EXISTENCE AND ASYMPTOTIC BEHAVIOR OF PROPER SOLUTIONS OF A CLASS OF SECOND-ORDER NONLINEAR DIFFERENTIAL EQUATIONS

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1. This paper deals with proper solutions of the second-order nonlinear differential equation

$$y'' = yF(y, x)\ , \tag{1.1}$$

where (i) $F(u, x)$ is continuous in u and x for $0 \leqq u < +\infty$ and $x \geqq x_0$,

(ii) $F(u, x) > 0$ for $u > 0$ and $x \geqq x_0$,

(iii) $F(u, x) < F(v, x)$ for each $x \geqq x_0$ and $0 < u < v < +\infty$.

By a proper solution we understand a real-valued solution y of (1.1) which is of class $C^2[a, \infty)$, where $x_0 \leqq a < +\infty$. An example of equations of this type is the Emden-Fower equation [2, chapter 7]

$$y'' = x^{\lambda} y^{n}\ . \tag{1.2}$$

Our interest is in the existence and asymptotic behavior of *positive proper solutions* of (1.1). Since $F(y, x) > 0$ for $y > 0$, all positive solutions of this equation are convex. They are therefore of two types: (1) those which are monotonically decreasing and tending to nonnegative limits as $x \to +\infty$, and (2) those which are ultimately increasing and becoming unbounded as x becomes infinite.

In this section we shall consider proper solutions which are of type (1), i.e., solutions which are confined to the semi-infinite strip $S = \{(x, y)\colon 0 \leqq y \leqq K, a \leqq x < +\infty\}$. We observe that in view of properties (i) and (iii) the function $yF(y, x)$ satisfies a Lipschitz condition

$$|\,uF(u, x) - vF(v, x)\,| \leqq H\,|\,u - v\,| \tag{1.3}$$

in every closed rectangle $R = \{(x, y)\colon 0 \leqq y \leqq K, a \leqq x \leqq b\}$, where $H = H(K, a, b)$. Before taking up the existence of such solutions, we first derive the following lemmas.

LEMMA 1.1. *Let $u(x)$ be a nonnegative solution of* (1.1) *passing*

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through two points (a, A) *and* (b, B), *where* $a < b$ *and* $A, B > 0$. *Then the solution is unique.*

Proof. Suppose that $v(x)$ is a second nonnegative solution such that $u(a) = v(a) = A$ and $u(b) = v(b) = B$. We first assume that (a, A) and (b, B) are two consecutive points of intersection of u and v and that $u(x) > v(x)$ for $a < x < b$. Using (1.1) and property (iii) we find that

$$\int_a^b (u''v - uv'')dx = \int_a^b uv[F(u, x) - F(v, x)]dx > 0 . \tag{1.4}$$

Since

$$\int_a^b (u''v - uv'')dx = B[u'(b) - v'\{(b)] - A[u'(a) - v'(a)] , \tag{1.5}$$

and since $u'(a) > v'(a)$ while $u'(b) < v'(b)$, the right-hand side of (1.5) is clearly negative which contradicts (1.4). If u and v should have other points of intersection on (a, b) we can partition the interval $[a, b]$ into several segments whose end points are the abscissas of the consecutive points of intersection of u and v. The same argument leads to a contradiction in each case. This proves the assertion.

LEMMA 1.2. *Let* $u(x)$ *be a nonnegative solution of* (1.1) *passing through* (a, A) *such that* $\lim_{x\to b} u'(b) = 0$, *where* b *may be finite or infinite. Then* $u(x)$ *is unique.*

The proof is identical with that of Lemma 1.1 since the right-hand side of (1.5) will also be negative under the present assumptions.

The next lemma guarantees the existence of solutions passing through two points, provided the abscissas of these points are sufficiently near each other.

LEMMA 1.3. *Let* (a, A) *and* (b, B) *be two points such that*

$$a < b, \qquad A, B > 0$$

$(b - a)$ *is small enough so that*

$$H(b - a)^2 < \rho < 1 \tag{1.6}$$

and

$$L(x) > \int_a^b g(x, t)L(t)F(L, t)dt , \tag{1.7}$$

where

$$L(x) = \frac{A(b-x) + B(x-a)}{(b-a)}, \tag{1.8}$$

and

$$g(x, t) = \begin{cases} \dfrac{(b-x)(t-a)}{(b-a)}, & t \leqq x \\ \dfrac{(b-t)(x-a)}{(b-a)}, & x \leqq t. \end{cases} \tag{1.9}$$

Then there exists exactly one positive solution $y \in C^2[a, b]$ *of* (1.1) *which passes through these points.*

Proof. In view of Lemma 1.1, a solution, if it exists, is necessarily unique. To establish the existence we replace the boundary value problem by the equivalent integral equation

$$y(x) = L(x) - \int_a^b g(x, t)y(t)F(y, t)dt, \tag{1.10}$$

where $L(x)$ and $g(x, t)$ are given by (1.8) and (1.9) respectively. To solve (1.10) by successive approximations, we introduce a sequence $\{y_k(x)\}$ of twice differentiable convex functions passing through (a, A) and (b, B) defined by

$$\begin{cases} y_0(x) = L(x) \\ y_{k+1}(x) = L(x) - \displaystyle\int_a^b g(x, t)y_k(t)F(y_k, t)dt \\ k = 0, 1, 2, \cdots. \end{cases} \tag{1.11}$$

Since both $g(x, t)$ and $L(x)$ are positive in (a, b), (1.7) shows that $0 < y_1(x) < L(x)$. If we assume that $0 < y_k(x) < L(x)$, then (1.7) and property (iii) implies

$$\begin{aligned} L(x) > y_{k+1}(x) &= L(x) - \int_a^b g(x, t)y_k(t)F(y_k, t)dt \\ &> L(x) - \int_a^b g(x, t)L(t)F(L, t)dt = y_1(x) > 0. \end{aligned}$$

It follows by induction that $0 < y_k(x) \leqq L(x) \leqq \max(A, B)$ for all k. The sequence $\{y_k(x)\}$ is thus positive and uniformly bounded.

Let $K = \max(A, B)$ and $M = \sup F(K, x)$, then

$$\begin{aligned} |y_1(x) - y_0(x)| &= \int_a^b g(x, t)L(t)F(L, t)dt \\ &\leqq KM \int_a^b g(x, t)dt \\ &\leqq KM(b-a)^2. \end{aligned}$$

If R denotes the closed rectangle defined by $0 \leqq y \leqq K$ and $a \leqq x \leqq b$, then by (1.3),

$$|uF(u, x) - vF(v, x)| \leqq H|u - v|$$

for all points of R. Moreover, (1.11) shows that

$$|y_{k+1}(x) - y_k(x)| \leqq H\int_a^b g(x, t)|y_k(t) - y_{k-1}(t)|dt$$

so that we have, by induction,

$$|y_{k+1}(x) - y_k(x)| \leqq (KM)H^k(b - a)^{2(k+1)} . \tag{1.12}$$

We thus obtain the estimate

$$|y_n(x)| \leqq K + H^{-1}KM\sum_1^n [H(b - a^2)]^{k+1} \tag{1.13}$$

which, in view of (1.6), implies the uniform convergence of $\{y_n(x)\}$. This proves the lemma.

