THE TIME-DOMAIN ANALYSIS OF A CONTINUOUS PARAMETER WEAKLY STATIONARY STOCHASTIC PROCESS

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1. Introduction. In this paper we shall give a new, spectral-free, method to obtain the differential innovations and the Wold decomposition of a univariate, continuous parameter, weakly stationary[1], mean-continuous, non-deterministic stochastic process $(f_t, \ -\infty < t < \infty)$. We shall affect a transition from the continuous to the discrete parameter case by systematic use of the infinitesimal generator iH of the shift group $(U_t, \ -\infty < t < \infty)$ of the process, and of the Cayley transform V of the self-adjoint operator H(§ 2). Our analysis will be purely in the time-domain.

With the f_t-process we shall associate the discrete parameter process $(f'_n)_{n=-\infty}^{\infty}$, where $f'_n = V^n(f_0)$. Since V is unitary, the f'_n-process is weakly stationary. Letting $\mathscr{M}_t$, $\mathscr{M}'_n$ be the past and present subspaces of the f_t- and f'_n-processes, respectively, and $\mathscr{M}_{-\infty}$, $\mathscr{M}'_{-\infty}$ be their remote pasts, we shall show that $\mathscr{M}_0 = \mathscr{M}'_0$ and $\mathscr{M}_{-\infty} = \mathscr{M}'_{-\infty}$ (§ 4). In the non-deterministic case we shall show that the subspace $\mathscr{N}_t = \mathscr{M}^{\perp}_{-\infty} \cap \mathscr{M}_t$ is the past and present of the process $(h_t, -\infty < t < \infty)$, where $h_t = U_t(h'_0)$, h'_0 being the 0th normalized innovation of the discrete f'_n-process (§ 5). We shall then show (§ 6) that the h_t-process is weakly Markovian[1] with covariance $e^{-|t|}$ for lag t, and that if

$$(1.1) \qquad \xi_t = T_t(h'_0), \qquad \text{where} \quad T_t = \frac{1}{\sqrt{2}}\left\{U_t - I + \int_0^t U_s ds\right\},$$

the process $(\xi_t, \ -\infty < t < \infty)$ has stationary, orthogonal increments such that $|\xi_b - \xi_a|^2 = |b - a|$. These increments are the "differential innovations" of our f_t-process; for we shall show (6.6) that the set of all convergent stochastic integrals $\int_{-\infty}^t c(s)d\xi_s$, $c \in L_2(-\infty, t)$, is identical with the subspace $\mathscr{N}_t$ mentioned above. Since

$$\mathscr{M}_t = \mathscr{N}_t + \mathscr{M}_{-\infty}, \quad \mathscr{N}_t \perp \mathscr{M}_{-\infty},$$

it follows at once that $f_t = u_t + v_t$, where the u_t form a one-sided moving

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[1] In this paper the term "weakly" has the same meaning as Doob's expression "in the wide sense" [5, p. 95].

average process, and the v_t a deterministic one:

$$u_t = \int_0^\infty c(s) d_s \xi_{t-s}, \quad v_t = \text{projection of } f_t \text{ on } \mathscr{M}_{-\infty} .$$

We thus get the Wold decomposition cf. 6.7 below.

In justification of this new approach we may mention its simplicity and coherence. With the time-domain analysis so completed, one can develop the spectral theory in an equally coherent way. One can also deal conveniently with the extension to vector-valued processes. In comparison, an approach in which spectral considerations are brought to bear on time-domain questions or vice versa seems cumbersome and roundabout. But quite apart from this, our approach is essentially more general than one based on the spectral resolution of the group $(U_t, -\infty < t < \infty)$ and is more suggestive of further research, although it does not yield any really new results on univariate stationary processes. As prediction theory has advanced, its connection with the theory of shift-invariant subspaces of the Hardy class H_2 initiated by Beurling [2] has been noticed; see especially Helson and Lowdenslager [10] and Lax [14]. Recently Halmos [8] has brought to light a result, which shows that underlying both theories is a semi-group of isometries on a Hilbert space (cf. also [15]). (In the case under discussion, this semi-group comprises the (isometric) restrictions of the unitary operators U_t^* to the subspace M_0.) One of us [16] has found that our approach based on use of the infinitesimal generator iH and of the operator T_t defined in (1.1) extends to general continuous parameter semi-groups of isometries to yield valuable results concerning their structure. But since in general these isometries will be non-normal, the generator H will not be self-adjoint and the usual spectral considerations will fail; cf. Cooper [3]. Thus *it seems worthwhile to try to dispense with spectral tools in the analysis of time-domain problems.*

Hanner [9] was the first to make a purely time-domain analysis in the continuous parameter case. By an ingenious construction he proved the existence of differential innovations and derived the Wold decomposition. His approach, somewhat *ad hoc* in nature, has not been pursued in the literature, and its points of contact with the earlier work of Cooper [3] have gone unnoticed. Our approach differs from that of Hanner and Cooper in the transition we make to the discrete parameter case by means of the infinitesimal generator and the Cayley transform.

It is reasonably clear that our approach will work in the case of processes for which the differential innovations can be had by Hanner's method. As an instance we cite the study of continuous parameter random distributions due to K. Ito, Gelfand, and Balagangadharan [12, 7, 1]. It is also possible that our ideas may apply to some of the non-stationary processes studied recently by Cramer [4, 4′, 4″].

2. The infinitesimal generator and Cayley transform. Let $(U_t, -\infty < t < \infty)$ be a *strongly continuous group of unitary operators acting on a complex Hilbert space* $\mathfrak{X}$; i.e. let

$$(2.1)\quad \begin{cases} \text{(a)} & U_t \text{ be a unitary operator on } \mathfrak{X} \text{ onto } \mathfrak{X}, \ -\infty < t < \infty. \\ \text{(b)} & U_s U_t = U_{s+t} = U_t U_s, \ -\infty < s, t < \infty. \\ \text{(c)} & U_{t+h} \to U_t (\text{strongly})^2 \text{ on } \mathfrak{X} \text{ as } h \to 0, \ -\infty < t < \infty. \end{cases}$$

