CENTERS OF PURITY IN ABELIAN GROUPS

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This note is a supplement to the paper [5][1] of J. D. Reid "On subgroups of an abelian group maximal disjoint from a given subgroup." Our main result is based on the observation that in the case of primary groups, a bit of extra information can be gleaned from Reid's Theorem 2.1. We are led to the following characterization of the "centers of purity" in a p-group.

THEOREM 1. *Let G be a p-group. For each integer $k \geqq 0$, define $P_k = G[p] \cap p^k G$. Let $P_\infty = G[p] \cap G^1$, and $P_{\infty+1} = P_{\infty+2} = 0$. Let H be a subgroup of G. Then H is a center of purity in G (that is, every subgroup of G which is maximal with respect to disjointness from H is pure) if and only if there exists k with $0 \leqq k \leqq \infty$ such that*

$$P_k \supseteqq H[p] \supseteqq P_{k+2} .$$

It is easy to see that if G is a torsion group and H is a subgroup of G, then H is a center of purity in G if and only if every p-component H_p of H is a center of purity in the corresponding p-component G_p of G. Thus, Theorem 1 can be used to determine the centers of purity in torsion groups. The following result shows that the centers of purity in arbitrary groups can also be characterized.

THEOREM 2. *A subgroup H of an abelian group G is a center of purity in G if and only if the following two conditions are satisfied:*

(i) *the torsion subgroup H_t of H is a center of purity in the torsion subgroup G_t of G;*

(ii) *either G/H is a torsion group, or else, for all primes p,*

$$H[p] \subseteqq \bigcap_{n=0}^{\infty} p^n G .$$

The problem of characterizing centers of purity in p-groups was first posed by J. M. Irwin in [2]. Irwin showed that any subgroup of a p-group G which is maximal disjoint from G^1 is pure in G. In [3], Irwin and Walker extended this result to arbitrary abelian groups. They also showed that if G is a torsion group and H is a subgroup

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of G^1, then H is a center of purity in G. Charles pointed out that the proof given in [1] of Erdélyi's theorem (see p. 81) shows that the subgroups $pG, p^2G, p^3G, \cdots$ of a p-group G are centers of purity. Khabbaz (in [4]) showed how the proof of Erdélyi's theorem could be modified to obtain a short proof of Irwin and Walker's result. Finally, Reid established the general sufficient condition 2.1 of [5] for a subgroup H of an arbitrary group G to be a center of purity. It was Reid who introduced the term "center of purity." In the lemma below, we show that Reid's condition is necessary as well as sufficient for H to be a center of purity in a p-group G. This lemma is then used to prove Theorem 1, from which Theorem 2 follows easily. The author is indebted to Professor Reid for sending him a pre-print of the paper [5]. It was the reading of this paper which inspired the present work.

The notation and terminology of [5] will be used in this paper. In addition, we let $O(x)$ denote the order of the element x.

LEMMA. *Let G be a p-group, and suppose that H is a subgroup of G. Then there is a subgroup M of G such that M is maximal with respect to disjointness from H, and M is not pure in G, if and only if the following conditon is satisfied.*

(*) *There exists $h \in H$ and $m \in G$ such that*

(i) $O(m) > O(h) = p$;

(ii) $h_p(m) = h_p(h) < h_p(m + h)$;

(iii) $\{m\} \cap H = 0$.

Proof. Suppose that M is a subgroup of G which is maximal disjoint from H and not pure in G. Using the fact that two subgroups of a p-group are disjoint if and only if their p-layers are disjoint, it is easy to see that M is maximal disjoint from $H[p]$. Therefore (*) is satisfied by Theorem 2.1 of [5]. (It is clear from the proof of 2.1 that $pm \neq 0$, so that $O(m) > O(h)$.)

Assume conversely that the condition (*) is satisfied. Let r be a natural number such that $h_p(m) < r \leqq h_p(m + h)$. Define $P_r = p^rG \cap G[p]$. Let $O(m) = p^j$, where, by (i), $j > 1$. Then by (i), $n = p^{j-1}m = p^{j-1}(m + h)$ has height $\geqq r + 1$. Thus, $n \in P_r$. However, by (iii), $n \notin H[p]$. Consequently, there is a vector space decomposition

$$P_r = S \oplus (P_r \cap H[p]) \,, \qquad n \in S \,.$$

By (i) and (ii), $h \in H[p]$ and $h \notin P_r \cap H[p]$. Therefore, there is a decomposition

$$H[p] = T \oplus (P_r \cap H[p]) \,, \qquad h \in T \,.$$

Clearly,

$$P_r + H[p] = S \oplus T \oplus (P_r \cap H[p]) .$$

Finally, choose a decomposition

$$G[p] = R \oplus (P_r + H[p]) .$$

Define

$$M_0 = R \oplus S .$$

Then we have

$$G[p] = M_0 \oplus H[p] , \qquad \text{with } n \in M_0 .$$

Let π be the projection mapping determined by this decomposition:

$$\pi\colon G[p] \rightarrow H[p] .$$

Note that by the construction, $\pi(P_r) = P_r \cap H[p]$. Define

$$K = \{M_0, m\} .$$

It is easy to see that since $p^{j-1}m = n \in M_0$, the p-layer of K is M_0. Thus, $K[p] \cap H[p] = M_0 \cap H[p] = 0$, and therefore $K \cap H = 0$. Let M be maximal containing K and disjoint from H. The proof of the lemma is completed by showing that $h_p^M(pm) \leqq r$. Indeed, this will imply that M is not pure, because

$$h_p(pm) = h_p(p(m + h)) \geqq h_p(m + h) + 1 \geqq r + 1 .$$

Suppose that $h_p^M(pm) \geqq r + 1$. Then $z \in M$ exists satisfying

$$p^{r+1}z = pm .$$

Consequently,

$$u = p^r z - m \in M \cap G[p] = M \cap (M_0 + H[p]) = M_0 + (M \cap H[p]) = M_0 .$$

