COMBINATORIAL FUNCTIONS AND REGRESSIVE ISOLS

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1. Introduction. *It is assumed that the reader is familiar with* the notions: regressive function, regressive set, regressive isol, cosimple isol, combinatorial function and its canonical extension. The first four are defined in [2], the last two in [3]. Denote the set of all numbers (nonnegative integers) by ε, the collection of all isols by Λ, the collection of all regressive isols by Λ_R and the collection of all cosimple isols by Λ_1. The following four propositions will be used.

$$(1) \quad \begin{cases} \text{Let } \tau = \rho t \text{ and } \tau^* = \rho t^*, \text{ where } t_n \text{ and } t_n^* \text{ are regressive} \\ \text{functions. Then } \tau \simeq \tau^* \Longleftrightarrow t_n \simeq t_n^* . \end{cases}$$

$$(2) \qquad B \leqq A \ \& \ A \in \Lambda_R \Longrightarrow B \in \Lambda_R .$$

$$(3) \quad \begin{cases} \text{Let } F(T) \text{ be the canonical extension to } \Lambda \text{ of the recursive,} \\ \text{combinatorial function } f(n). \text{ Then } T \in \Lambda_R \Longrightarrow F(T) \in \Lambda_R . \end{cases}$$

$$(4) \qquad B \leqq A \ \& \ A \in \Lambda_1 \Longrightarrow B \in \Lambda_1 .$$

The first three are Propositions 3, 9(b) and Theorem 3(a) of [2] respectively. The fourth is Theorem 56(b) of [1].

DEFINITION. Let $f(n)$ be a one-to-one function from ε into ε and let $T \in \Lambda_R - \varepsilon$. Then

$$\phi_f(T) = \text{Req}\, \rho t_{f(n)} ,$$

where t_n is any regressive function ranging over any set in T.

Using (1) it is readily seen that ϕ_f is a well defined function from $\Lambda_R - \varepsilon$ into $\Lambda - \varepsilon$. The main result of this paper is as follows: *Let $f(n)$ be a strictly increasing, recursive, combinatorial function; let $F(X)$ be its canonical extension to Λ, and let $T \in \Lambda_R - \varepsilon$; then $\phi_f(F(T)) = T$.*

2. The operation ϕ_f.

PROPOSITION 1. *Let $f(n)$ be a strictly increasing, recursive function and let $T \in \Lambda_R - \varepsilon$. Then*

$$\phi_f(T) \leqq T \quad \textit{and} \quad \phi_f(T) \in \Lambda_R .$$

February 15, 1963. This paper was written while the author was a National Defense Graduate Fellow at Rutgers University and is a portion of a thesis directed by Professor J. C. E. Dekker to submitted as partial fulfillment of the requirements for the degree of Doctor of Philosophy.

If in addition $T \in \Lambda_1$, *then* $\phi_f(T) \in \Lambda_R \cdot \Lambda_1$.

Proof. In view of (2) and (4), it suffices to show only that $\phi_f(T) \leqq T$. Let t_n be a regressive function such that $\rho t = \tau \in T$. Put $\alpha = \rho f$ and suppose $p(x)$ is a regressing function of t_n. Define

$$p^*(x) = (\mu y)[p^{y+1}(x) = p^y(x)] \quad \text{for } x \in \delta p \ .$$

Then $p^*(t_n) = n$ and

$$\rho t_f \subset \{x \in \delta p^* \mid p^*(x) \in \alpha\} \ ,$$
$$\tau - \rho t_f \subset \{x \in \delta p^* \mid p^*(x) \notin \alpha\} \ .$$

Since α is recursive it follows that ρt_f is separable from $\tau - \rho t_f$. Hence $\phi_f(T) \leqq T$.

It is known (by an unpublished result of Dekker) that Λ_R is neither closed under addition nor under multiplication. We do, however, have some closure properties for isols of the type $\phi_f(T)$, where $T \in \Lambda_R - \varepsilon$ and $f(n)$ is a strictly increasing, recursive function.

PROPOSITION 2. *Let* $f(n)$ *and* $g(n)$ *be strictly increasing, recursive function and let* $T \in \Lambda_R - \varepsilon$. *Then*

(a) $\phi_f(\phi_g(T)) \in \Lambda_R - \varepsilon$,
(b) $\phi_f(T) \cdot \phi_g(T) \in \Lambda_R - \varepsilon$,
(c) $\phi_f(T) + \phi_g(T) \in \Lambda_R - \varepsilon$.

Proof. In view of Proposition 1,

$$\phi_f(\phi_g(T)) \leqq \phi_g(T) \leqq T \ .$$

This implies (a). To verify (a) one could also observe that $\phi_f(\phi_g(T)) = \phi_{gf}(T)$. Combining $\phi_f(T) \leqq T$ and $\phi_g(T) \leqq T$, we obtain by [1, Cor. of Thm. 77]

$$\phi_f(T) \cdot \phi_g(T) \leqq T^2 \ .$$

However, $T^2 \in \Lambda_R - \varepsilon$ by (3). Hence (b) follows by (2). Finally, it is readily seen that

$$\phi_f(T) + \phi_g(T) \leqq \phi_f(T) \cdot \phi_g(T) \ ,$$

since $\phi_f(T)$ and $\phi_g(T)$ are $\geqq 2$ (in fact, infinite). Thus (c) follows from (2) and (b).

3. The main result. We first state and prove two lemmas which might be of interest for their own sake. Let $\rho_0, \rho_1, \cdots$ be the canonical enumeration of the class Q of all finite sets defined by

$$\rho_0 = o$$
$$\rho_{x+1} = \begin{cases}(y_1, \cdots, y_k) \text{ where } y_1, \cdots, y_k \text{ are the distinct numbers} \\ \text{such that } x + 1 = 2^{y_1} + \cdots + 2^{y_k} .\end{cases}$$

We denote the cardinality of ρ_x by r_x.