As pointed out before, a positive proper solution of (1.1) is either monotonically decreasing or monotonically increasing. As the following theorem shows there always exists exactly one solution of the former type which passes through a given point (a, A).

THEOREM 1.1. *For any given point* (a, A) *where* $A > 0$, *there exists exactly one positive proper solution* y *of the class* $C^2[a, \infty)$ *which passes through* (a, A) *and is monotonically decreasing in* $[a, \infty)$.

To prove this result we consider the variational problem of minimizing the functional

$$J(y) = \int_a^\infty [(y')^2 + 2h(y, x)]dx , \tag{1.14}$$

where

$$h(y, x) = \int_0^y tF(t, x)dt , \tag{1.15}$$

within the class Ω of all nonnegative functions $y \in D^1[a, \infty)$ such that $y(a) = A$ and that the integral (1.14) exists. Since (1.1) is the Euler-Lagrange equation of problem (1.14), the solution y of (1.14) will be a solution of (1.1), provided, of course, y exists and is of class $C^2[a, \infty)$.

Since the functional $J(y)$ is positive-definite, $J(y)$ has the trivial lower bound 0. We next remark that we may restrict our attention to positive functions $y \in \Omega$ which are convex in $[a, \infty)$. To show this, we assume that the positive function y is concave in an interval (c, d), i.e.,

$$y(x) \geqq \frac{y(c)(d-x) + y(d)(x-c)}{(d-c)} \equiv L(x) .$$

In view of hypothesis (iii) and the definition of $h(y, x)$, we then have

$$h(L, x) \leqq h(y, x), \qquad c \leqq x \leqq d ,$$

and, by a variational argument,

$$\int_c^d [L'(x)]^2 dx < \int_c^d [y'(x)]^2 dx \tag{1.16}$$

unless $y(x) \equiv L(x)$ in (c, d). Hence, if y^* denotes the function obtained from y by substituting $L(x)$ for $y(x)$ in (c, d),

$$J(y^*) < J(y) .$$

Also, we need only consider positive convex functions y which are nonincreasing in $[a, \infty)$, since, as (1.16) shows, the functional $J(y)$ becomes infinite for convex increasing functions. Finally, the problem $J(y) = \min$ is not vacuous, since the function v defined by

$$v(x) = \begin{cases} A\left(\dfrac{b-x}{b-a}\right), & a \leqq x \leqq b \\ 0 , & b \leqq x \end{cases}$$

is in Ω and evidently $J(v) < C < +\infty$.

The proof of the theorem depends on the validity of an analogous result for a finite interval $[a, b]$ and the performing of a suitable passage to the limit $b \to \infty$. The result in question is the following:

LEMMA 1.4. *There exists a unique positive solution* $u(x)$ *of equation* (1.1) *which passes through the two points* (a, A) *and* (b, B), *where* $b > a$ *and* $A, B > 0$. *If* v *denotes any other positive function of* $D^1[a, b]$ *for which* $v(a) = A$, $v(b) = B$, *and if* $J(y; b)$ *denotes the functional*

$$J(y; b) = \int_a^b [(y')^2 + 2h(y, x)] dx , \tag{1.14'}$$

then

$$J(u; b) < J(v; b) \tag{1.17}$$

unless $v(x) \equiv u(x)$ *in* $[a, b]$.

We first assume that the interval $[a, b]$ is short enough so that conditions (1.6) and (1.7) are satisfied. Lemma 1.3 will then guarantee the existence of the unique positive solution u of (1.1) through the two points, and all we have to prove is inequality (1.17). To do so, we note that the solution $w(x)$ of the linear differential system

$$\begin{cases} w'' = p(x)w, & p(x) > 0 \\ w(a) = A \\ w(b) = B \end{cases} \tag{1.18}$$

satisfies the inequality

$$\int_a^b [(w')^2 + p(x)w^2]dx < \int_a^b [(v')^2 + p(x)v^2]dx\,, \tag{1.19}$$

where v is any other function of $D^1[a, b]$ which satisfies the same boundary conditions and does not coincide with $w(x)$. Inequality (1.19) is an obvious consequence of the identity

$$\begin{aligned}&\int_a^b [(v' - w')^2 + p(x)(v - w)^2]dx \\ &= \int_a^b [(v')^2 + p(x)v^2]dx - \int_a^b [(w')^2 + p(x)w^2]dx\end{aligned}$$

which is obtained by expanding the left-hand side and observing that, in view of (1.18) and the boundary conditions,

$$\int_a^b v'w'dx = [vw']_a^b - \int_a^b vw''dx = [vw']_a^b - \int_a^b pvwdx$$

and

$$[vw']_a^b = [ww']_a^b = \int_a^b (w'^2 + ww'')dx = \int_a^b (w'^2 + pw^2)dx\,.$$

Setting, in particular, $p(x) = F(u, x)$, we have $w(x) = u(x)$ and thus, by (1.19)

$$\int_a^b [(u')^2 + u^2F(u, x)]dx < \int_a^b [(v')^2 + v^2F(u, x)]dx\,. \tag{1.20}$$

Since $F(s, x)$ is a nondecreasing function of s for $s > 0$, the function $h(u, x)$ defined by (1.15) is convex in u. Hence, for nonnegative u and v,

$$2[h(u, x) - h(v, x)] \leqq (u^2 - v^2)F(u, x)\,.$$

Combining this with (1.20), we obtain

$$\int_a^b [(u')^2 + 2h(u, x)]dx < \int_a^b [(v')^2 + 2h(v, x)]dx$$

unless u and v coincide. This establishes (1.17) in the case in which the interval $[a, b]$ is short enough so as to satisfy conditions (1.6) and (1.7).

