CAPACITY DIFFERENTIALS ON OPEN RIEMANN SURFACES

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1. Introduction. We study in this report some orthogonal decompositions of the space Γ_h of harmonic differentials of finite norm, on a Riemann surface W. We obtain generalizations of the known decompositions (I)

$$\Gamma_h = \Gamma_{hm} \dot{+} \Gamma^*_{hse}$$
$$\Gamma_h = \Gamma_{h0} \dot{+} \Gamma^*_{he} .$$

We then prove some existence theorems for differentials on W harmonic except for the singularity $dz/(z-\zeta)$, of finite norm on $W - \Delta$, where Δ is a disk about $z = \zeta$.

A necessary and sufficient condition for their existence is the existence on $W - \Delta$ of a differential in $\Gamma_h(W - \delta)$ with nonzero period about the boundary β of W.

We then construct "Green's differential", "Capacity differentials", and prove some of their properties on compact bordered Riemann surfaces. The orthogonal property of Green's differential is extended to open hyperbolic Riemann surfaces.

2. Some subspaces of Γ_h.

2A. Let $\bar{W}$ be a compact bordered Riemann surface, with boundary β. Partition β into γ and $\delta = \beta - \gamma$ where γ is a union of contours γ_i. We shall define the following subspaces of Γ_h:

$$\Gamma_{h(0\gamma)} = \{\omega : \omega \in \Gamma_h, \omega = 0 \text{ on } \gamma\} .$$
$$\Gamma_{h(se\gamma)} = \left\{\omega : \omega \in \Gamma_h, \int_{\gamma_i} \omega = 0\right\} .$$

Those subspaces are clearly closed. We shall denote by $\Gamma_{h(m\gamma)}$ the subspace $\Gamma_{he} \cap \Gamma_{h(0\gamma)}$. We shall prove some orthogonal decomposition theorems.

THEOREM. $$\Gamma_h = \Gamma_{h(m\gamma)} \dot{+} \Gamma^*_{h(se\gamma)} \cap \Gamma^*_{h(o\delta)} .$$

Proof. Let $\omega \in \Gamma_h$ and $df^* \in \Gamma^*_{h(m\gamma)}$. Then $(\omega, df^*) = \int_\beta \omega\bar{f} = \sum_i \bar{f}_{\gamma_i} \int_{\gamma_i} \omega + \int_\delta \omega\bar{f}$ where $\bar{f}_{\gamma_i}$ is the constant value of $\bar{f}$ on γ_i. Now, if

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$\omega \in \Gamma_{h(se\gamma)} \cap \Gamma_{h(o\delta)}$, $\int_{\gamma_i} \omega = 0$, and $\int_\delta \omega \bar{f} = 0$. It follows that $(\omega, df^*) = 0$. Conversely, if $(\omega, df^*) = 0$, then $\sum_i \bar{f}_{\gamma_i} \int_{\gamma_i} \omega + \int_\delta \omega \bar{f} = 0$.

Select $f = 1$ on *one* of the γ_i, say γ_{i_0}, $f = 0$ an δ and all other γ_i.

It follows that $\int_{\gamma_{i_0}} \omega = 0$. This is true for any contour γ_{i_0}. Hence $\omega \in \Gamma_{h(se\gamma)}$. Now take $f = 0$ on γ; then $\int_\delta \omega \bar{f} = 0$ for all such f. This readily implies $\omega = 0$ on δ, which proves the theorem.

2B, Define $\hat{W}_\gamma$ to be the double of $\bar{W}$ with respect to γ. It is obtained by partial welding of $\bar{W}$ along γ. It can be shown by a method analogous to the one in (I. Chapter V. § 14) that the harmonic differentials which can be continued to $\hat{W}_\gamma$ form the subspace $\Gamma_{h(0\gamma)} \dotplus \Gamma^*_{h(0\gamma)}$.