It is known [17, p. 385] that the group has an *infinitesimal generator*

$$(2.2)\qquad iH = \lim_{h \to 0} \frac{1}{h}\{U_t - I\} \qquad \text{on } \mathscr{D},$$

where H is a self-adjoint operator with domain $\mathscr{D}$, and $\mathscr{D}$ is a linear manifold everywhere dense in $\mathfrak{X}$. Also, cf. [19, p. 142 and 6, p. 622]

$$(2.3)\quad \begin{cases} \text{(a)} & H + iI \text{ is one-to-one on } \mathscr{D} \text{ onto } \mathfrak{X}, \\ \text{(b)} & (H + iI)^{-1} = \dfrac{1}{i}\displaystyle\int_0^\infty e^{-t} U_t dt \text{ is bounded and one-to-one} \\ & \text{on } \mathfrak{X} \text{ onto } \mathscr{D}, \text{ and } |(H + iI)^{-1}|_B \leqq 1^{(3)}. \end{cases}$$

Now let V be the *Cayley transform* of H:

$$(2.4)\qquad V = c(H) = (H - iI)(H + iI)^{-1} \text{ on } \mathfrak{X}.$$

Then [19, p. 304]

$$(2.5)\quad \begin{cases} \text{(a)} & V \text{ is unitary on } \mathfrak{X} \text{ onto } \mathfrak{X}, \\ \text{(b)} & I - V = 2i(H + iI)^{-1} \text{ is one-to-one on } \mathfrak{X} \text{ onto } \mathscr{D}, \\ \text{(c)} & H = i(I + V)(I - V)^{-1} \text{ on } \mathscr{D}, \\ \text{(d)} & U_t V^n = V^n U_t \text{ on } \mathfrak{X}, \ -\infty < n, t < \infty, \ n = \text{integer}. \end{cases}$$

In this section we shall establish the relationship between U_t and V^n for arbitrary t and n on which will hinge the subsequent development.

The U_t are expressible in terms of H by the Hille-Yosida exponential formula, cf. [17, p. 403],

$$(2.6)\quad \begin{cases} U_t = \lim\limits_{n \to \infty} \exp(tiHJ_n), \ (\text{strong})^2, \qquad t \geqq 0 \\ J_n = \left(I - \dfrac{1}{h} iH\right)^{-1}. \end{cases}$$

One sees trivially that J_n is a bounded operator and that so therefore is $iHJ_n = n(J_n - I)$. Hence the term $\exp(tiHJ_n)$ in (2.6) is definable

[2] It is to be understood in the sequel that all operator-limits are in the strong sense.

[3] $|T|_B$ refers to the Banach norm of the operator T.

by the usual power-series. We now assert two lemmas:

2.7 Lemma. (*Expression of U_t in terms of V^k*).

$$U_{\pm t} = e^{-t}I + \lim_{n\to\infty}\sum_{k=1}^{\infty}\frac{1}{k!}\left(\frac{-nt}{n+1}\right)^k\{(I + A_{\pm n})^k - I\}\,, \qquad t \geqq 0\,,$$

where

$$A_{\pm n} = \frac{2n}{n+1}\sum_{j=1}^{\infty}\left(\frac{n-1}{n+1}\right)^{j-1}V^{\pm j}\,, \qquad n \geqq 0.$$

Proof. Let $t \geqq 0$. Then by (2.6)

(1) $$U_t = \lim_{n\to\infty}\sum_{k=0}^{\infty}\frac{t^k}{k!}(iHJ_n)^k\,.$$

Using (2.5)(c) we can express the R.H.S. of (1) in terms of V:

$$\begin{aligned} iHJ_n &= -(I+V)(I-V)^{-1}\left\{I + \frac{1}{n}(I+V)(I-V)^{-1}\right\}^{-1} \\ &= -\frac{n}{n+1}(I + A_n) \end{aligned}$$

after some simplification. Thus

$$(iHJ_n)^k = \left(-\frac{n}{n+1}\right)^k I + \left(-\frac{n}{n+1}\right)^k\{(I + A_n)^k - I\}\,, \qquad k \geqq 0\,.$$

Hence from (1)

(2) $$U_t = \lim_{n\to\infty}\exp\left(\frac{-nt}{n+1}\right)I + \lim_{n\to\infty}\sum_{k=1}^{\infty}\frac{1}{k!}\left(\frac{-nt}{n+1}\right)^k\{(I + A_n)^{-1} - I\}\,.$$

Since the first term on the R.H.S. is $e^{-t}I$, we have the desired expression for U_t, $t \geqq 0$.

To obtain the expression for U_{-t}, $t \geqq 0$, we note that $U_{-t} = U_t^*$, $V^* = V^{-1}$ and so $A_n^* = A_{-n}$, $n \geqq 0$. Thus, taking adjoints on both sides of (2), we get the desired result.

2.8 Lemma. (*Expression of V^n in terms of U^t*).

$$V^{\pm n} = I + 2\int_0^{\infty} L_n'(2t)e^{-t}U_{\pm t}dt\,, \qquad n \geqq 0$$

where

$$L_n(t) = \sum_{k=0}^{n}\frac{(-1)^k}{k!}\binom{n}{k}t^k\,, \qquad n \geqq 0, \text{ (nth } \textit{Laguerre polynomial}).$$

Proof. The result obviously holds for $n = 0$. We establish it for $n > 0$ by induction. For $n = 1$, the desired result reduces to the equality

$$V = I - 2\int_0^\infty e^{-t} U_t dt \ , \tag{1}$$

the correctness of which is clear from (2.5)(b) and (2.5)(b). Next, assuming the result for n, we find using (1) that

$$V^{n+1} = I + 2\int_0^\infty \{L'_n(2t) - 1\} e^{-t} U_t dt - 4\int_0^\infty \int_0^\infty L'_n(2t) e^{-(s+t)} U_{s+t} ds dt \ .$$

Putting $\sigma = s + t$, and using Dirichlet's formulae we find that

$$4\int_0^\infty \int_0^\infty L'_n(2t) e^{-(s+t)} U_{s+t} ds dt = 2\int_0^\infty \{L_n(2\sigma) - 1\} e^{-\sigma} U_\sigma d\sigma \ .$$

Hence

$$V^{n+1} = I + 2\int_0^\infty \{L'_n(2t) - L_n(2t)\} e^{-t} U_t dt \ .$$

Since the Laguerre polynomials satisfy the recurrence relation $L_n = L'_n - L'_{n+1}$, cf. [18, p. 299, (10)] we get

$$V^{n+1} = I + 2\int_0^\infty L'_{n+1}(2t) e^{-t} U_t dt \ ,$$

as desired. The result thus holds for all $n \geqq 1$. Its validity for $n \leqq -1$ follows on taking adjoints and noting that $V^{-n} = (V^n)^*$ and $U_{-t} = U_t^*$.

