A NOTE ON HYPONORMAL OPERATORS

STERLING K. BERBERIAN

The last exercise in reference [4] is a question to which I did not know the answer: does there exist a hyponormal ($TT^* \leqq T^*T$) completely continuous operator which is not normal? Recently Tsuyoshi Andô has answered this question in the negative, by proving that every hyponormal completely continuous operator is necessarily normal ([1]). The key to Andô's solution is a direct calculation with vectors, showing that a hyponormal operator T satisfies the relation $\| T^n \| = \| T \|^n$ for every positive integer n (for "subnormal" operators, this was observed by P. R. Halmos on page 196 of [6]). It then follows, from Gelfand's formula for spectral radius, that the spectrum of T contains a scalar μ such that $|\mu| = \| T \|$ (see [9], Theorem 1.6.3.).

The purpose of the present note is to obtain this result from another direction, via the technique of approximate proper vectors ([3]); in this approach, the nonemptiness of the spectrum of a hyponormal operator T is made to depend on the elementary case of a self-adjoint operator, and a simple calculation with proper vectors leads to a scalar μ in the spectrum of T such that $|\mu| = \| T \|$. This is the Theorem below, and its Corollaries 1 and 2 are due also to Andô. In the remaining corollaries, we note several applications to completely continuous operators.

We consider operators (=continuous linear mappings) defined in a Hilbert space. As in [3], the spectrum of an operator T is denoted $s(T)$, and the approximate point spectrum is $a(T)$. We note for future use that every boundary point of $s(T)$ belongs to $a(T)$; see, for example, ([4], hint to Exercise VIII. 3.4).

LEMMA 1. *Suppose T is a hyponormal operator, with $\| T \| \leqq 1$, and let $\mathscr{M}$ be the set of all vectors which are fixed under the operator TT^*. Then,*

(i) *$\mathscr{M}$ is a closed linear subspace,*

(ii) *the vectors in $\mathscr{M}$ are fixed under T^*T,*

(iii) *$\mathscr{M}$ is invariant under T, and*

(iv) *the restriction of T to $\mathscr{M}$ is an isometric operator in $\mathscr{M}$.*

Proof. Since $\mathscr{M} = \{x : TT^*x = x\}$ is the null space of $I - TT^*$, it is a closed linear subspace. The relation $TT^* \leqq T^*T \leqq I$ implies $0 \leqq I - T^*T \leqq I - TT^*$, and from this it is clear that the null space of $I - TT^*$ is contained in the null space of $I - T^*T$. That is, $TT^*x = x$

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implies $T^*Tx = x$. This proves (ii). (Alternatively, given $TT^*x = x$, one can calculate directly that $\| T^*Tx - x \|^2 \leqq 0$.) If $x \in \mathscr{M}$, that is if $TT^*x = x$, then the calculation $TT^*(Tx) = T(T^*Tx) = Tx$ shows that $Tx \in \mathscr{M}$; moreover, $\| Tx \|^2 = (T^*Tx | x) = \| x \|^2$.

LEMMA 2. *Every isometric operator has an approximate proper value of absolute value* 1.

Proof. Let U be an isometric operator in a nonzero Hilbert space. Suppose first that the spectrum of U contains 1; since $\| U \| = 1$, it follows that 1 is a boundary point of $s(U)$ (see [4], part (ix) of Exercise VII. 3. 12), hence 1 is an approximate proper value for U.

If the spectrum of U does not contain 1, that is if $I - U$ is invertible, we may form the Cayley transform A of U; thus,

$$A = i(I + U)(I - U)^{-1} = i(I - U)^{-1}(I + U) .$$

Using the hypothesis $U^*U = I$, let us show that A is self-adjoint. Left-multiplying the relation $(I - U)A = i(I + U)$ by U^*, we have $(U^* - I)A = i(U^* + I)$, thus $(I - U)^*A = -i(I + U)^*$. Since $(I - U)^*$ is invertible, with inverse $[(I - U)^{-1}]^*$, we have

$$A = -i[(I - U)^{-1}]^*(I + U)^* = -i[(I + U)(I - U)^{-1}]^* = A^* .$$

It follows that the operators $A + iI$ and $A - iI$ are invertible, and solving the relation $(I - U)A = i(I + U)$ for U, we have

$$U = (A - iI)(A + iI)^{-1} = (A + iI)^{-1}(A - iI) .$$

Incidentally, since U is the product of invertible operators, we conclude that U is unitary.