2C. We shall consider here the subspace:

$$\Gamma_{he(0\delta)} = \{\omega : \omega \in \Gamma_h, \omega = df, f = 0 \text{ on } \delta\} .$$

The following theorem gives an orthogonal decomposition of Γ_h involving $\Gamma^*_{h(se\gamma)}$:

THEOREM. $$\Gamma_h = \Gamma^*_{h(se\gamma)} \dotplus \Gamma_{he(0\delta)} \cap \Gamma_{h(0\gamma)} .$$

Proof. Let $df^* \in \Gamma_{h(0\gamma)} \cap \Gamma_{he(0\delta)}$, $\omega \in \Gamma_h$. Then $(\omega, df) = \int_\beta \omega \bar{f} = \Sigma \bar{f}_{\gamma_i} \int_{\gamma_i} \omega + \int_\delta \omega \bar{f} = \Sigma \bar{f}_{\gamma_i} \int_{\gamma_i} \omega$. If $\omega \in \Gamma_{h(se\gamma)}$, then $\int_{\gamma_i} \omega = 0$, and $(\omega, df^*) = 0$. Conversely if $(\omega, df^*) = 0$, then $\Sigma \bar{f}_{\gamma_i} \int_{\gamma_i} \omega + \int_\delta \omega \bar{f} = 0$. Take $f = 1$ on $\gamma_{i_0}, f = 0$ elsewhere. Then $\int_{\gamma_{i_0}} \omega = 0$ for any γ_{i_0} and $\omega \in \Gamma_{h(se\gamma)}$.

2D. The next theorem gives an orthogonal decomposition of Γ_h, involving $\Gamma_{h(0\gamma)}$.

THEOREM. $$\Gamma_h = \Gamma_{h(0\gamma)} \dotplus \Gamma^*_{he(0\delta)} .$$

Proof. Let $df^* \in \Gamma^*_{he(0\delta)}$, $\omega \in \Gamma_h$. Then $(\omega, df^*) = \int_\beta \omega \bar{f} = \int_\gamma \omega \bar{f}$. If $\omega \in \Gamma_{h(0\gamma)}$, $\int_\gamma \omega f = 0 = (\omega, df^*)$. Conversely, if $(\omega, df^*) = 0$, then $\int_\gamma \omega \bar{f} = 0$. This readily implies $\omega = 0$ on γ, hence $\omega \in \Gamma_{h(0\gamma)}$.

2E. We shall now extend our results to open Riemann surfaces. Let W be an open Riemann surface. Consider a closed partition of the ideal boundary β into γ and $\delta = \beta - \gamma$. Consider a neighborhood of δ, say $N_0(\delta)$, bounded by a set of contours δ_0. δ_0 divides W into $N_0(\delta)$

and $W - N_0(\delta)$. We shall exhaust $W_0 = W - N_0(\delta)$, using a regular exhaustion $\{\Omega_n\}$. Let $\omega_{h(0\gamma)} \in \Gamma_{h(0\gamma)}$. The restriction of $\omega_{h(0\gamma)}$ to Ω has a decomposition:

$$\omega_{h(0\gamma)}|_\Omega = \omega_{h(0\gamma)\Omega} \dotplus \omega^*_{he(0\delta_0)\Omega} .$$

Where $\omega_{h(0\gamma)\Omega} \in \Gamma_{h(0\gamma)}(\Omega)$ and $\omega_{he(0\delta_0)\Omega} \in \Gamma_{he(0\delta_0)}(\Omega)$. If $\Omega' \supset \Omega$, $\omega_{h(0\gamma)\Omega} - \omega_{h(0\gamma)\Omega} - \omega_{h(0\gamma)\Omega'} = \omega^*_{he(0\delta_0)\Omega'} - \omega^*_{he(0\delta_0)\Omega}$ where the right hand side is an element of $\Gamma_{he(0\delta_0)}(\Omega)$ and therefore is orthogonal to $\omega_{h(0\gamma)\Omega}$ on Ω. It follows that

$$\| \omega_{h(0\gamma)\Omega} - \omega_{h(0\gamma)\Omega'} \|^2_\Omega = \| \omega_{h(0\gamma)\Omega'} \|^2_\Omega - \| \omega_{h(0\gamma)\Omega'} \|^2_\Omega .$$