We shall denote by $\mathfrak{S}(X_\lambda)_{\lambda \in \Lambda}$ the (closed) subspace spanned by the subsets X_λ of $\mathfrak{X}$, for $\lambda \in \Lambda$. We now assert the following lemma:

2.9 LEMMA. *For any $X \subseteqq \mathfrak{X}$, we have*

(a) $$\mathfrak{S}\{V^{-n}(X)\}_{n \geqq 0} = \mathfrak{S}\{U_{-t}(X)\}_{t \geqq 0}$$

(b) $$\mathfrak{S}\{V^{n}(X)\}_{n \geqq 0} = \mathfrak{S}\{U_{t}(X)\}_{t \geqq 0} \ .$$

Proof (a). Lemma 2.8 asserts that for $n \geqq 0$ V^{-n} is the strong limit of a linear combination of the U_{-t} for $t \geqq 0$. It follows that for any $X \subseteqq \mathfrak{X}$,

$$V^{-n}(X) \subseteqq \mathfrak{S}\{U_{-t}(X)\}_{t \geqq 0} \ ,$$

whence

$$\mathfrak{S}\{V^{-n}(X)\}_{n \geqq 0} \subseteqq \mathfrak{S}\{U_{-t}(X)\}_{t \geqq 0} \ .$$

On the other hand, Lemma 2.7 asserts that for $t \geqq 0$, U_{-t} is the strong limit of a linear combination of the V^{-n}, for $n \geqq 0$, so that

$$U_{-t}(X) \subseteqq \mathfrak{S}\{V^{-n}(X)\}_{n\geqq 0}\,.$$

From this follows the inclusion reverse to (1), thereby yielding (a).

(b) can be derived similarly from Lemmas 2.7, 2.8 taking V^n, U_t, instead of V^{-n}, U_{-t}, with $n, t \geqq 0$.

3. **Weakly stationary stochastic processes.** In this section we shall recall the basic notions and results on weakly stationary stochastic processes.

By a *weakly stationary stochastic process* (S.P.) is meant a function f on $(-\infty, \infty)$ to a complex Hilbert space X such that the inner product

$$(3.1) \qquad (f_s, f_t) = \gamma_{s-t}$$

depends only on the difference $s-t$ and not on s and t separately. The complex-valued function γ on $(-\infty, \infty)$ is called the *covariance function* of the S.P. It is convenient to denote the values of f and γ at t by f_t and γ_t rather than by $f(t)$ and $\gamma(t)$, and to denote the S.P. itself by $(f_t, -\infty < t < \infty)$ rather than by f.

We shall be especially interested in the subspaces

$$(3.2) \qquad \begin{cases} \mathscr{M}_t = \mathfrak{S}(f_s)_{s\leqq t}, & -\infty < t < \infty \\ \mathscr{M}_{-\infty} = \bigcap_{-\infty<t<\infty} \mathscr{M}_t\,, & \mathscr{M}_\infty = \mathfrak{S}(f_s)_{s<\infty}\,. \end{cases}$$

We shall call $\mathscr{M}_t$ *the past and present of* f_t, $\mathscr{M}_{-\infty}$ *the remote past of the* S.P., *and* $\mathscr{M}_\infty$ *the space spanned by the* S.P. Obviously

$$(3.3) \qquad \begin{cases} \mathscr{M}_{-\infty} \subseteqq \mathscr{M}_s \subseteqq \mathscr{M}_t \subseteqq \mathscr{M}_\infty\,, & -\infty < s < t < \infty \\ \mathscr{M}_{-\infty} = \bigcap_{t\leqq 0} \mathscr{M}_t\,. \end{cases}$$

It is known, cf. Karhunen [13, p. 55], that if $(f_t, -\infty < t < \infty)$ is a weakly stationary S.P., then there exists a group of unitary operators U_t on $\mathfrak{X}$, $-\infty < t < \infty$, such that

$$(3.4) \qquad f_{s+t} = U_t(f_s)\,, \qquad -\infty < s, t < \infty\,.$$

The operators U_t are uniquely determined on the subspace $\mathscr{M}_\infty$ but not on X. We shall call $(U_t, -\infty < t < \infty)$ *the shift group of the* S.P. $(f_t, -\infty < t < \infty)$. It follows easily, cf. Hanner [9, p.162], that

$$(3.5) \quad U_t(\mathscr{M}_s) = \mathscr{M}_{s+t},\ U_t(\mathscr{M}_{-\infty}) = \mathscr{M}_{-\infty},\ U_t(\mathscr{M}_\infty) = \mathscr{M}_\infty,\ -\infty < s, t < \infty\,.$$

We call a S.P. $(f_t, -\infty < t < \infty)$ *mean-continuous*, if the function f is continuous on $(-\infty, \infty)$ with respect to the metric induced by the norm of the Hilbert space $\mathfrak{X}$. From the stationarity condition (3.1) we readily infer the following:

3.6 LEMMA. *For a weakly stationary* S.P. $(f_t, -\infty < t < \infty)$ *with covariance function* γ *mean-continuity is equivalent to each of the conditions*:

(i) f *is continuous at* 0,

(ii) γ *is continuous at* 0,

(iii) γ *is continuous on* $(-\infty, \infty)$,

(iv) *the shift group* $(U_t, -\infty < t < \infty)$ *is strongly continuous on* $\mathscr{M}_\infty$.

The following result is known:

3.7 LEMMA. *If the* S.P. *is mean-continuous, then*

(a) $\mathscr{M}_\infty$ *is a separable subspace of* $\mathfrak{X}$,

(b) $\mathscr{M}_{t-} = \mathscr{M}_t = \mathscr{M}_{t+}$, $-\infty < t < \infty$, *where* $\mathscr{M}_{t-} = \text{clos.} \bigcup_{s<t} \mathscr{M}_s$, $\mathscr{M}_{t+} = \bigcap_{s>t} \mathscr{M}_s$.

4. The associated discrete parameter process. Let $(f_t, -\infty < t < \infty)$ be a weakly stationary, *mean-continuous* S.P. with shift group $(U_t, -\infty < t < \infty)$. Let V be the Cayley transform of H, where iH is the infinitesimal generator of the shift group, cf. (2.2), (2.4). Let

$$f'_n = V^n(f_0)\,. \tag{4.1}$$

Then the bisequence $(f'_n)_{-\infty}^{\infty}$ is a discrete-parameter, weakly stationary S.P. with shift operator V. We shall call it *the discrete* S.P. *associated with* $(f_t, -\infty < t < \infty)$.