If b is an arbitrary value in (a, ∞), it is sufficient to consider the problem

$$\begin{cases} J(y;b) = \int_a^b [(y')^2 + 2h(y,x)]dx = \min \\ y(a) = A, \qquad y(b) = B \end{cases} \tag{1.21}$$

in the class Ω_b of nonnegative convex functions $y \in D^1[a, b]$. We thus may assume

$$0 \leqq y(x) \leqq \max(A, B) = K, \qquad a \leqq x \leqq b .$$

Now we divide the interval $[a, b]$ into a finite number of subintervals $[a_k, a_{k+1}]$ $(a = a_0 < a_1 \cdots < a_m = b)$ in each of which the assumptions of Lemma 1.3 are satisfied. If $y(a_k) = A_k$, where $y \in \Omega_b$, the conditions restricting the length of these subintervals will be

$$H(a_{k+1} - a_k)^2 < \rho < 1 \tag{1.22}$$

and

$$\int_{a_k}^{a_{k+1}} g_k(x, t)L_k(t)F(L_k, t)dt < L_k(x) , \tag{1.23}$$

where

$$L_k(x) = \frac{A_k(a_{k+1} - x) + A_{k+1}(x - a_k)}{(a_{k+1} - a_k)} , \tag{1.24}$$

and

$$g_k(x, t) = \begin{cases} \dfrac{(a_{k+1} - x)(t - a_k)}{(a_{k+1} - a_k)} , & t \leqq x \\ \dfrac{(a_{k+1} - t)(x - a_k)}{(a_{k+1} - a_k)} , & x \leqq t . \end{cases} \tag{1.25}$$

Since $A_k \leqq \max(A, B) = K$, we have $F[L_k(t), t] < F(K, t)$. Hence, if $M = \max F(K, x)$ in $[a, b]$, condition (1.23) will be satisfied if

$$M \int_{a_k}^{a_{k+1}} g_k(x, t)L_k(t)dt < L_k(x) .$$

In view of (1.24), this will be true if both the inequalities

$$M \int_{a_k}^{a_{k+1}} g_k(x, t)(a_{k+1} - t)dt < (a_{k+1} - x)$$

and

$$M \int_{a_k}^{a_{k+1}} g_k(x, t)(t - a_k)dt < (x - a_k) \tag{1.26}$$

hold. Since these inequalities are equivalent, it is sufficient to consider one of them. A computation shows that

$$\int_{a_k}^{a_{k+1}} g_k(x, t)(t - a_k)dt = \frac{1}{6}(x - a_k)(a_{k+1} - x)(x + a_{k+1} - 2a_k) ,$$

and (1.26) will therefore follow if

$$\frac{M}{6}(a_{k+1} - x)(x + a_{k+1} - 2a_k) < 1 .$$

Since

$$(a_{k+1} - x) \leqq (a_{k+1} - a_k)$$

and $(x + a_{k+1} - 2a_k) = (x - a_k) + (a_{k+1} - a_k) \leqq 2(a_{k+1} - a_k)$, the length of the interval is thus restricted by the condition

$$M(a_{k+1} - a_k) < 2$$

and inequality (1.22). Since $H = H(K, a, b)$, this shows that a finite partition of the type indicated is indeed possible.

In each of these subintervals we now replace $y, y \in \Omega_b$, by the solution of (1.1) having the same values at the ends of the interval. If the new function so obtained is y^*, it follows from the result just proved that

$$J(y^*; b) < J(y; b) .$$

In the treatment of the minimum problem (1.21) it is therefore sufficient to consider curves y consisting of a finite number of arcs each of which is a solution of (1.1). Moreover, the abscissas of the points where two adjacent arcs meet may be taken to be the same for all functions of a sequence $\{y_n\}$ minimizing the functional $J(y; b)$.

Since in each of the subintervals $[a_k, a_{k+1}]$ the functions y_n are solutions of (1.1), elementary considerations show that we can select a subsequence $\{y_{n'}\}$ which converges in each subinterval $[a_k, a_{k+1}]$ to a solution $y_{(k)}$ of (1.1) and that, moreover, $y_{(k)}(a_{k+1}) - y_{(k+1)}(a_{k+1})$. The function y defined by $y(x) = y_{(k)}(x)$ for $a_k \leqq x \leqq a_{k+1}$ is therefore of class $D^1[a, b]$, and it is thus a solution of the minimum problem (1.21).

To show that $y(x)$ coincides in all these intervals with the *same* solution of (1.1), we have to show that y' is continuous at the points a_k. To do so, we choose a positive ε such that

$$(a_{k-1} + \varepsilon) < a_k, \ (a + \varepsilon) < a_{k+1}$$

and ε is small enough so that Lemma 1.3 applies to the interval $[a_k - \varepsilon, a_k + \varepsilon]$. There will then exist a solution u of (1.1) for which $u(a_k - \varepsilon) = y(a_k - \varepsilon)$, $u(a_k + \varepsilon) = y(a_k + \varepsilon)$ and, as shown above, we have the inequality

$$\int_{a_k-\varepsilon}^{a_k+\varepsilon}[u'^2 + 2h(u, x)]dx < \int_{a_k-\varepsilon}^{a_k+\varepsilon}[y'^2 + 2h(y, x)]dx ,$$

unless $y(x) \equiv u(x)$ in this interval. Hence if y' is discontinuous at $x = a_k$, it is possible to replace y by another function which yields a smaller value of $J(y; b)$. But this contradicts the minimum property of y, and we have thus proved that y' must be continuous throughout $[a, b]$. This completes the proof of Lemma 1.4.

We are now in a position to complete the proof of Theorem 1.1. As pointed out above, it is sufficient to consider positive admissible functions $y \in \Omega$ which are convex and decreasing in $[a, \infty)$. If y is any such function, we choose a value b in (a, ∞) and define a function $u \in \omega$, $\omega \subset \Omega$, as follows: $u(x) = y(x)$ in $[b, \infty)$ and $u(x) = y_b(x)$, where $y_b(x)$ denotes the solution of (1.1)—whose existence is established in Lemma 1.4—which satisfies $y_b(a) = A$ and $y_b(b) = y(b)$. In view of Lemma 1.4, we have

$$J(y_b) < J(y) ,$$

and it is clear that $0 \leqq y_b(x) \leqq A$ in $[a, \infty)$.

We now take a sequence $\{y_n\}$ in Ω for which

$$\lim_{n\to\infty} J(y_n) = \inf J(y) , \tag{1.27}$$

and we choose a sequence of values $b_m(a < b_1 < b_2 < \cdots)$ for which $\lim b_m = +\infty$. For each of these values b_m we construct the corresponding function $y_{n,b_m} \in \omega$. As just shown, we have

$$J(y_{n,b_{m+1}}) \leqq J(y_{n,b_m}) .$$

Hence, the diagonal sequence $J(y_{n,b_n})$ cannot have a larger limit than the sequence $J(y_n)$, and (1.27) shows that $J(y_{n,b_n})$ is likewise a minimizing sequence.