Therefore $\| \omega_{h(0\gamma)\Omega} \|_\Omega$ increases with Ω. But it is also bounded, for the orthogonal decomposition $\omega_{h(0\gamma)}|_\Omega = \omega_{h(0\gamma)\Omega} \dotplus \omega^*_{he(0\delta_0)\Omega}$ shows that $\| \omega_{h(0\gamma)\Omega} \|_\Omega \leqq \| \omega_{h(0\gamma)} \|_\Omega \leqq \| \omega_{h(0\gamma)} \|$. We find that $\| \omega_{h(0\gamma)\Omega} \|_\Omega$ has a finite limit and this implies that

$$\| \omega_{h(0\gamma)\Omega} - \omega_{h(0\gamma)\Omega'} \|_\Omega \to 0 \text{ as } \Omega \text{ and } \Omega' \to W_0 .$$

For a fixed Ω_0, the triangle inequality gives: $\| \omega_{h(0\gamma)\Omega'} - \omega_{h(0\gamma)\Omega''} \|_{\Omega_0} \to 0$ as $\Omega', \Omega'' \to W_0$ independently of each other. We conclude (I. Chapter II. Theorem 13C) that $\omega_{h(0\gamma)\Omega}$ tends to a harmonic limit differential $\omega_{h(0\gamma)W_0}$. Furthermore:

$$\| \omega_{h(0\gamma)\Omega} - \omega_{h(0\gamma)W_0} \|_\Omega \to O \text{ as } \Omega \to W_0 .$$

Let now $\sigma^* \in \Gamma^*_{he(0\delta_0)}$. Then $(\omega_{h(0\gamma)W_0}, \sigma^*)_\Omega = (\omega_{h(0\gamma)W_0} - \omega_{h(0\gamma)\Omega}, \sigma^*)$; as $\Omega \to W_0$. Then for $\delta_\nu \subset \Omega$

$$(\omega, \sigma^*) = \lim_{\nu\to\infty} (\omega, \sigma^*_\nu) = \lim_{\nu\to\infty} \left[\lim_{\Omega\to W} (\omega - \omega_{h(0\gamma)\Omega}, \sigma^*_\nu)_\Omega \right]$$

or

$$|(\omega, \sigma^*)|^2 \leqq \lim_{\nu\to\infty} \left[\lim_{\Omega\to W} \| \omega - \omega_{h(0\gamma)\Omega} \|^2_\Omega \right] .$$

$$\| \sigma^*_\nu \|^2_\Omega \leqq \lim_{\nu\to\infty} \left[\lim_{\Omega\to W} \| \omega - \omega_{h(0\gamma)\Omega} \|^2_\Omega \| \sigma^*_\nu \|^2_W \right]$$

$$= \lim_{\Omega\to W} \| \omega - \omega_{h(0\gamma)\Omega} \|^2_\Omega \cdot \lim_{\nu\to\infty} \| \sigma^*_\nu \|^2_W .$$

The last limit being finite, it follows that $(\omega, \sigma^*) = 0$. We conclude that $\omega \in \Gamma_{h(0\delta)}(W)$. Thus $\Gamma_{h(0\delta)}(W)$ is formed precisely by those differentials which can be approximated by differential of class $\Gamma_{h(0\delta)}(\Omega)$.

We state this result as a theorem.

THEOREM. *$\Gamma_{h(0\gamma)}(W)$ is the limit of $\Gamma_{h(0\gamma)}(\Omega)$ for $\Omega \to W$ in the sense that $\omega \in \Gamma_{h(0\gamma)}(W) \Longleftrightarrow$ there exists differentials $\omega_{h(0\gamma)\Omega} \in \Gamma_{h(0\gamma)}(\Omega)$ such that $\| \omega - \omega_{h(0\gamma)\Omega} \|_\Omega \to 0$.*

2F. We shall now extend Theorem 2C to open surfaces.

THEOREM. *On an arbitrary Riemann surface*

$$\Gamma_h = \Gamma^*_{h(se\gamma)} \dotplus \Gamma_{h(0\gamma)} \cap \Gamma_{he(0\delta)} .$$

Proof. It is easy to see that $\Gamma_{h(se\gamma)} \perp \Gamma^*_{h(0\gamma)} \cap \Gamma^*_{he(0\delta)}$. Let $\sigma \in \Gamma_{h(se\gamma)}$ and $\omega \in \Gamma_{h(0\gamma)} \cap \Gamma_{he(0\delta)}$. Consider a canonical exhaustion $\{\Omega\}$. Let ω be approximated in norm by $\omega_\Omega \in \Gamma_{h(0\gamma)}(\Omega) \cap \Gamma_{he(0\delta)}(\Omega)$. Then, Ω being canonical, $(\sigma, \omega^*_\Omega)_\Omega = 0$ thus $(\sigma, \omega^*)_\Omega = (\sigma, \omega^* - \omega^*_\Omega)$ and the inner product can be made arbitrarily small, while Ω is arbitrarily large. Hence $(\sigma, \omega^*) = 0$ and the orthogonality is proved.