We shall denote the past and present of f'_n, the remote past, and the subspace spanned by the S.P. $(f'_n)_{-\infty}^{\infty}$ by $\mathscr{M}'_n$, $\mathscr{M}'_{-\infty}$, and $\mathscr{M}'_\infty$, respectively; thus

$$\mathscr{M}'_n = \mathfrak{S}(f'_k)_{k=-\infty}^{n}, \qquad \mathscr{M}'_{-\infty} = \bigcap_{n=\infty}^{\infty} \mathscr{M}'_n\,, \qquad \mathscr{M}'_\infty = \mathfrak{S}(f'_k)_{k=-\infty}^{\infty}\,. \tag{4.2}$$

It follows that

$$\begin{cases} \mathscr{M}'_{-\infty} \subseteq \mathscr{M}'_m \subseteq \mathscr{M}'_n \subseteq \mathscr{M}'_\infty, \ -\infty < m < n < \infty \\ \mathscr{M}'_{-\infty} = \bigcap\limits_{n=-\infty}^{\circ} \mathscr{M}'_n\,. \end{cases} \tag{4.3}$$

As far as we know the associated discrete parameter S.P. $(f'_n)_{-\infty}^{\infty}$ has been defined in the literature, not by (4.1), but as the process whose spectral distribution is the Cayley transform (in the complex plane) of the spectral distribution of the given continuous parameter process, cf. e.g. Doob [5, p. 583]. It can be shown that the two definitions are equivalent. But as indicated in § 1 there are advantages in adopting a purely time-domain and spectral-free definition. For instance, in the light of Lemma 2.9 we can assert the following theorem, which reveals the close relationship between the two processes. Variants of parts (a), (b) of this theorem are know, cf. e.g. Doob [5, p. 583–84]; part (c) is new as for as we know.

4.4 THEOREM. (a) $\mathscr{M}_0 = \mathscr{M}'_0$, (b) $\mathscr{M}_\infty = \mathscr{M}'_\infty$, (c) $\mathscr{M}_{-\infty} = \mathscr{M}'_{-\infty}$.

Proof. (a) Take $X = \{f_0\}$ in 2.9(a). We then get

$$\mathscr{M}'_0 = \mathfrak{S}(V^{-n}(f_0))_{n\geqq 0} = \mathfrak{S}(U_{-t}(f_0))_{t\geqq 0} = \mathscr{M}_0 .$$

(b) Now take $X = \{f_0\}$ in 2.9(b). We then get $\mathfrak{S}(V^n(f_0))_{n\geqq 0} = \mathfrak{S}(U_t(f_0))_{t\geqq 0}$. Hence,

$$\begin{aligned} \mathscr{M}'_\infty &= \text{clos.}\,\{\mathscr{M}'_0 + \mathfrak{S}(V^n(f_0))_{n\geqq 0}\} \\ &= \text{clos.}\,\{\mathscr{M}_0 + \mathfrak{S}(U_t(f_0))_{t\geqq 0}\} \qquad \text{(by (a))} \\ &= \mathscr{M}_\infty . \end{aligned}$$

(c) Take $X = \mathscr{M}_{-\infty}$ in 2.9(b). Then using (3.5) we get

$$V^k(\mathscr{M}_{-\infty}) \subseteqq \mathfrak{S}(V^n(\mathscr{M}_{-\infty}))_{n\geqq 0} = \mathfrak{S}(U_t(\mathscr{M}_{-\infty}))_{t\geqq 0} = \mathscr{M}_{-\infty}, \quad k \geqq 0 .$$

Applying V^{-k} to both sides, and using (a),

$$\mathscr{M}_{-\infty} \subseteqq V^{-k}(\mathscr{M}_{-\infty}) \subseteqq V^{-k}(\mathscr{M}_0) = V^{-k}(\mathscr{M}'_0) = \mathscr{M}'_{-k}, \quad k \geqq 0 .$$

Hence, cf. (4.3),

$$(1) \qquad \mathscr{M}_{-\infty} \subseteqq \bigcap_{k=0}^{\infty} \mathscr{M}'_{-k} = \mathscr{M}'_{-\infty} .$$

Next taking $X = \mathscr{M}'_{-\infty}$ in 2.9(b), we get

$$U_s(\mathscr{M}'_{-\infty}) \subseteqq \mathfrak{S}(U_t(\mathscr{M}'_{-\infty}))_{t\geqq 0} = \mathfrak{S}(V^n(\mathscr{M}'_{-\infty}))_{n\geqq 0} = \mathscr{M}'_{-\infty}, \quad s \geqq 0.$$

Proceeding as before, we derive the inclusion relation reverse to that in (1). Thus (c).

5. Non-deterministic S.P. Pre-Wold decomposition.

We shall say that a S.P. $(f_t, -\infty < t < \infty)$ is *deterministic*, if and only if $\mathscr{M}_{-\infty} = \mathscr{M}_\infty$; otherwise *non-deterministic*. From the stationarity condition (3.1) we infer the following lemma, cf. Hanner [9, p. 163]:

5.1 LEMMA. *For a weakly stationary* S.P. *the following conditions are equivalent*:

(i) *the* S.P. *is deterministic*

(ii) $\mathscr{M}_s = \mathscr{M}_t$ *for all* s, t, $-\infty < s, t < \infty$

(iii) $\mathcal{M}_s = \mathcal{M}_t$ *for some* $s, t \quad -\infty < s < t < \infty$

(iv) $f_t \in \mathcal{M}_s$ *for some* $s, t, \quad -\infty < s < t < \infty$.

Let the S.P. be non-deterministic. Then by 5.1 (iii) for any t and any $s < t$, $\mathcal{M}_{-\infty} \subseteqq \mathcal{M}_s \subset \mathcal{M}_t$. Hence

$$\mathcal{N}_t = \mathcal{M}_{-\infty}^{\perp} \cap \mathcal{M}_t \neq \{0\}, \quad -\infty < t < \infty ,$$

and we get the decomposition

$$\mathcal{M}_t = \mathcal{M}_{-\infty} + \mathcal{N}_t, \quad \mathcal{M}_{-\infty} \perp \mathcal{N}_t \neq \{0\}, \quad -\infty < t < \infty . \tag{5.2}$$

Moreover from (3.5)

$$U_t(\mathcal{N}_s) = \mathcal{N}_{s+t}, \; -\infty < s, t < \infty . \tag{5.3}$$

If in the preceding paragraphs of this section we interpret s, t as integers rather than as real numbers, we get the definition and properties of non-deterministic processes in the discrete parameter case. But in the discrete case, additional results are readily available. We recall some of these in the next paragraph.