Since $h_p(m + h) \geqq r$, we can write

$$m + h = p^r y$$

for some $y \in G$. Thus,

$$p^r(y - z) = h - u \in G[p] \cap p^r G = P_r ,$$

and therefore since $u \in M_0$,

$$h = \pi(h - u) \in \pi(P_r) = P_r \cap H[p] \subseteqq P_r .$$

However, $h_p(h) < r$ by the choice of r. This contradiction shows that $h_p^M(pm) > r$ is impossible, so that the proof of the lemma is complete.

We can now prove Theorem 1. Suppose that $P_k \supseteqq H[p] \supseteqq P_{k+2}$. If $k = \infty$, there cannot be any $h \in H[p]$ satisfying condition (ii) of the lemma. Suppose therefore that k is finite. Assume that $h \in H$ and $m \in G$ exist satisfying conditions (i), (ii) and (iii) of (*) in the lemma. Let $h_p(h) = j$. Then $k \leqq j < h_p(m + h) \leqq \infty$. Let $O(m) = p^f$, where $f \geqq 2$ by (i). Write $x = m + h$. Then $h_p(x) \geqq k + 1$. Consequently, $h_p(p^{f-1}x) \geqq k + 2$. Therefore, $p^{f-1}m \in P_{k+2} \subseteqq H[p]$. This is contrary to (iii). It follows that H is a center of purity in G. Conversely, suppose that $P_k \supseteqq H[p] \supseteqq P_{k+2}$ is not satisfied for any k. Then in particular, $H[p] \not\subseteqq P_\infty$. Since $P_\infty = \bigcap_{k<\omega} P_k$, it follows that $H[p] \not\subseteqq P_j$ for some finite j. Let $k \geqq 0$ be the largest natural number such that $H[p] \subseteqq P_k$. The maximality of k and the fact that $P_k \supseteqq H[p] \supseteqq P_{k+2}$ is false implies that

$$H[p] \not\subseteqq P_{k+1} , \quad H[p] \subseteqq P_k , \quad \text{and} \quad P_{k+2} \not\subseteqq H[p] .$$

Therefore, there is an element $h \in H[p]$ such that $h_p(h) = k$, and there exists $u \in P_{k+2}$ such that $u \notin H[p]$. Let $u = pv$, where $v \in G$ and $h_p(v) \geqq k + 1$. Define $m = v - h$. Then $O(m) = p^2 > p = O(h)$, $h_p(m) = k$, $h_p(m + h) = h_p(v) \geqq k + 1$, and $\{m\} \cap H = 0$, since $pm = pv = u \notin H[p]$. It follows from the lemma that H is not a center of purity in G. The proof of Theorem 1 is therefore complete.

Theorem 2 is obtained with the help of Theorem 1, by refining the proof of Lemmas 3.5 and 3.7 in [5]. Suppose that G/H is a torsion group, and H_t is a center of purity in G_t. If M is maximal disjoint from H, then $M \subseteqq G_t$, and a short calculation shows that M is maximal disjoint from H_t in G_t. Therefore M is pure in G_t, and hence also in G. Next, suppose that $H[p] \subseteqq \bigcap_{n=0}^{\infty} p^n G$ for all primes p. If H is not a center of purity in G, then by Theorem 2.1 in [5], there exists $h \in H_p$ such that $h_p(h) < \infty$. Let $O(h) = p^r$. Using the same argument that was given in the last paragraph of the proof of 2.1 in [5], we can show that $h_p(p^{r-1}h) < \infty$. This contradiction proves that H must be a center of purity. Suppose conversely that H is a center of purity in G. It is a routine exercise to show that H_t is a center of purity in G_t. Assume that G/H is not a torsion group and for some prime p, $H[p] \not\subseteqq \bigcap_{n=0}^{\infty} p^n G$. Let k be the largest integer such that $p^k G \supseteqq H[p]$. Then by Theorem 1

$$p^k G \cap G[p] \supseteqq H[p] \supseteqq p^{k+2} G \cap G[p] , \qquad p^{k+1} G \cap G[p] \not\supseteqq H[p] .$$

Let $t \in H[p]$ satisfy $h_p(t) = k$. Since G/H is not a torsion group, an element $x \in G$ exists satisfying $O(x) = \infty$ and $\{x\} \cap H = 0$. Consequently $\{p^{k+2}x + t\} \cap H = 0$. Let M be maximal disjoint from H, with $p^{k+2}x + t \in M$. Then $p^{k+3}x = p(p^{k+2}x + t) \in M$. Since H is a center of purity, M is pure. Consequently, $m \in M$ exists satisfying $p^{k+3}m =$

$p^{k+3}x$. Thus, $p^{k+2}(x-m) \in p^{k+2}G \cap G[p] \subseteqq H[p]$. Therefore,

$$p^{k+2}x + t - p^{k+2}m = p^{k+2}(x-m) + t \in H \cap M = 0 ,$$

so that $h_p(t) \geqq k+2$. However, $h_p(t) = k$ by choice. The contradiction shows that the condition (ii) must hold.

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