Conversely, if $\omega \in \Gamma_h$ and $\omega \perp \Gamma^*_{h(se\gamma)}(W)$, for a canonical Ω let $\omega_{1\Omega}$ be the projection of ω, restricted to Ω on $\Gamma_{h(0\gamma)} \cap \Gamma_{he(0\delta)}$. Then $\omega - \omega_{1\Omega} \in \Gamma^*_{h(se\gamma)}(\Omega)$. For $\Omega' \supset \Omega$, we conclude that $\omega_{1\Omega} - \omega_{1\Omega'} \in \Gamma^*_{h(se\gamma)}(\Omega)$, hence $\omega_{1\Omega} - \omega_{1\Omega'} \perp \omega_{1\Omega}$. Therefore $\| \omega_{1\Omega} - \omega_{1\Omega'} \|^2_\Omega = \| \omega_{1\Omega'} \|^2_\Omega - \| \omega_{1\Omega} \|^2_\Omega \leqq \| \omega_{1\Omega'} \|^2_{\Omega'} - \| \omega_{1\Omega} \|^2_\Omega$. It follows that $\|\omega_{1\Omega}\|^2_\Omega$ increases with Ω. But $\| \omega_{1\Omega} \| \leqq \| \omega \|$. Therefore $\omega_1 = \lim_{\Omega \to W} \omega_{1\Omega}$ exists and lies in $\Gamma_{h(0\gamma)} \cap \Gamma_{he(0\delta)}$. Furthermore because $\omega - \omega_{1\Omega} \in \Gamma_{h(se\gamma)}(\Omega)$ and every dividing cycle lies in an Ω, it follows that $\omega - \omega_1 \in \Gamma^*_{h(se\gamma)}(W)$. On the other hand, $\omega \perp \Gamma^*_{h(se\gamma)}$ by assumption and $\omega \perp \Gamma^*_{h(se\gamma)}$. We conclude that $\omega - \omega_1$ and $\Gamma_{h(0\gamma)} \cap \Gamma_{he(0\delta)} \perp \Gamma^*_{h(se\gamma)}$. $\Gamma_{h(se\gamma)}$ being closed, $\Gamma^*_{h(se\gamma)}$ is $\perp \Gamma_{h(0\delta)} \cap \Gamma_{he(0\delta)}$.

3. Existence theorem.

3A. We shall now prove some existence theorems for harmonic differentials with a singularity of the type $dz/(z - \zeta)$. Let W be an open Riemann surface, $z = \zeta$ a point of W. Let us consider a disk Δ mapped on $|z| < 1$ such that $\zeta \in \Delta$. Select r_1 and r_2 positive such that $|\zeta| < r_1 < r_2 < 1$. Construct a function $e_1(z) \in C^2$ which has value 1 for $|z| < r_1$ and value 0 for $|z| > r_2$, and the function $e_2(z)$ such that $e_1 + e_2 = 1$ on W.

Let $\underline{W} = W - \{z: |z| < r_1\}$. We shall call α_0 the contour $|z| = r_1$. Let us *assume* that on $\underline{W}$ there exists a reproducing differential for α_0, say $\sigma(\alpha_0)$. To $\sigma(\alpha_0)$ corresponds an analytic differential on $\underline{W}$: $\omega = \sigma(\alpha_0) + i\sigma^*(\alpha_0)$. Denoting by q the period of ω around α_0, we consider $\varphi = (2\pi i/q)\omega$. In the annulus $r_1 < |z| < r_2$, $dz/(z - \zeta) - \varphi$ is exact; let Φ be an analytic function such that $d\Phi = dz/(z - \zeta) - \varphi$ in the annulus. Notice that Φ is defined up to an additive constant. We now construct the following differential:

$$\Theta = e_1 dz/(z - \zeta) + \Phi de_1 + e_2 \varphi$$

Θ is an element of C^1 and is closed on W punctured at $z = \zeta$. Moreover $\Theta - i\Theta^* = 0$ near the singularity and in a boundary neighborhood. Hence