Let $(f'_n)_{-\infty}^{\infty}$ be any weakly stationary, non-deterministic S.P. with shift operator V. Denote by $(f'_n \mid \mathcal{M}'_{n-1})$ the orthogonal projection of f'_n on the subspace $\mathcal{M}'_{n-1}$, cf. (4.2). Then

$$g'_n = f'_n - (f'_n \mid \mathcal{M}'_{n-1}) \neq 0, \quad -\infty < n < \infty . \tag{5.4}$$

The vectors g'_n and $h'_n = g'_n/|g'_n|$ are called the nth *innovation* and *normalized innovation vectors*, respectively, of the process $(f'_n)_{-\infty}^{\infty}$. It is easily seen that

$$(h'_m, h'_n) = \delta_{mn}, \quad h'_{m+n} = V^m(h'_n), \quad -\infty < m, n < \infty , \tag{5.5}$$

so that $(h'_n)_{-\infty}^{\infty}$ is an orthonormal S.P. with the same shift operator V as $(f'_n)_{-\infty}^{\infty}$. It is an important fact that in the discrete analogue of (5.2), viz.

$$\mathcal{M}'_n = \mathcal{M}'_{-\infty} + \mathcal{N}'_n, \quad \mathcal{M}'_{-\infty} \perp \mathcal{N}'_n \neq \{0\} , \tag{5.6}$$

the subspace N'_n is the past and present of h'_n:

$$\mathcal{N}'_n = \mathfrak{S}(h'_k)_{k=-\infty}^{n} = \mathfrak{S}(V^k(h'_0))_{k=-\infty}^{n} . \tag{5.7}$$

The relations (5.6), (5.7) constitute the *Wold decomposition* of M'_n. From this decomposition follows at once the canonical decomposition of f'_n into a one-sided moving-average part and a deterministic part:

$$\begin{cases} f'_n = u'_n + v'_n, \qquad -\infty < n < \infty \\ u'_n = (f'_n \mid \mathcal{N}'_{-\infty}) = \sum_{k=0}^{\infty} c_k h'_{n-k}, \quad c_0 = |g'_0|, \qquad \sum_{k=0}^{\infty} |c_k|^2 < \infty \\ v'_n = (f'_n \mid \mathcal{M}'_n) \\ (u'_n)_{-\infty}^{\infty},\ (v'_n)_{-\infty}^{\infty} \text{ have the same shift operator } V \text{ as } (f'_n)_{-\infty}^{\infty}. \end{cases} \tag{5.8}$$

To revert to the continuous parameter case, let $(f_t, -\infty < t < \infty)$ be a weakly stationary, mean-continuous, non-deterministic S.P. with shift group $(U_t, -\infty < t < \infty)$. It is clear from the equalities in 3.7(b) that attempts to define "innovation vectors" g_t for this process by an equation analogous to (5.4) will fail. Indeed, since there is no atomic time unit in the continuous parameter case, all that we may expect our f_t-process to possess are "differential innovations."

Now let $(f'_n)_{-\infty}^{\infty}$ be the discrete S.P. associated with $(f_t, -\infty < t < \infty)$. Since the latter process is non-deterministic, it follows from Theorem 4.4 that so is the former. Let h'_0 be its 0th normalized innovation vector, and let

$$h_t = U_t(h'_0), \quad -\infty < t < \infty . \tag{5.9}$$

The resulting process $(h_t, -\infty < t < \infty)$ plays an important role in the theory. In § 6 we shall show that it is weakly Markovian, and explain how the differential innovations of the f_t-process can be had from it.[4] Here we shall show that the subspaces $\mathscr{N}_t$ of (5.2) are its past and present subspaces:

5.10 THEOREM. *(Pre-Wold Decomposition) Let* $(f_t, -\infty < t < \infty)$ *be a weakly stationary, mean-continuous, non-deterministic* S.P. *with shift group* $(U_t, -\infty < t < \infty)$, *so that cf.* (5.2)

$$\mathscr{M}_t = \mathscr{M}_{-\infty} + \mathscr{N}_t, \quad \mathscr{M}_{-\infty} \perp \mathscr{N}_t ,$$

Then $\mathscr{N}_t = \mathfrak{S}(h_s)_{s \leq t}$ *is the past and present of* h_t, $-\infty < t < \infty$.

Proof. By Theorem 4.4, $\mathscr{M}'_0 = \mathscr{M}_0$, $\mathscr{M}'_{-\infty} = \mathscr{M}_{-\infty}$. Hence, taking $t = 0 = n$ in (5.2), (5.6) we see that $\mathscr{N}'_0 = \mathscr{N}_0$. But taking $X = \{h'_0\}$ in 2.9(a), where h'_0 is the 0th normalized innovation of the associated discrete process, we find on using (5.7) that

$$\mathscr{N}_0 = \mathscr{N}'_0 = \mathfrak{S}(V^k(h'_0))_{k=-\infty}^{0} = \mathfrak{S}(U_s(h'_0))_{s \leq 0} . \tag{5.11}$$

Hence by (5.3) and (5.9)

$$\mathscr{N}_t = U_t(\mathscr{N}_0) = \mathfrak{S}(U_s(h'_0))_{s \leq t} = \mathfrak{S}(h_s)_{s \leq t} .$$

6. Differential innovations and the Wold decomposition. Let

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[4] The physical significance of the h_t-process has been indicated by Wiener and Wintner [20]. When $\mathfrak{X}$ is the class of L_2-functions on a probability space $(\Omega, \mathscr{B}, P)$, and t is the time, h_t provides the weak (or wide sense) version of "random time", i.e. time as measured by a perfect clook which is subjected to Brownian fluctuations. More precisely, if

$$y_t(\omega) = \exp[i\lambda\{t + ax_t(\omega)\}], \qquad \omega \in \Omega$$

where $(x_t, -\infty < t < \infty)$ is the Brownian movement S.P., and λ, a are constants such that $a\lambda = \sqrt{2}$, then the y_t- and h_t-processes have the same wide sense properties.

$(f_t, -\infty < t \in \infty)$ be a weakly stationary, mean-continuous, non-deterministic S.P. with shift group $(U_t, -\infty < t < \infty)$, and let h_0' be the 0th normalized innovation vector of the associated discrete process $(f_n')_{-\infty}^{\infty}$. In the next lemma we study the S.P. $(h_t, -\infty < t < \infty)$ defined by (5.9), the present and past subspaces $\mathscr{N}_t$ of which have been mentioned in the Pre-Wold decomposition 5.10.

6.1 LEMMA. (a) *The h_t-process is weakly (or wide sense) Markovian; more fully,*

$$(h_t \mid \mathscr{N}_s) = e^{s-t} h_s, \quad -\infty < s < t < \infty$$

depends only on the terminal vector h_s of $\mathscr{N}_s = \mathfrak{S}(h_\sigma)_{\sigma \leqq s}$.

