(k + 2) + (k + 3) = 2k + 5$$

vertices. It is easy to construct simple paths containing exactly this many vertices, and thus the path-length of G_k is $2k + 5$. Since $n(G_k) = 3k + 2$, it follows that if $n \equiv 2 (\mathrm{mod}\, 3)$,

$$p(n) \leqq \frac{2n + 11}{3}$$
$$m(n) \geqq \frac{n - 8}{3} .$$

To get bounds for $n \equiv 1 \,(\mathrm{mod}\ 3)$, consider the graph G_k^- obtained by omitting one white vertex from G_k. For $n \equiv 0 \,(\mathrm{mod}\ 3)$, consider the graph G_k^+ obtained by adjoining to G_k a vertex connected to c_1, d_1, and e. It follows that

$$\left.\begin{aligned} p(n) &\leqq \frac{2n + 13}{3} \\ m(n) &\geqq \frac{n - 10}{3} \end{aligned}\right\} n \equiv 1 \,(\mathrm{mod}\ 3) \qquad \left.\begin{aligned} p(n) &\leqq \frac{2n + 13}{3} \\ m(n) &\geqq \frac{n - 9}{3} \end{aligned}\right\} n \equiv 0 \,(\mathrm{mod}\ 3) .$$

Grünbaum and Motzkin asked if $n(G) = p(G)$ provided all of the faces of the polyhedron representing G were triangles, and our examples show that this is not the case. They further conjectured that

$$\max_{n(G)=n} m(G)\cdot p(G) \geqq n^{1+\gamma} \qquad \text{for some } \gamma > 0 .$$

Our examples show that

$$\max_{n(G)=n} m(G)\cdot p(G) \geqq \frac{2n^2 - 7n\,130}{9} .$$

Thus for any $\gamma < 1$ we can find an N_γ such that

$$\max_{n(G)=n} m(G)\cdot p(G) > n^{1+\gamma} \qquad \text{for all } n \geqq N_\gamma .$$

Furthermore, this result is the best possible in a sense; for since $m(G) < n$ and $p(G) \leqq n$, it follows that

$$\max_{n(G)=n} m(G)\cdot p(G) < n^2 \qquad \text{for all } n .$$

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