Θ is square integrable and by de Rham's decomposition theorem:

$$\Theta - i\Theta^* = \omega_{e0} + \omega_h + \omega_{e0}^* .$$

Then $\tau = \Theta - \omega_{e0} = i\Theta^* + \omega_h + \omega_{e0}^*$ is closed and coclosed in any region which does not contain $z = \zeta$. τ is therefore harmonic on W except for the singularity $dz/(z - \zeta)$. Such a differential is necessarily unique; in fact, let τ and τ' be 2 solutions corresponding to the same Θ. Then $\tau - \tau'$ is harmonic and $\tau - \tau' \in \Gamma_{e0}$. Therefore $\tau - \tau' = 0$. We shall remark that two different functions Φ, differing by a constant C will yield the same τ: for in Θ, Cde, is an element of Γ_{e0}, hence immaterial for the definition of τ.

3B. Let us consider a closed partition of the ideal boundary β of W into 2 parts γ and δ, and the corresponding partition into $\gamma' = \alpha_0 \cup \gamma$ and δ for $\underline{W}$. On W we perform the decomposition:

$$\omega_h = \omega_1^* + \omega_2$$

where $\omega_1^* = \Gamma_{h(se\delta)}^*(W)$ and $\omega_2 \in \Gamma_{he(0\gamma)}(W) \cap \Gamma_{h(0\delta)}(W)$. Then $\tau = i(e_1 dz/(z - \zeta) + \Phi de_1)^* + e_2\varphi + \omega_1^* + \omega_2 + \omega_{e0}^*$ and $\tau - \omega_2 = i(e_1 dz/(z - \zeta) + \Phi de_1)^* + e_2\varphi + \omega_1^* + \omega_{e0}^*$. The left hand side has the same periods about δ as Θ, and so does the right hand side. It follows that $\tilde{\tau} = \tau - \omega_2$ and $\tilde{\tau}^*$ have the same periods about δ as the given Θ. (They have actually on $\underline{W}$ the same periods as Θ).

In particular, *if there exists* on $\underline{W}$ a differential φ' analytic with zero period along δ, we can repeat the construction outlined in § 3A and get differentials $\tilde{\tau}$ and $\tilde{\tau}^*$ with zero periods about δ.

3C. We may write the decomposition

$$\tilde{\tau} = \tilde{\psi} + \bar{\tilde{\chi}}$$

where χ is analytic and ψ is analytic except for the singularity at $z = \zeta$. If $\tilde{\tau}$ and $\tilde{\tau}^*$ have zero period about δ, the same is true for $\tilde{\psi}$ and $\tilde{\chi}$ for:

$$\tilde{\psi} = \tfrac{1}{2}(\tilde{\tau} + i\tilde{\tau}^*)$$
$$\tilde{\chi} = \tfrac{1}{2}(\tilde{\tau} - i\tilde{\tau}^*) .$$

Notice that $\tilde{\tau} = \tau$ for $\gamma = \beta$.

3D. Let Δ be the disk $|z| < r_1$. On $\underline{W}$, $(\varphi + \bar{\varphi})/2 \in \Gamma_{he} \cap \Gamma_{h0}$. We shall call $dg = \frac{1}{2}(\varphi + \bar{\varphi})$, where g is harmonic and constant on every component of the boundary of $\underline{W}$. In Δ, $\frac{1}{2}[dz/(z - \zeta) + d\bar{z}/(\bar{z} - \bar{\zeta})]$ is the differential of $\log|z - \zeta|$. To sum up we have here:

$$(\Theta + \bar{\Theta})/2 = d(e_1 \log|z - \zeta|) + d(e_2 g) .$$

By the procedure outlined in § 3A we obtain a differential $(\tau + \bar{\tau})/2$, which is harmonic exact. Putting $(\tau + \bar{\tau})/2 = dh$, h is constant on every component of $\beta(W)$.