(b) *Its convariance function γ is given by $\gamma_t = e^{-|t|}$, $-\infty < t < \infty$.*

Proof. (a) Let $t \geqq 0$. Then by 2.7

$$h_t = e^{-t} h_0' + \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k!} \left(\frac{-nt}{n+1}\right)^k \{(I + A_n)^k - I\}(h_0')$$

where

$$A_n(h_0') = \frac{2n}{n+1} \sum_{j=1}^{\infty} \left(\frac{n-1}{n+1}\right)^{j-1} V^j(h_0') .$$

Since by (5.5) the $h_j' = V^j(h_0')$ constitute an orthonormal process, we see that

$$h_t = e^{-t} h_0' + \eta_t, \quad \text{where} \quad \eta_t \perp h_0', h_{-1}', \cdots, t \geqq 0 .$$

It follows from (5.10) that $\eta_t \perp \mathscr{N}_0 = \mathfrak{S}(h_s)_{s \leqq 0}$. Hence

$$e^{-t} h_0' = (h_t \mid \mathscr{N}_0), \ t \geqq 0.$$

On applying U_s to both sides we get

$$(1) \qquad e^{-t} h_s = (h_{s+t} \mid \mathscr{N}_s) , \quad -\infty < s < \infty , \quad t \geqq 0 .$$

This reduces to the desired relation on changing the index.

(b) From (1) it follows at once that

$$\gamma_t = (h_{s+t}, h_s) = e^{-t} , \quad t \geqq 0.$$

This in turn entails that $\gamma_t = e^{-|t|}$, $-\infty < t < \infty$.

We shall now study the ξ_t-process mentioned in (1.1). By definition

$$(6.2) \qquad \xi_t = \frac{1}{\sqrt{2}} \left\{ h_t - h_0 + \int_0^t h_s ds \right\} , \quad -\infty < t < \infty .$$

It follows at once that $\xi_0 = 0$ and

$$(6.3) \qquad \xi_b - \xi_a = \frac{1}{\sqrt{2}}\left\{h_b - h_a + \int_a^b h_s ds\right\}, \quad -\infty < a, b < \infty .$$

6.4 THEOREM. (a) *The ξ_t-process has increments which are stationary under the group $(U_t, -\infty < t < \infty)$, i.e.*

$$U_t(\xi_b - \xi_a) = \xi_{b+t} - \xi_{a+t}, \quad -\infty < a, b, t < \infty .$$

(b) *The ξ_t-process has orthogonal increments, i.e.*

$$\xi_b - \xi_a \perp \xi_d - \xi_c, \quad \textit{if} \quad -\infty < a < b \leqq c < d < \infty .$$

(c) $|\xi_b - \xi_a|^2 = |b - a|, \quad -\infty < a, b < \infty .$

(d) $(\xi_b - \xi_a, \xi_d - \xi_c) = |\xi_b - \xi_c|^2 = b - c$, *if* $-\infty < a < c < b < d < \infty$.

Proof. (a) follows at once from (6.3) since $U_t(h_s) = h_{s+t}$.

(b) Let $a < b \leqq c < d$. Then from (6.3) and 6.1(b),

$$\begin{aligned} 2(\xi_b - \xi_a, \xi_d - \xi_c) &= \left(h_b - h_a + \int_a^b h_s ds, \quad h_d - h_c + \int_c^d h_t dt\right) \\ &= \left\{e^{b-d} - e^{b-c} + \int_c^d e^{b-t}dt\right\} - \left\{e^{a-d} - e^{a-c} + \int_c^d e^{a-t}dt\right\} \\ &\quad + \int_a^b \left\{e^{s-d} - e^{s-c} + \int_c^d e^{s-t}dt\right\}ds . \end{aligned}$$

Since the expression in each { } on the R.H.S. is zero, the result follows.

(c) First let $0 = a < b$. Then from (6.3) and 6.1(b)

$$\begin{aligned} 2|\xi_b - \xi_0|^2 &= \left(h_b - h_0 + \int_0^b h_s ds, \quad h_b - h_0 + \int_0^b h_t dt\right) \\ &= \left\{1 - e^{-b} + \int_0^b e^{t-b}dt\right\} - \left\{e^{-b} - 1 + \int_0^b e^{-t}dt\right\} \\ &\quad + \int_0^b \left\{e^{s-b} - e^{-s} + \int_0^b e^{-|s-t|}dt\right\}ds \\ &= 2(1 - e^{-b}) + 0 + \int_0^b\int_0^b e^{-|s-t|}dtds . \end{aligned}$$

Since the last integral equals

$$\int_0^b \left\{\int_0^s e^{t-s}dt + \int_s^b e^{s-t}dt\right\}ds = 2b + 2(e^{-b} - 1) ,$$

it follows that $|\xi_b - \xi_0|^2 = b$.

Next, let $-\infty < a < b < \infty$. Then by (a) $\xi_b - \xi_a = U_a(\xi_{b-a} - \xi_0)$, $b - a > 0$, and so

$$|\xi_b - \xi_a|^2 = |\xi_{b-a} - \xi_0|^2 = b - a .$$

(c) is a simple consequence of (a), (b), the vertification of which we leave to the reader.

In view of the last theorem, the stochastic integral $\int_{-\infty}^{\infty} c(s)d\xi_s$ will exist for any complex-valued function $c \in L_2(-\infty, \infty)$, cf. Doob [5, Ch. IX, § 2]. In the next lemma we shall show that the vector h_t is expressible in terms of the ξ_s by means of such an integral. In effect we shall invert the relation expressed in (6.2):

6.5 LEMMA. (*Inversion formula*)

$$h_t = \sqrt{2}\left\{\xi_t - \int_{-\infty}^{t} e^{s-t}\xi_s ds\right\} = \sqrt{2}\int_{-\infty}^{t} e^{s-t} d\xi_s\,, \quad -\infty < t < \infty\,.$$

Proof. Since $h_t = U_t(h_0')$ and U (as a function of t) is strongly continuous on $(-\infty, \infty)$, it follows that the vector-valued function h *is* continuous for $t \in (-\infty, \infty)$, and therefore by (6.2) so is the function ξ. Hence the Riemann integral $\int_a^b e^{s-t}\xi_s ds$ exist for $-\infty < a < b \leqq t$. Moreover, since

$$\left|\int_a^b e^{s-t}\xi_s ds\right| \leqq \int e^{s-t}\,|\,\xi_s\,|\,ds = \int_a^b e^{s-t}\sqrt{s}\cdot ds \text{ or } \sqrt{s}\cdot ds\,,$$

the infinite integral $\int_{-\infty}^{t} e^{s-t}\xi_s ds$ converges.