Since $0 \leqq y_{n,b_n} \leqq A$, and since y_{n,b_n} is a solution of (1.1) in $[a, b_N]$ if $n \geqq N$, an elementary argument shows that this sequence contains a limit function y which is a solution of (1.1) in $[a, b_N]$. But N is arbitrary, and y is thus a solution of (1.1) throughout $[a, \infty)$, the function y—being necessarily convex—must be decreasing for $a \leqq x < +\infty$. This completes the proof of Theorem 1.1.

Such a solution separates those solutions which are convex and increasing to $+\infty$ from those which are decreasing and becoming ultimately negative.

We add here a property of the positive decreasing solutions whose existence is established in Theorem 1.1.

LEMMA 1.5. *If y is a decreasing, positive proper solution of* (1.1), *then*

$$\lim_{x\to\infty} xy'(x) = 0 \ . \tag{1.28}$$

Since $(xy' - y)' = xy'' = xyF(y, x) > 0$, the negative quantity $\phi(x) \equiv xy' - y$ is increasing for $x > a$. Let $\lim y(x) = c$, $c \geqq 0$, then clearly $\phi(x) \leqq -c$. If $\lim \phi(x) = -c$, the lemma is proved. If $\lim \phi(x) = -(c + A)$, where $A > 0$, we have $xy' - y \leqq -(A + c)$ for x in (a, ∞), i.e.,

$$\frac{y - A - c}{x} \leqq 0, \qquad a < x < +\infty \ .$$

This, however, implies a contradiction, since the expression $x^{-1}(y - A - c)$ is negative for large x and tends to zero for $x \to +\infty$. This completes the proof.

THEOREM 1.2. *Equation* (1.1) *has solutions which ultimately decrease monotonically to positive constants if, and only if, there is some* $\beta > 0$ *such that*

$$\int^{\infty} xF(\beta, x)dx < +\infty \ . \tag{1.29}$$

Proof. If y is such a solution, it is easily confirmed that

$$y(x) = y(b) + y'(b)(x - b) + \int_x^b (t - x)y(t)F(y, t)dt \ .$$

Since $y(b) > 0$ and $y'(b) < 0$, it follows that

$$y(a) \geqq y(x) \geqq \int_x^b (t - x)y(t)F(y, t)dt \geqq \alpha \int_x^b (t - x)F(\alpha, t)dt \ ,$$

where $\lim y(x) = \alpha > 0$. This shows that condition (1.29) is necessary.

To show sufficiency, we consider the integral equation

$$y(x) = \alpha + \int_x^{\infty} (t - x)y(t)F(y, t)dt \ , \tag{1.30}$$

and suppose that β is a positive constant such that (1.29) holds. Then we can find a point $a \geqq x_0$ such that for all $x \geqq a$, we have

$$\int^{\infty} (t - x)F(\beta, t)dt < \frac{1}{2} \ .$$

We define a sequence of functions $\{y_k(x)\}$ by

$$\begin{cases} y_0(x) = \alpha \\ y_{k+1}(x) = \alpha + \displaystyle\int_x^{\infty} (t - x)y_k(t)F(y_k, t)dt \\ k = 0, 1, 2, \cdots . \end{cases} \tag{1.31}$$

If we choose α such that $0 < \alpha < \beta/2$, we see that $0 < \alpha < y_k(x)$ and $y_0(x) = \alpha < \beta$. By assuming $y_k(x) < \beta$ we find that

$$\begin{aligned} y_{k+1}(x) &= \alpha + \int_x^\infty (t - x)y_k(t)F(y_k, t)dt \\ &< \frac{\beta}{2} + \beta\int_x^\infty (t - x)F(\beta, t)dt < \beta . \end{aligned}$$

Hence induction shows that $0 < \alpha \leqq y_k(x) < \beta$ for all k. Moreover, if x_1 and x_2 are any two points such that $a \leqq x_1 < x_2 < \infty$, then, from (1.31), we have

$$\begin{aligned} | y_{k+1}(x_2) - y_{k+1}(x_1) | &\leqq | x_2 - x_1 | \left\{\int_{x_1}^{x_2} y_kF(y_k, t)dt + \int_{x_1}^\infty y_kF(y_k, t)dt\right\} \\ &= | x_2 - x_1 | \int_{x_1}^\infty y_k(t)F(y_k, t)dt . \end{aligned}$$

In view of the uniform boundedness of $\{y_k\}$ and (1.29), it follows that the sequence is equi-continuous also. Since $F(u, x) < F(v, x)$ whenever $0 < u < v < \infty$, it follows from the assumption $y_{k+1} > y_k$ that

$$y_{k+2}(x) - y_{k+1}(x) = \int_x^\infty (t - x)[y_{k+1}F(y_{k+1}, t) - y_kF(y_k, t)]dt > 0 .$$

This, together with the fact that $y_1 > y_0$, shows that $\{y_k(x)\}$ is a monotonically increasing sequence. We can therefore find a uniformly converging subsequence whose limit function $y(x)$ is the solution of equation (1.30).

It remains to show that the solution of (1.31) so obtained is indeed of class $C^2[a, \infty)$ and satisfies (1.1). To this end, we observe that, for $h > 0$,

$$\begin{aligned} \left| \frac{y(x + h) - y(x)}{h} + \int_x^\infty y(t)F(y, t)dt \right| &\leqq \int_x^{x+h} \left|\frac{x - t}{h}\right| y(t)F(y, t)dt \\ &\leqq \int_x^{x+h} y(t)F(y, t)dt . \end{aligned}$$

A corresponding inequality holds for $h < 0$. The solution y of (1.31) being continuous in $[a, \infty)$, it follows that

$$y'(x) = -\int_x^\infty y(t)F(y, t)dt .$$

In a similar manner, we can show that $y'' = yF(y, x)$, and the conclusion follows.

COROLLARY. *Equation* (1.1) *has proper solutions which ultimately decrease monotonically to zero if, and only if, for each* $\beta > 0$

$$\int^{\infty} xF(\beta, x)dx = +\infty \, . \tag{1.32}$$

Proof. We note that Theorem 1.1 assures the existence of a positive solution of (1.1) which is asymptotically equivalent to either a positive constant or zero, and that Theorem 1.2 gives a condition which is both necessary and sufficient for the former to hold, it follows that (1.32) is both necessary and sufficient for a solution to decrease to zero. The necessity can also be shown directly by the following simple argument.