3E. We show here that one may get a function h which is constant along β. Let $\sigma(\alpha_0)$ be defined as in § 3A. $\sigma(\alpha_0)^* \in \Gamma^*_{h0}(\underline{W})$, therefore $\sigma(\alpha_0)^* \notin \Gamma_{he}(\underline{W})$. Then $\sigma(\alpha_0)^*$ has a nonzero period along α_0 and $\sigma(\alpha_0)^* \notin \Gamma_{h(se\alpha_0)}(\underline{W})$. It follows that $\sigma(\alpha_0) \notin \Gamma^*_{h(se\alpha_0)}$ and the orthogonal projection of $\sigma(\alpha_0)$ on $\Gamma_{he(0\beta)} \cap \Gamma_{h(0\alpha_0)}$ is not zero. (Theorem 2C.) Let $\sigma'(\alpha_0)$ be that projection; using $\sigma'(\alpha_0)$ instead of $\sigma(\alpha_0)$ in the previous construction one gets a function h with the required property, say h_0. We suggest for dh_0 the name of *Green's differential*, and for the corresponding τ, say τ_0, the name of *capacity differential.*

3F. Let us now consider a closed partition of β into γ and δ; put $\alpha_0 \cup \gamma = \gamma'$. We consider here instead of $\sigma^*(\alpha_0)$ the projection of $\sigma^*(\alpha_0)$ on $\Gamma_{h(se\delta)}$. This is equivalent to subtracting from $\sigma^*(\alpha_0)$ a quantity which is an element of $\Gamma^*_{he(0\delta)} \cap \Gamma^*_{h(0\gamma')}$: (This means that the remaining part of $\sigma(\alpha_0)$ is still an element of $\Gamma_{he} \cap \Gamma_{h0}$.) We get a nonzero projection if and only if $\sigma(\alpha_0) \notin \Gamma_{h(0\gamma)} \cap \Gamma_{he(0\gamma')}$ i.e. putting $\sigma(\alpha_0) = df$, f should have different constant values on α_0 and γ. We shall call the differential τ thus obtained a *capacity differential for the boundary part* γ. If γ is a component of β, we get the capacity differential of the boundary component γ.

4. Reproducing properties.

4A. We shall assume first that W is the interior of a compact bordered surface. Let us call α the circle $|z - \zeta| = r$ and set $W_0 = W - \{|z - \zeta| < r\}$. Let τ_0 be Green's differential, and Θ_0 the corresponding singularity. For $\omega = df \in \Gamma_{he}$ we write down the generalized Green's formula on W_0:

$$(\omega, (\tau_0 + \bar{\tau}_0)/2) - (\omega^*, (\tau_0 + \bar{\tau}_0)^*/2) = 0.$$

or

$$\int_{\beta - \alpha} f(\tau_0 + \bar{\tau}_0)^*/2 - h_0 df^* = 0 \,.$$

First, h_0 being 0 on β, $\int_\beta h_0 df^* = 0$. Therefore:

$$\int_\beta f(\tau_0 + \bar{\tau}_0)^*/2 = \int_\alpha f(\tau_0 + \bar{\tau}_0)^*/2 - h_0 df^* \,.$$

Let now $W_0 \to W$, or $r \to 0$. For $r = \varepsilon$ on $|z| = r$, $h_0 = \log|z - \zeta| + \eta_1(z)$.

where $\eta_1(z)$ is bounded. It follows that $\lim_{r\to 0}\int_\alpha h_0 df^* = 0$. Now on $|z| < r$,

$$\tfrac{1}{2}(\tau_0 + \bar{\tau}_0)^* = (\Theta + \bar{\Theta}/2)^* + \eta_2(z) ,$$

where $\eta_2(z)$ is bounded. Moreover:

$$(\Theta + \bar{\Theta})^*/2 = (-i\Theta + i\bar{\Theta})/2 = -i(\Theta - \bar{\Theta})/2 = d\arg(z - \zeta) .$$

Therefore:

$$\lim_{r\to 0}\int_\alpha f(\tau_0 + \bar{\tau}_0)^*/2 = \lim_{r\to 0}\int_\alpha f d\arg(z-\zeta) = 2\pi f(\zeta) .$$

We now may state the following theorem:

THEOREM. *For all harmonic functions f or W, the differential $\tau_0 + \bar{\tau}_0/2$ has the following reproducing property*:

$$\int_\beta f(\tau_0 + \bar{\tau}_0)^*/2 = 2\pi f(\zeta) .$$

4B. If we now use h instead of h_0 we need to restrict df to the class $\Gamma_{he} \cap \Gamma^*_{hse}$ and state:

THEOREM. For all harmonic functions f on W whose conjugate periods vanish along all dividing cycles, the differential $\tau + \bar{\tau}/2$ satisfies:

$$\int_\beta f(\tau + \bar{\tau})/2 = 2\pi f(\zeta) .$$

4C. Green's differential enjoys another important property:

THEOREM. *Let $df \in \Gamma_{he}$, and τ_0 be Green's differential. Then*:

$$(df, (\tau_0 + \bar{\tau}_0)^*/2) = 0 .$$

Proof.

$$\begin{aligned}(df, (\tau_0 + \bar{\tau}_0)^*/2) &= (df, (\Theta_0 + \bar{\Theta}_0)^*/2) \\ &= -\lim_{r\to 0}\int_{\beta-\alpha} f(\Theta_0 + \bar{\Theta}_0)/2 = \lim_{r\to 0}\int_\alpha f(\Theta_0 + \bar{\Theta}_0)/2 .\end{aligned}$$

4D. We shall now extend Theorem 4C to open Riemann surfaces. Let W be an open Riemann surface and $\{\Omega\}$ a canonical exhaustion. Let $dF_\Omega = (\varphi_{0\Omega} + \bar{\varphi}_{0\Omega})/2$; we know that $dF_\Omega \in \Gamma_{he(0\beta)} \cap \Gamma_{h(0\alpha)}$ on $\Omega - \delta$. If $dF = (\varphi_0 + \bar{\varphi}_0)/2$, we obtain easily by a reasoning analogous to the one in (I, Chapter V. § 14. C) that

$$\lim_{\Omega\to W} \| dF - dF_\Omega \|_{\Omega-\delta} = 0 .$$

We recall that $(\Theta + \bar{\Theta})/2 = d(e_1 \log|z - \zeta|) + d(e_2 F)$. We now have:

$$\begin{aligned}(df, (\tau_0 + \bar{\tau}_0)^*/2) &= (df, (\Theta_0 + \bar{\Theta}_0)^*/2) \\ &= \lim_{\Omega \to W} (df, (\Theta_0 + \bar{\Theta}_0)^*/2)_\Omega \\ &= \lim_{\Omega \to W} (df, \tfrac{1}{2}(\Theta_0 + \bar{\Theta}_0)^* - \tfrac{1}{2}(\Theta_{0\Omega} + \bar{\Theta}_{0\Omega})^*)_\Omega \\ &= \lim_{\Omega \to W} (df, d(e_2 F)^* - d(e_2 F)^*)_\Omega \\ &= \lim_{\Omega \to W} (df, d(e_2 F)^* - d(e_2 F_\Omega)^*)_{\Omega - \delta} .\end{aligned}$$

Now let A be the compact set $\{z : r_1 \leqq |z| \leqq r_2\}$ and let $\Omega - \delta = A \cup A'$. We have:

$$\begin{aligned}&\| d(e_2 F)^* - d(e_2 F_\Omega)^* \|_{\Omega - \delta} \\ &\qquad = \| d(e_2 F) - d(e_2 F_\Omega) \|_{\Omega - \delta} \\ &\qquad = \| de_2(F - F_\Omega) \|_A + \| dF - dF_\Omega \|_{A'} .\end{aligned}$$

Because $\| dF - dF_\Omega \|_A \to 0$ as $\Omega \to W$, $F \to F_\Omega$ uniformly on A hence $\lim_{\Omega \to W} \| de_2(F - F_\Omega) \|_A = 0$. Now on A'

$$\lim_{\Omega \to W} \| dF - dF_\Omega \|_{A'} \leqq \lim_{\Omega \to W} \| dF - dF_\Omega \|_{\Omega - \delta} = 0 .$$

It follows that $\lim_{\Omega \to W} \| d(e_2 F)^* - d(e_2 F_\Omega)^* \|_{\Omega - \delta} = 0$ and $|(df, (\tau_0 + \bar{\tau}_0)/2)| \leqq \lim_{\Omega \to W} \| df \|_{\Omega - \delta} \| d(e_2 F)^* - d(e_2 F_\Omega)^* \|_{\Omega - \delta} = 0$, which proves the theorem.

Bibliography

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