Now consider the case $t = 0$. We have from (6.2)

$$\begin{aligned}\sqrt{2}\int_{-\infty}^{0} e^s(\xi_0 - \xi_s)ds &= -\int_{-\infty}^{0} e^s\left\{h_s - h_0 + \int_0^s h_\sigma d\sigma\right\}ds \\ &= -\int_{-\infty}^{0} e^s h_s ds + h_0 + \int_{-\infty}^{0}\int_s^0 e^s h_\sigma d\sigma ds\,.\end{aligned}$$

Now by Dirichlet's formula the last integral equals

$$\int_{-\infty}^{0}\int_{-\infty}^{0} e^s h_\sigma ds d\sigma = \int_{-\infty}^{0}\left\{\int_{-\infty}^{0} e^s ds\right\}h_\sigma d\sigma = \int_{-\infty}^{0} e^\sigma h_\sigma d\sigma\,.$$

Hence

$$(1) \qquad \sqrt{2}\int_{-\infty}^{0} e^s(\xi_0 - \xi_s)ds = h_0\,.$$

Since for any real t, $U_t(\xi_0 - \xi_s) = \xi_t - \xi_{s+t}$, $U_t(h_0) = h_t$, we get the first equality in the lemma by applying U_t to both sides of (1) and then changing variables.

The second equality follows on integrating by parts:

$$\int_{-\infty}^{t} e^{s-t}d\xi_s = [e^{s-t}\xi_s]_{s\to-\infty}^{s=t} - \int_{-\infty}^{t}\xi_s d_s(e^{s-t}) = \xi_t - \int_{-\infty}^{t} e^{s-t}\xi_s ds\,.$$

The use of integration by parts is justified as follows. In the first place, for $-\infty < a < t < \infty$ we have

$$(2) \qquad \int_a^t e^{s-t} d\xi_s = [e^{s-t}\xi_s]_{s=a}^{s=t} - \int_a^t \xi_s d_s(e^{s-t}) \ .$$

This follows from the fact that for a continuous integrand the stochastic integral is a Riemann-Stieltjes integral (with vector-valued integrator ξ_s) and that for the latter, integration by parts is valid, cf. [5, p. 429 (2.6)] and [11, p. 63 (3.31)]]. Next, the last integral in (2) is obviously equal to $\int_a^t \xi_s e^{s-t} ds$. Finally, since both $\int_{-\infty}^t e^{s-t} d\xi_s$, $\int_{-\infty}^t \xi_s e^{s-t} ds$ are known to exist, we can let $a \to -\infty$ in (2), cf. [5, p. 428 (2.4)].

The formulae (6.2) and 6.5 together entail the following important result:

6.6 LEMMA. *For any real t, the past and present subspace $\mathscr{N}_t$ of h_t is the set of all (convergent) stochastic integrals $\int_{-\infty}^t c(s)d\xi_s$, with complex-valued functions $c \in L_2(-\infty, t)$, i.e. $\mathscr{N}_t = \mathfrak{S}(\xi_\sigma - \xi_\tau)_{\sigma,\tau \leqq t}$.*

Proof. Denote by $\mathscr{N}_t^{(\xi)}$ the set of all such stochastic integrals. Let $-\infty < \tau \leqq t < \infty$. Then by 6.5

$$h_\tau = \sqrt{2}\int_{-\infty}^{\tau} e^{s-\tau} d\xi_s = \int_{-\infty}^{\tau} c(s) d\xi_s \ ,$$

where $c(s) = \sqrt{2}e^{s-\tau}$ on $(-\infty, \tau]$ and $c(s) = 0$ on $(\tau, t]$. Since $c \in L_2(-\infty, t]$, it follows that $h_\tau \in \mathscr{N}_t^{(\xi)}$. Hence $\mathscr{N}_t = \mathfrak{S}(h_\tau)_{\tau \leqq t} \subseteqq \mathscr{N}_t^{(\xi)}$.

To prove the reverse inclusion, let

$$g = \int_{-\infty}^{t} c(s) d\xi_s, \text{ where } c \in L_2(-\infty, t] \ .$$

Suppose first that c is a step-function:

$$c(s) = \sum_{k=1}^{n} c_k \chi_{J_k}(s)$$

χ_{J_k} being the indicator function of the interval $J_k = [a_k, b_k] \subseteqq (-\infty, t]$. Then by definition (cf. Doob [5, p. 427 (2.1)])[5]

$$g = \sum_{k=1}^{n} c_k(\xi_{b_k} - \xi_{a_k}) \ .$$

From (6.3) it is clear that $g \in \mathscr{N}_t$. Next suppose $c \in L_2(-\infty, t]$, and $c = \lim_{n\to\infty} c^{(n)}$, where $c^{(n)}$ is a step-function. Then by definition

[5] We note that from 6.4(c) it follows that $\xi_{t-} = \xi_t = \xi_{t+}$, $-\infty < t < \infty$.

$$g = \lim_{n\to\infty} \int_{-\infty}^{t} c^{(n)}(s) d\xi_s \in \mathscr{N}_t ,$$

since $\mathscr{N}_t$ is closed. Thus $\mathscr{N}_t^{(\xi)} \subseteqq \mathscr{N}_t$.

We may sum up the main results established so far as follows:

6.7 THEOREM. (*Wold Decomposition I*) *Let* $(f_t, -\infty < t < \infty)$ *be a weakly stationary, mean-continuous, non-deterministic* S.P. *with shift group* $(U_t, -\infty < t < \infty)$. *Let* h_0' *be the* 0*th normalized innovation of the associated discrete process, and let*

$$h_t = U_t(h_0') , \quad \xi_t = h_t - h_0 + \int_0^t h_s ds , \quad -\infty < t < \infty .$$

Then (a) $\mathscr{M}_t = \mathscr{N}_t + \mathscr{M}_{-\infty}$, $\mathscr{N}_t \perp \mathscr{M}_{-\infty}$, $-\infty < t < \infty$, *where* $\mathscr{N}_t = \mathfrak{S}(h_s)_{s \leqq t}$ *is the past and present of* h_t;

(b) *the* ξ_t*-process has stationary, orthogonal increments such that* $|\xi_t - \xi_s|^2 = |t - s|$; *moreover,* $\mathscr{N}_t = \mathfrak{S}(\xi_\sigma - \xi_\tau)_{\sigma,\tau \leqq t}$, *i.e.* $\mathscr{N}_t$ *is the set of all stochastic integrals* $\int_{-\infty}^t c(s) d\xi_s$ *with* $c \in L_2(-\infty, t]$.