If $y(x) \to 0$ as $x \to \infty$, we can choose a value $a \geqq x_0$ such that $y(x) < \lambda$ if $x > a$ and $y(a) = \lambda$, where λ is a positive constant. Writing (1.1) in the form

$$\begin{aligned} \lambda = y(a) &= y(b) + y'(b)(a-b) + \int_a^b (t-a)y(t)F(y, t)dt \\ &\leqq y(b) + y'(b)(a-b) + \lambda \int_a^b (t-a)F(\lambda, t)dt \, , \end{aligned}$$

where b is a number in (a, ∞). By Lemma 1.5, we can make $|y'(b)(a-b)|$ arbitrarily small by taking b large enough. Since $y(b) \to 0$ for $b \to \infty$, we can thus choose a b such that

$$|y(b) + y'(b)(a-b)| < \frac{\lambda}{2} \, .$$

Hence,

$$\frac{1}{2} < \int_a^b tF(\lambda, t)dt < \int_a^\infty tF(\lambda, t)dt \, .$$

Since a can be taken arbitrarily large, the result follows.

2. In this section we consider positive proper solutions of (1.1) which are convex and increasing. We begin with a necessary condition for the existence of such a solution, which is valid if hypothesis (iii) is replaced by the nonlinearity condition (iv) $u^{-2\varepsilon}F(u, x)$ is a strictly increasing function of u for each $x \geqq x_0$ and some positive constant ε.

THEOREM 2.1. *If $F(y, x)$ satisfies hypothesis* (iv) *instead of* (iii), *and if* (1.1) *has positive, convex increasing proper solutions, then*

$$\int^{\infty} [x^{-2\varepsilon}F(\beta x, x)]^{1/2+\varepsilon}dx < +\infty \tag{2.1}$$

for some $\beta > 0$.

Proof. Let y be a positive, convex increasing proper solution of (1.1), then $y(x) > \beta x$ for $\beta > 0$ and some $x \geqq x_0$. Let

$$w(x) = y(x)y'(x) \tag{2.2}$$

so that by (iv)

$$\begin{aligned} w' &= (y')^2 + y^2 F(y, x) \\ &= (y')^2 + y^2 \frac{F(y, x)}{F(\beta x, x)} F(\beta x, x) \\ &> (y')^2 + y^{2+2\varepsilon} G(x) \\ &= yy'[y'y^{-1} + G(x)y^{1+2\varepsilon}(y')^{-1}] , \end{aligned} \tag{2.3}$$

where $G(x) = (\beta x)^{-2\varepsilon} F(\beta x, x)$. If we set $r = (1 + \varepsilon)/(2 + \varepsilon)$ and $s = (2 + \varepsilon)^{-1}$, then, $r, s > 0$ and $r + s = 1$. With the help of the inequality [4, p. 37]

$$rA + sB > A^r B^s , \tag{2.4}$$

where we have set

$$y'y^{-1} = rA$$

and

$$G(x)y^{1+2\varepsilon}(y')^{-1} = sB ,$$

we find that

$$w'w^{-\alpha-1} > \rho[x^{-2\varepsilon} F(\beta x, x)]^{1/2+\varepsilon} , \tag{2.5}$$

where $\rho =$ constant and $0 < \alpha = \varepsilon(2 + \varepsilon)^{-1} < 1$. We now define

$$h(x) = \rho \int_{x0}^{x} [x^{-2\varepsilon} F(\beta x, x)]^{1/2+\varepsilon} dx , \tag{2.6}$$

and

$$H(x) = \frac{1}{\alpha}[w(x)]^{-\alpha} + h(x) ,$$

then (2.5) becomes

$$H'(x) < 0 .$$

The positive function H is thus necessarily decreasing for sufficiently large x and must ultimately tend to some finite limit $\lambda^2 \geqq 0$. Since $w^{-\alpha}$ is bounded for all $x \geqq x_0$, we conclude that $h(x)$ must ultimately be bounded also. This proves our assertion.

In the case of the special equation

$$y'' = Q(x)y^{2n+1} , \tag{2.7}$$

where Q is a nonnegative continuous function in $[x_0, \infty)$, Theorem 2.1 reduces to

COROLLARY 3.1. *A necessary condition for equation* (2.7) *to have positive convex increasing proper solutions is that*

$$\int^{\infty} [Q(x)]^{1/n+2} dx < +\infty \ . \tag{2.8}$$

With slight changes, the technique used in the proof of Theorem 2.1 will yield the following more general result:

THEOREM 2.2. *If $F(y, x)$ satisfies hypothesis* (iv) *instead of* (iii), *and if equation* (1.1) *has positive, convex increasing proper solutions, then there is some constant $\beta > 0$ such that*

$$\int^{\infty} x^{2s-\delta-1}[F(\beta x, x)]^s dx < +\infty \ , \tag{2.9}$$

where δ and s are any two positive constants which satisfy

$$\begin{cases} 0 < s < 1 \\ \delta + 2s \leqq 1 \\ \delta + 1 \leqq 2s(1+\varepsilon) \ . \end{cases} \tag{2.10}$$

Proof. If y is a positive, convex increasing proper solution of (1.1), then there is some $\beta > 0$ such that

$$y(x) > \beta x \ , \qquad y'(x) > \beta \ , \qquad \text{for all } x > x_0 \ . \tag{2.11}$$

From (2.3) and inequality (2.4), we see that

$$\begin{aligned} \frac{w'}{w} &> y'y^{-1} + G(x)y^{1+2\varepsilon}(y')^{-1} \\ &> \left(\frac{1}{r}\right)^r \left(\frac{1}{s}\right)^s [G(x)]^s (y')^{r-s} y^{s(1+2\varepsilon)-r} \ . \end{aligned}$$

Hence, for any $\delta > 0$,

$$w^{-1-\delta}w' > k[G(x)]^s (y')^{1-2s-\delta} y^{2s(1+\varepsilon)-\delta-1} \ , \tag{2.12}$$

where $k =$ constant. If moreover, s and δ are so chosen as to satisfy condition (2.10), then the exponents of y and y' in the inequality (2.12) above are both nonnegative. Combining this inequality with (2.11), and using the fact that $G(x) = (\beta x)^{-2\varepsilon}F(\beta x, x)$, we obtain

$$w^{-1-\delta}w' > \rho x^{2s-\delta-1}[F(\beta x, x)]^s \ , \tag{2.13}$$

for all $x \geqq x_0$, and $\rho =$ constant.

As in Theorem 2.1, we now define

$$h^*(x) = \rho \int_{x_0}^{x} x^{2s-\delta-1}[F(\beta x, x)]^s dx$$

and

$$H^*(x) = \frac{1}{\delta}[w(x)]^{-\delta} + h^*(x) .$$

It follows from (2.13) that

$$\frac{d}{dx} H^*(x) < 0 ,$$

and we thus conclude, as in Theorem 2.1, that $h^*(x)$ is necessarily bounded. This completes the proof.