Since A is self-adjoint, we know from an elementary argument that the approximate point spectrum of A is non empty ([7], Theorem 34.2). Let $\alpha \in a(A)$, and let x_n be a sequence of unit vectors such that $\| Ax_n - \alpha x_n \| \to 0$. Define $\mu = (\alpha + i)^{-1}(\alpha - i)$; since α is real, μ has absolute value 1. It will suffice to show that μ is an approximate proper value for U; indeed, $\| (U - \mu I)x_n \| \to 0$ results from the calculation

$$\begin{aligned} U - \mu I &= (A + iI)^{-1}(A - iI) - (\alpha + i)^{-1}(\alpha - i)I \\ &= (\alpha + i)^{-1}(A + iI)^{-1}[(\alpha + i)(A - iI) - (\alpha - i)(A + iI)] \\ &= 2i(\alpha + i)^{-1}(A + iI)^{-1}(A - \alpha I) , \end{aligned}$$

the fact that $\| (A - \alpha I)x_n \| \to 0$, and the continuity of the operator $2i(\alpha + i)^{-1}(A + iI)^{-1}$.

Incidentally, if U is an isometric operator such that the spectrum of U excludes some complex number μ of absolute value 1, then $\mu^{-1}U$

is an isometric operator whose spectrum excludes 1. The proof of Lemma 2 then shows that $\mu^{-1}U$ is unitary, hence so is U. In other words: the spectrum of a nonnormal isometry must include the unit circle $|\mu| = 1$; indeed, Putnam has shown that the spectrum is the unit disc $|\mu| \leqq 1$ ([8], Corollary 1). The latter result is also an immediate consequence of ([5], Lemma 2.1), and the fact that the spectrum of any unilateral shift operator is the unit disc.

THEOREM. (Andô) *Every hyponormal operator T has an approximate proper value μ such that $|\mu| = \|T\|$.*

Proof. We may assume $\|T\| = 1$ without loss of generality. Since $TT^* \geqq 0$ and $\|TT^*\| = 1$, we know that 1 is an approximate proper value for TT^*. Since the property of hyponormality is preserved under $*$-isomorphism, we may assume, after a change of Hilbert space, that 1 is a proper value for TT^* ([3], Theorem 1). Form the nonzero closed linear subspace $\mathscr{M} = \{x : TT^*x = x\}$; according to Lemma 1, $\mathscr{M}$ is invariant under T, and the restriction of T to $\mathscr{M}$ is an isometric operator U in the Hilbert space $\mathscr{M}$. By Lemma 2, U has an approximate proper value μ of absolute value 1. Let x_n be any sequence of unit vectors in $\mathscr{M}$ such that $\|Ux_n - \mu x_n\| \to 0$. Since $Ux_n = Tx_n$, obviously μ is an approximate proper value for T, and $|\mu| = 1 = \|T\|$.

COROLLARY 1. *A generalized nilpotent hyponormal operator is necessarily zero.*

Proof. If T is hyponormal, then $s(T)$ contains a scalar μ such that $|\mu| = \|T\|$. For every positive integer n, it follows that $s(T^n)$ contains μ^n (see [7], Theorem 33.1); then $\|T\|^n = |\mu|^n = |\mu^n| \leqq \|T^n\| \leqq \|T\|^n$, and so $\|T^n\| = \|T\|^n$. If moreover T is a generalized nilpotent, that is if $\lim \|T^n\|^{1/n} = 0$, then $\|T\| = 0$.

COROLLARY 2. *If T is a completely continuous hyponormal operator, then T is normal.*

Proof. The proof to be given is essentially the same as Andô's. The proper subspaces of T are mutually orthogonal, and reduce T ([4], Exercise VII. 2.5). Let $\mathscr{M}$ be the smallest closed linear subspace which contains every proper subspace of T, and let $\mathscr{N} = \mathscr{M}^\perp$; clearly $\mathscr{N}$ reduces T, and the restriction $T/\mathscr{N}$ is a completely continuous hyponormal operator in $\mathscr{N}$ ([4], Exercise VI. 9.18). If the spectrum of $T/\mathscr{N}$ were different from $\{0\}$, it would have a nonzero boundary point μ, hence μ would be a proper value for $T/\mathscr{N}$ (see [4], Theorem VIII. 3.2); this is impossible since $\mathscr{N}^\perp = \mathscr{M}$ already contains every proper vector for T.

We conclude from the Theorem that $T/\mathscr{N} = 0$, and this forces $\mathscr{N} = \{0\}$ (recall that $\mathscr{N}^{\perp}$ contains the null space of T). Thus, the proper subspaces of T are a total family, hence T is normal by ([4], Exercise VII. 2.5).