6.8 UNIQUENSES THEOREM. *Let* $(\eta_t, -\infty < t < \infty)$ *be any process with the following properties*:

(i) *it has orthogonal increments such that*

$$|\eta_b - \eta_a|^2 = |b - a| , \quad -\infty < a, b < \infty , \quad \text{and} \quad \eta_0 = 0$$

(ii) $U_t(\eta_b - \eta_a) = \eta_{b+t} - \eta_{a+t}$, $-\infty < a, b, t < \infty$

(iii) $\mathfrak{S}(\eta_\sigma - \eta_\tau)_{\sigma,\tau \leqq 0} = \mathscr{M}_{-\infty}^{\perp} \cap \mathscr{M}_0$.

Then $\eta_t = e^{i\alpha}\xi_t$, *where* ξ_t *is as in* 6.7, *and* α *is some real number.*

Proof. Our proof of this result is essentially that given by Hanner [9, p. 175–176]. Since our treatments and notations differ, we may indicate the main steps. We first show that

$$\mathfrak{S}(\eta_\sigma - \eta_\tau)_{\sigma,\tau \leqq b} = \mathscr{N}_b , \qquad \mathfrak{S}(\eta_\sigma - \eta_\tau)_{a \leqq \sigma,\tau \leqq b} = \mathscr{N}_a^{\perp} \cap \mathscr{N}_b ,$$

where $\mathscr{N}_b$ is an in 6.7(a). It follows from 6.7(b) that $\xi_b - \xi_a = \int_a^b f_{a,b}(s) d\eta_s$. By piecing together the functions $f_{n,n+1}$, $-\infty < n < \infty$, we can define a function f on $(-\infty, \infty)$ such that

$$\xi_b - \xi_a = \int_a^b f(s) d\eta_s , \qquad a < b .$$

Using the fact that $\xi_b - \xi_a = U_h(\xi_{b-h} - \xi_{a-h})$, we can show that f is essentially constant-valued on $(-\infty, \infty)$. From this the desired result is immediate.

An immediate corollary of Theorem 6.7 is the cannonical decomposition of the vector f_t itself:

6.9 COROLLARY. (*Wold Decomposition II*) *With the hypothesis of Theorem* 6.7 *we have*

(a) $f_t = u_t + v_t$, $u_t = (f_t \mid \mathscr{N}_t)$, $v_t = (f_t \mid \mathscr{M}_{-\infty})$;

(b) *the u_t-process in* (a) *is a one-sided moving average, i.e.*

$$u_t = \int_0^\infty c(s) d_s \xi_{t-s}\,, \quad -\infty < t < \infty, \quad \textit{where} \quad c \in L_2[0, \infty)\,.$$

and $\mathfrak{S}(u_s)_{s \leq t} = \mathscr{N}_t$, $-\infty < t < \infty$;

(c) *the v_t-process is deterministic, and* $\mathfrak{S}(v_s)_{s \leq t} = \mathscr{M}_{-\infty}$, *for* $-\infty < t < \infty$.

7. **Purely non-deterministic stochastic processes.** We call a weakly stationary S.P. *purely non-deterministic*, if and only if $\mathscr{M}_{-\infty} = \{0\}$. For completeness we state here the anologue of a theorem given by Kolmogorov for discrete parameter processes:

7.1 THEOREM. *For any weakly stationary, mean-continuous stochastic process* (f_t, $-\infty < t < \infty$) *the following conditions are equivalent*:

(i) (f_t, $-\infty < t < \infty$) *is purely non-deterministic*;

(ii) (f_t, $-\infty < t < \infty$) *is a one-sided moving average*:

$$f_t = \int_0^\infty c(s) d_s \xi_{t-s}\,, \quad c \in L_2[0, \infty]\,,$$

(ξ_s, $-\infty < s < \infty$) *being a process with stationary and orthogonal increments such that* $|\xi_b - \xi_a|^2 = |b - a|$;

(iii) $\lim_{t \to \infty} (f_0 \mid \mathscr{M}_{-t}) = 0$.

Proof. The proof runs parallel to that in the discrete case and is omitted.

It follows from Corollary 6.9 and Theorem 7.1 that every weakly stationary, mean-continuous, non-deterministic S.P. (f_t, $-\infty < t < \infty$) can be decomposed in the form $f_t = u_t + v_t$, where the u_t-process is purely non-deterministic, the v_t-process is deterministic, and all three processes have the same shift group (U_t, $-\infty < t < \infty$). We shall refer to the u_t-and v_t-processes as the *purely non-deterministic part* and the *deterministic part* of the f_t-process. With an obvious notation, we have

$$\mathscr{M}_t = \mathscr{M}_t^{(u)} + \mathscr{M}_t^{(v)}\,, \qquad \mathscr{M}_\infty^{(u)} \perp \mathscr{M}_\infty^{(v)}$$
$$\mathscr{M}_t^{(u)} = \mathscr{N}_t\,, \qquad \mathscr{M}_t^{(v)} = \mathscr{M}_{-\infty}\,.$$

Now let $(u'_n)_{-\infty}^{\infty}$, $(v'_n)_{-\infty}^{\infty}$ be the purely non-deterministic and deterministic parts of the discrete process $(f'_n)_{-\infty}^{\infty}$ associated with $(f_t, -\infty < t < \infty)$. Then by 6.9(a), 4.4(c), and (5.8)

$$v_0 = (f_0 \,|\, \mathscr{M}_{-\infty}) = (f'_0 \,|\, \mathscr{M}'_{-\infty}) = v'_0 ,$$

and therefore

$$u_0 = f_0 - v_0 = f'_0 - v'_0 = u'_0 .$$

Moreover, the shift operator V of the u'_n-, v'_n-processes is the Cayley transform of H, where iH is the infinitesimal generator of the shift group $(U_t, -\infty < t < \infty)$ of the u_t-, v_t-processes. We can thus assert the following:

7.2 COROLLARY. *If $(f'_n)^{\infty}$ is the discrete process associated with the weakly stationary, mean-continuous, non-deterministic S.P. $(f_t, -\infty < t < \infty)$, then the purely non-deterministic and deterministic parts of $(f'_n)_{-\infty}^{\infty}$ are the discrete processes associated with the deterministic and purely non-deterministic parts of $(f_t, -\infty < t < \infty)$.*

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