It is easily confirmed that for $\delta = \varepsilon(2+\varepsilon)^{-1}$ and $s = (2+\varepsilon)^{-1}$, condition (2.10) is satisfied, and (2.9) reduces to (2.1) so that Theorem 2.1 is indeed a special case of Theorem 2.2. If we apply Theorem 2.2 to equation (2.7), we obtain the following extension of Corollary 2.1:

COROLLARY 2.2. *If δ and s are any two positive constants for which condition* (2.10) *holds, and if equation* (2.7) *has positive, convex increasing proper solutions, then*

$$\int^{\infty} x^{\lambda}[Q(x)]^s dx < +\infty , \tag{2.14}$$

where $\lambda = 2s(n+1) - \delta - 1$.

We will now consider the problem of existence of positive increasing proper solutions of (1.1) having specified asymptotic forms. The simplest case is that of finding a solution y such that $y(x) \sim \alpha x$, where $\alpha > 0$.

THEOREM 2.3. *Equation* (1.1) *has positive proper solutions y of the form*

$$y(x) \sim \alpha x , \qquad \alpha > 0 , \tag{2.15}$$

if, and only if, there exists a positive constant β such that

$$\int^{\infty} xF(\beta x, x)dx < +\infty . \tag{2.16}$$

We write $y(x) = xu(t)$, where $t = 1/x$. The function $u(t)$ will then have a constant limit if t decreases to zero. Making the necessary substitutions in equation (1.1) we obtain

$$\frac{d^2u}{dt^2} = ut^{-4}F\left(\frac{u}{t}, \frac{1}{t}\right) = uG(u, t) \tag{2.17}$$

for $u(t)$. Since

$$\int_0^a tG(\beta, t)dt = \int_0^a F\left(\frac{\beta}{t}, \frac{1}{t}\right)\frac{dt}{t^3} = \int_{1/a}^\infty xF(\beta x, x)dx ,$$

Theorem 2.3 will be a consequence of the following result (which we formulate in terms of x, y and F rather t, u and G):

THEOREM 2.4. *If $F(y, x)$ is continuous for $0 < x < b$ and otherwise satisfies hypotheses* (i), (ii) *and* (iii), *then equation* (1.1) *will have solutions which are continuous in some interval* $[0, a)$ $(0 < a < b)$ *and decrease to a positive constant as x decrease to zero if, and only if, there exists a constant $\beta > 0$ such that*

$$\int_0^a xF(\beta, x)dx < +\infty . \tag{2.18}$$

Theorem 2.4 is in many respects analogous the Theorem 1.2, and its proof depends likewise on our solving a suitable integral equation. The integral equation in question is

$$y(x) = A + Bx - \int_0^a g(x, t)y(t)F(y, t)dt , \tag{2.19}$$

where $g(x, t)$ is the Green's function

$$g(x, t) = \begin{cases} x , & x \leqq t \leqq a , \\ t , & 0 \leqq t \leqq x . \end{cases}$$

To show that condition (2.18) is necessary for the existence of a solution y with the required properties, we note that $y(x)$ must satisfy the integral equation

$$y(x) = A_1 + B_1x - \int_\varepsilon^a g(x, t)y(t)F(y, t)dt , \tag{2.20}$$

where $0 < \varepsilon < a$, and A_1 and B_1 are determined from the conditions

$$y(\varepsilon) = A_1 + B_1\varepsilon$$
$$y'(a) = B_1 - \int_\varepsilon^a y(t)F(y, t)dt .$$

Since $y'(a) > 0$, B_1 must be positive. In view of the fact that

$$y(x) \geqq \lim_{x\to 0} y(x) = A > 0 ,$$

it thus follows from (2.20) that

$$A\int_{\varepsilon}^{x} tF(A, t)dt \leqq A_1 + B_1 a ,$$

and this implies (2.18).

To show that (2.18) is also sufficient, we solve the integral equation (2.19) by the iteration

$$(2.21) \qquad \begin{cases} y_0(x) = A \\ y_{k+1}(x) = A + Bx - \int_0^a g(x, t) y_k(t) F(y_k, t) dt \\ k = 0, 1, 2, \cdots , \end{cases}$$

where $A = \beta/2$, $B = \beta/2a$, and the value a is chosen so that

$$\int_0^a xF(\beta, x) \leqq \frac{1}{2} .$$

The possibility of choosing such a value of a follows from (2.18). If $0 \leqq y_k(x) \leqq \beta$, we have

$$\int_0^a g(x, t) y_k(t) F(y_k, t) dt \leqq \beta \int_0^a tF(\beta, t) dt \leqq \frac{\beta}{2}$$

and thus, by (2.21),

$$y_{k+1}(x) \geqq A + Bx - \frac{\beta}{2} = \frac{\beta x}{2a} \geqq 0$$

Moreover,

$$y_{k+1}(x) \leqq A + Bx = \frac{\beta}{2}\left(1 + \frac{x}{a}\right) \leqq \beta .$$

It follows that $0 \leqq y_{k+1}(x) \leqq \beta$. Since $y_0(x) = \beta/2$, all functions $y_k(x)$ of the sequence (2.21) satisfy these inequalities.

The rest of the convergence proof for the iteration (2.21) is exactly the same as the corresponding argument used in the proof of Lemma 1.3.

COROLLARY 2.4. *Under the hypotheses of Theorem* 2.4, *equation* (1.1) *will have solutions which are continuous in* $[0, a)$ *and decrease to zero for* $x \to 0$ *if, and only if, there exists a positive constant* β *such that*

$$(2.22) \qquad \int_0^a xF(\beta x, x) dx < +\infty .$$

With the help of the transformation $y(x) = xu(t)$, where $t = 1/x$, and equation (2.17), we have

$$\int_{1/a}^{\infty} tG(\beta, t)dt = \int_{1/a}^{\infty} F\left(\frac{\beta}{t}, \frac{1}{t}\right)\frac{dt}{t^3}$$
$$= \int_0^a xF(\beta x, x)dx .$$

Hence, if $u(t)$ is any positive solution of (2.17) which decreases monotonically to a positive constant as $t \to \infty$, then $y(x) = xu(x)$ will be the desired solution of (1.1) in $[0, a)$. By Theorem 1.2, a necessary and sufficient condition for (2.17) to have such solutions is that

$$\int^{\infty} tG(\beta, t)dt < +\infty ,$$

for some $\beta > 0$, and the result follows from (2.23).