Suppose T is a normal operator whose spectrum (a) has empty interior, and (b) does not separate the complex plane. Wermer has shown that the invariant subspaces of T reduce T ([10], Theorem 7). It is well known that the conditions (a) and (b) are fulfilled by the spectrum of any completely continuous operator. In particular: if T is a completely continuous normal operator, then every invariant subspace of T reduces T. A more elementary proof of this may be based on Corollary 2:

COROLLARY 3. *If T is a completely continuous normal operator, and $\mathscr{N}$ is a closed linear subspace invariant under T, then $\mathscr{N}$ reduces T.*

Proof. Indeed, it suffices to assume that T is hyponormal and $\mathscr{N}$ is an invariant subspace such that $T/\mathscr{N}$ is completely continuous. Since $T/\mathscr{N}$ is hyponormal ([4], Exercise VI. 9.10), it follows from Corollary 2 that $T/\mathscr{N}$ is normal, hence $\mathscr{N}$ reduces T by ([4], Exercise VI. 9.9).

Quoting ([4], Theorem VII. 3.1), we have:

COROLLARY 4. *If T is a hyponormal operator, then*

$$\| T \| = LUB\{|(Tx \mid x)| : \| x \| \leqq 1\} .$$

Incidentally, if T is hyponormal, it is clear from Corollary 4 that $\| T^* \| = LUB\{|(T^*x \mid x)| : \| x \| \leqq 1\}$.

COROLLARY 5. *If the completely continuous operator T is semi-normal in the sense of* [8], *then T is normal.*

Proof. The definition of semi-normality is that either $TT^* \leqq T^*T$ or $TT^* \geqq T^*T$, in other words, either T or T^* is hyponormal; since both are completely continuous (see [4], Exercise VIII. 1.6), our assertion follows from Corollary 2.

Let us say that an operator T is *nearly normal* in case T commutes with T^*T. The structure of nearly normal operators has been determined by Brown, and it is a consequence of his results that a completely continuous nearly normal operator is in fact normal (see the concluding remarks in [5]). This may also be proved as follows. An elementary calculation with square roots shows that a nearly normal operator is hyponormal (see [2], proof of Corollary 1 of Theorem 8); assuming also complete continuity and citing Corollary 2, we have:

COROLLARY 6. *If T is a completely continuous nearly normal operator, then T is normal.*

Finally,

COROLLARY 7. *If $S = T + \lambda I$, where T is a completely continuous operator, and if S is hyponormal, then S is normal.*

Proof. Since S is hyponormal, so is T ([4], hint to Exercise VII. 1.6), hence T is normal by Corollary 2; therefore S is normal. So to speak, the C^*-algebra of all operators of the from $T + \lambda I$, with T completely continuous, is of "finite class".

We close with an elementary remark about the adjoint of a hyponormal operator: if T is hyponormal, then $s(T^*) = a(T^*)$. For, suppose λ does not belong to $a(T^*)$, and let $\mu = \lambda^*$. Then, $(T - \mu I)^* = T^* - \lambda I$ is bounded below ([4], Exercise VII. 3.8), and since $T - \mu I$ is also hyponormal, the relation $(T - \mu I)(T - \mu I)^* \leqq (T - \mu I)^*(T - \mu I)$ shows that $T - \mu I$ is also bounded below. Then $T - \mu I$ is invertible ([4], Exercise VI. 8.11), hence so is $T^* - \lambda I$, thus λ does not belong to $s(T^*)$.

REFERENCES

1. T. Andô, Forthcoming paper in Proc. Amer. Math. Soc.
2. S. K. Berberian, *Note on a theorem of Fuglede and Putnam*, Proc. Amer. Math. Soc., **10** (1959), 175-182.
3. ———, *Approximate proper vectors*, Proc. Amer. Math. Soc., **13** (1962), 111-114.
4. ———, *Introduction to Hilbert space*, Oxford University Press, New York, 1961.
5. A. Brown, *On a class of operators*, Proc. Amer. Math. Soc., **4** (1953), 723-728.
6. P. R. Halmos, *Commutators of operators, II.*, Amer. J. Math., **76** (1954), 191-198.
7. ———, *Introduction to Hilbert space and the theory of spectral multiplicity*, Chelsea, New York, 1951.
8. C. R. Putnam, *On semi-normal operators*, Pacific J. Math., **7** (1957), 1649-1652.
9. C. E. Rickart, *General theory of Banach algbras*, D. van Nostrand, New York, 1960.
10. J. Wermer, *On invariant subspaces of normal operators*, Proc. Amer. Math. Soc., **3** (1952), 270-277.

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