We will now consider the following more general question: Let v be a given positive convex increasing function of class $C^2[a, \infty)$. The problem is to determine whether equation (1.1) has positive proper solutions which are asymptotically equivalent to v. To answer this question we introduce a Liouville type transformation

$$\begin{cases} y = uv \\ x = x(t) , \end{cases} \tag{2.24}$$

where the new independent variable t is defined by

$$t = \int_x^{\infty} [v(s)]^{-2} ds . \tag{2.25}$$

Under this transformation, the interval $[a, \infty)$ is mapped onto $(0, b]$, and a computation shows that u must satisfy the equation

$$\frac{d^2u}{dt^2} = u\left[\dot{x}^2 F(uv, t) - \frac{1}{2}\{x, t\}\right] = uG(u, t) , \tag{2.26}$$

where $\{x, t\}$ denotes the Schwarzian differential operator

$$\{x, t\} = \frac{d}{dt}\left(\frac{\ddot{x}}{\dot{x}}\right) - \frac{1}{2}\left(\frac{\ddot{x}}{\dot{x}}\right)^2 .$$

In order that $y(x) \sim cv(x)$, $u(x)$ must therefore be a positive solution of (2.26) which decreases to a positive constant for $t \to 0$.

We observe that if the given function v were convex decreasing rather than convex increasing, the problem of determining whether (1.1) has proper solutions of this type can be treated in the same way. However, the new variable t in the Liouville transformation will now be given by

$$t = \int_a^x [v(s)]^{-2} ds ,$$

where v is now a positive, convex decreasing function of class $C^2[a, \infty)$. Since the procedure is the same in either case, we need only consider the convex increasing case.

To simplify matters we shall further restrict ourselves to those convex functions $v(x)$ for which the positive continuous function $p(x)$ defined by

$$p(x) = \frac{v''(x)}{v(x)} \tag{2.27}$$

is such that $[F(uv, x) - p(x)]$ is ultimately of one sign. That is to say, we assume that either (1) $G(u, x) < 0$ for all $u > 0$ and $0 < t \leqq a < b$, or (2) $G(\beta, t) > 0$ for some $\beta > 0$ and all sufficiently small t.

If case (1) holds, then the Atkinson-Nehari criterion [6, Theorem I] shows that a necessary and sufficient condition for the existence of a positive solution $u(t)$ which decreases to a positive constant as t decreases to zero, is that

$$0 \leqq -\int_0^a tG(\mu, t)dt < +\infty \tag{2.28}$$

for some constant $\mu > 0$.

On the other hand, if (2) holds, then by Theorem 2.4, the corresponding necessary and sufficient condition is the existence of some positive constant β for which

$$\int_0^a tG(\beta, t)dt < +\infty \, . \tag{2.29}$$

Expressed in terms of x and $v(x)$, both (2.28) and (2.29) may be combined into a single condition:

$$\int^\infty v^2(x)\int_x^\infty \frac{ds}{v^2(s)}\left| F(\beta v, x) - \frac{v''}{v}\right| dx < +\infty \, . \tag{2.30}$$

If we regard (2.27) as a linear homogeneous equation with $p(x)$ given, and that u and v are two linearly independent positive solutions whose Wronskian is negative, then one solution must be convex increasing and the other is convex decreasing. Moreover, if v denotes the increasing solution, then

$$u(x) = v(x)\int_x^\infty [v(s)]^{-2}ds$$

so that (2.30) may be written as

$$\int^\infty u(x)v(x)\,|\,F(\beta v, x) - p(x)\,|\,dx < +\infty \, .$$

We can now state the following result:

THEOREM 2.5. *Let $p(x)$ be a positive continuous function in $[x_0, \infty)$ and u and v be two linearly independent positive solutions of* (2.27). *If, moreover, $[F(\mu v, x) - p(x)]$ is either negative for all $\mu > 0$ or positive for some $\mu > 0$, then a necessary and sufficient condition for equation* (1.1) *to have positive, convex proper solutions y of the form*

$$y(x) \sim cv(x) , \qquad c > 0 , \tag{2.31}$$

is that there is some $\beta > 0$ such that

$$\int^{\infty} u(x)v(x) \,|\, F(\beta v, x) - p(x) \,|\, dx < +\infty . \tag{2.32}$$

COROLLARY 2.51. *If $F(ux^{\alpha}, x) - \alpha(\alpha - 1)x^{-2}$ is ultimately of one sign, where $\alpha > 1$, then a necessary and sufficient condition for equation* (1.1) *to have positive proper solutions of the form*

$$y(x) \sim cx^{\alpha} , \qquad c > 0 , \qquad \alpha > 1 , \tag{2.33}$$

is that, for some $\beta > 0$,

$$\int^{\infty} x \,|\, F(\beta x^{\alpha}, x) - \alpha(\alpha - 1)x^{-2} \,|\, dx < +\infty . \tag{2.34}$$

Proof. If we let $p(x) = \alpha(\alpha - 1)x^{-2}$, then $u(x) = x^{1-\alpha}$ and $v(x) = x^{\alpha}$, and the result follows from (2.32).

COROLLARY 2.52. *If $F(ue^{\alpha x}, x) - \alpha^2$ is ultimately of one sign, where $\alpha > 0$, then equation* (1.1) *has positive proper solutions of the form*

$$y(x) \sim ce^{\alpha x} , \qquad \alpha, c > 0 , \tag{2.35}$$

if, and only if, there exists some constant $\beta > 0$ such that

$$\int^{\infty} |\, F(\beta e^{\alpha x}, x) - \alpha^2 \,|\, dx < +\infty . \tag{2.36}$$

As pointed out before, the Emden-Fowler equation

$$y'' = x^{\lambda} y^{n} , \qquad n > 1 , \tag{1.2}$$

is a particular example of equation (1.1) with $F(y, x) = x^{\lambda}y^{n-1}$. We can therefore apply the results obtained here to investigate the existence and asymptotic behavior of proper solutions of this equation.

From Theorem 1.2, we see that a necessary and sufficient condition for equation (1.2) to have positive proper solutions which ultimately decrease to positive constants is that

$$\int^{\infty} x^{\lambda+1} dx < +\infty .$$

It follows that we must have

$$\lambda + 2 < 0 . \tag{2.37}$$

From Theorem 2.3 we find that equation (1.2) has positive proper solutions y of the form $y(x) \sim cx$, $c > 0$, if, and only if

$$\int^{\infty} x^{\lambda+n} dx < +\infty .$$

Hence, we obtain the condition

$$\lambda + n + 1 < 0 . \tag{2.38}$$

Corollary 2.51 shows that a necessary and sufficient condition for (1.2) to have positive proper solutions of the form $y(x) \sim cx^{\alpha}$, $\alpha > 1$, is that

$$\int^{\infty} x \,|\, \beta^{n-1} x^{\alpha(n-1)+\lambda} - \alpha(\alpha - 1)x^{-2} \,|\, dx < +\infty .$$

This condition will be satisfied if, and only if $\beta^{n-1} = \alpha(\alpha - 1)$ and $\alpha(n-1) + \lambda = -2$. Thus, the required condition in this case will be

$$\alpha = -\frac{\lambda + 2}{n - 1} > 1 . \tag{2.39}$$

From Corollary 2.52, it is easy to see that equation (1.2) cannot have any proper solution which is exponential. Finally, suppose that $u(x)$ is *any* positive, convex increasing proper solution of the Emden-Fowler equation, then, by Corollary 2.1, it is necessary that

$$\int^{\infty} [Q(x)]^{2/n+3} dx = \int^{\infty} x^{2\lambda/n+3} dx < +\infty .$$

In other words, we must have

$$2\lambda + n + 3 < 0 . \tag{2.40}$$

Applying this inequality to the special equation

$$y'' = x^{-2} y^{n} , \qquad n > 1 ,$$

we find that it cannot have any proper solution which is convex and increasing. Moreover,

$$\int^{\infty} x^{\lambda+1} dx = \int^{\infty} x^{-4/n+3} dx = +\infty$$

so that, by the Corollary of Theorem 1.2, this equation has a decreasing proper solution through every point (a, A), $A > 0$, which decreases to zero as $x \to \infty$. (cf. [2], Chapter 7, Theorems 1 to 5).

An elementary example of the Emden-Fowler equation for which an explicit solution is known is the equation

$$y'' = 2x^{-6}y^3 .$$

It is easily confirmed that x^2 is a solution of this equation. If we set $\alpha = 2$ and $\beta = 1$ in (2.34) we find that the integral vanishes so that the condition of Corollary 2.51 is indeed satisfied.

If we assume moreover that $v(x)$ is a proper solution of (1.1), then (2.30) may be used to determine the possible existence of a second proper solution y distinct from v such that their ratio is asymptotically constant. Without loss of generality we may assume that $y(x) \geqq v(x)$ for each $x \geqq x_0$. A necessary and sufficient condition for the existence of such solutions is the boundedness of

$$\int^{\infty} v^2(x) \int \frac{dx}{v^2(x)} [F(\beta v, x) - F(v, x)]dx$$

for some $\beta > 1$.

A condition for the difference of two proper solutions to be asymptotically constant may be obtained as follows: Let $w(x)$ be a positive proper solution of (1.1), and we let a second proper solution y be of the form

$$y(x) = u(x) + w(x) ,$$

where $u \in C^2[a, \infty)$ and $u(x) \sim k$, $k > 0$. Differentiation shows that u must satisfy the equation

$$\begin{cases} u'' = G(u, x) , \\ G(u, x) = uF(u + w, x) + [F(u + w, x) - F(w, x)] . \end{cases}$$

In view of Theorem 1.2, this equation will have proper solutions which ultimately decrease to positive constants if, and only if, there exists some $\beta > 0$ such that

$$\int^{\infty} xG(\beta, x)dx < +\infty .$$

THEOREM 2.6. *Let $w(x)$ be a positive proper solution of* (1.1).

1. *A necessary and sufficient condition for the existence of a second positive proper solution y such that $y(x)/w(x) \sim k$, $k > 0$, is that*

$$\int^{\infty} w^2(x) \int^{x} \frac{ds}{w^2(s)} [F(\beta w, x) - F(w, x)]dx < +\infty$$

for some $\beta > 1$.

2. *A necessary and sufficient condition for the existence of a second solution* y *such that* $y(x) - w(x) \sim c$, $c > 0$, *is that for some* $\mu > 0$

$$\int^{\infty} x[(\mu + w)F(\mu + w, x) - F(w, x)]dx < +\infty .$$

All results obtained thus far concern the asymptotic behavior of proper solutions, but the question of positive convex solutions having finite asymptotes is also of interest. As the following result shows, equation (1.1) always has such discontinuous solutions.

THEOREM 2.7. *If* F *satisfies hypothesis* (iv) *instead of* (iii), *and if* A *is an arbitrary real number and* a *and* δ *are positive; then there exists a solution* y *of* (1.1) *with* $y(a) = A$, *which is not continuous in* $(a, a + \delta)$.

Proof. Since $y(x)$ is convex, the value of $y(a + \delta)$ can be made arbitrarily large by a sufficiently large choice of $y'(a)$. We may accordingly assume that $y(a + \delta) > 1$. Let c be the point in $(a, a + \delta)$ where $y(c) = 1$, and we recall that, for $y > 1$, $F(y, x) > y^{2\varepsilon}F(1, x)$. It follows from (1.1) that

$$[y'(x)]^2 = \alpha^2 + 2\int_a^x y(t)F(y, t)y'(t)dt , \tag{2.41}$$

where $y'(a) = \alpha$, and $a < x < a + \delta$. If $0 < \rho \leqq F(1, x)$ for $x \in [a, a + \delta]$ and $x > c$, we then have

$$\begin{aligned} 2\int_a^x yy'F(y, t)dt &> 2\int_c^x yy'F(y, t)dt \\ &\geqq 2\int_0^x y^{1+2\varepsilon}y'F(1, t)dt \\ &\geqq 2\rho\int_c^x y^{1+2\varepsilon}y'dt \\ &= \frac{\rho}{1+\varepsilon}\{[y(x)]^{2+2\varepsilon} - 1\} . \end{aligned}$$

For $x \in [a, c]$, this holds trivially. Choosing α^2 large enough so that $\alpha^2 > \rho(1 + \varepsilon)^{-1}$, we conclude from (2.41) that

$$[y'(x)]^2 \geqq \alpha^2 + \frac{\rho}{1+\varepsilon}[y^{2+2\varepsilon} - 1] ,$$

or, with $\beta^2 = \alpha^2 - \rho(1 + \varepsilon)^{-1}$ and $\lambda^{2+2\varepsilon} = \rho(1 + \varepsilon)^{-1}$,

$$[y'(x)]^2 \geqq \beta^2 + (\lambda y)^{2+2\varepsilon} .$$

If $y(x)$ is continuous in $[a, a+\delta]$, $y'(x)$ necessarily remains positive. Hence

$$\int_0^\infty \frac{dt}{\sqrt{\beta^2 + (\lambda t)^{2+2\varepsilon}}} = \int_{y(a)}^{y(b)} \frac{dt}{\sqrt{\beta^2 + (\lambda t)^{2+2\varepsilon}}} \geqq (b-a) ,$$

where $b = a + \delta$, and this reduces to

$$(b-a) \leqq \beta^{-\varepsilon/1+\varepsilon} \int_0^\infty \frac{dt}{\sqrt{1 + (\lambda t)^{2+2\varepsilon}}} .$$

Since the integral exists, this provides a bound for the right end point of the interval of continuity. In view of the fact that $\beta^2 = \alpha^2 - \rho(1+\varepsilon)^{-1}$, it is also obvious that $(b-a)$ can be made arbitrarily small by a sufficiently large choice of $\alpha = y'(a)$. This completes the proof.

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