LEMMA 1. *Let $f(n)$ be any combinatorial function and let C_i be the function from ε into ε such that $f(n) = \sum_{i=0}^{n} c_i\binom{n}{i}$. Then*

$$f(n) = \sum_{x=0}^{2^n-1} c_{r(x)} .$$

Proof. Since every n-element set has $\binom{n}{i}$ subsets of cardinality i, we have

$$(5) \qquad f(n) = \text{card}\,\{j(x, y) \mid \rho_x \subset (0, 1, \cdots, n-1) \;\&\; y < c_{r(x)}\} .$$

It follows from the definition of ρ_x that

$$\begin{aligned}\rho_x \subset (0, 1, \cdots, n-1) &\Longleftrightarrow x \leqq 2^0 + 2^1 + \cdots + 2^{n-1} \\ &\Longleftrightarrow x \leqq 2^n - 1 .\end{aligned}$$

Combining this with (5) we obtain

$$f(n) = \text{card}\,\{j(x, y) \mid x \leqq 2^n - 1 \;\&\; y < c_{r(x)}\} = \sum_{x=0}^{2^n-1} c_{r(x)} .$$

DEFINITION. Let $a(n)$ be a one-to-one function from ε into ε. Then

$$a'(n) = l_{n0}\cdot 2^{a(0)} + \cdots + l_{nn}\cdot 2^{a(n)} ,$$

where $l_{n0}, \cdots, l_{nn}$ is the sequence of zeros and ones such that

$$n = l_{n0}\cdot 2^0 + \cdots + l_{nn}\cdot 2^n .$$

LEMMA 2. (Dekker) *Let $a(n)$ be a one-to-one function from ε into ε with range α and let $A = \text{Req}\,(\alpha)$. Then $a'(n)$ is also a one-to-one function from ε into ε. Moreover,*

$$a'(2^n) = 2^{a(n)} , \quad \rho_{a'(n)} = a(\rho_n) \textit{ and } \rho a' \in 2^A .$$

Finally, if $a(n)$ is regressive, so is $a'(n)$.

Proof. It is clear that $a'(n)$ is a one-to-one function such that $a'(2^n) = 2^{a(n)}$. We have $\rho_{a'(0)} = \rho_0 = o$ while $a(\rho_0) = a(o) = o$; for $n \geqq 1$

$$\rho_n = \{i \mid 0 \leqq i \leqq n \;\&\; l_{ni} = 1\} .$$

Hence for every number n

$$\begin{aligned}\rho_{a'(n)} &= \{a(i) \mid 0 \leqq i \leqq n \ \& \ l_{ni} = 1\} \\ &= a\{i \mid 0 \leqq i \leqq n \ \& \ l_{ni} = 1\} = a(\rho_n) \, .\end{aligned}$$

Thus, if n ranges over ε, ρ_n ranges over the class Q of all finite sets, $\rho_{a'(n)} = a(\rho_n)$ over the class of all finite subsets of α. We conclude that $\rho a' \in 2^A$. Finally, assume that $a(n)$ is a regressive function. Using the three facts that

$$\begin{aligned}&a'(n+1) = l_{n+1,0} \cdot 2^{a(0)} + \cdots + l_{n+1,n+1} \cdot 2^{a(n+1)} \, , \\ &a'(n) = l_{n0} \cdot 2^{a(0)} + \cdots + l_{nn} \cdot 2^{a(n)} \, , \\ &\max \{i \mid l_{ni} = 1\} \leqq \max \{i \mid l_{n+1,i} = 1\} \, ,\end{aligned}$$

we infer that $a'(n)$ is a regressive function.

THEOREM. *Let $f(n)$ be a strictly increasing, recursive combinatorial function, let $F(X)$ be its canonical extension to Λ and let $T \in \Lambda_R - \varepsilon$. Then $\phi_f(F(T)) = T$.*

Proof. Let $f(n) = \sum_{i=0}^{n} c_i \binom{n}{i}$ be the strictly increasing, recursive, combinatorial function. Then $c_1 > 0$ since $f(n)$ is strictly increasing, and c_i is a recursive function of i, since $f(n)$ is recursive. Let $\tau \in T \in \Lambda_R - \varepsilon$ and assume that t_n is a regressive function ranging over τ. Put $g(n) = t'(n)$. By Lemma 2 we have $\rho_{g(n)} = t(\rho_n)$; thus, if n assumes successively the values $0, 1, 2, 3, 4, 5, 6, 7, \cdots, \rho_{g(n)}$ assumes successively the "values"

$$o, (t_0), (t_1), (t_0, t_1), (t_2), (t_0, t_2), (t_1, t_2), (t_0, t_1, t_2), \cdots \, .$$

We have by definition

$$F(T) = \operatorname{Req} \{j(x, y) \mid \rho_x \subset \tau \ \& \ y < c_{r(x)}\} \, .$$

Since $g(n)$ ranges without repetitions over $\{n \mid \rho_n \subset \tau\}$, it follows that

$$F(T) = \operatorname{Req} \{j(g(x), y) \mid y < c_{r(x)}\} \, . \tag{6}$$

We shall use w_n to denote the function which for $0, 1, \cdots$ takes on the values of the array

$$\begin{array}{l} j(g(0), 0), \ \cdots, j(g(0), c_{r(0)} - 1) \\ j(g(1), 0), \ \cdots, j(g(1), c_{r(1)} - 1) \\ j(g(2), 0), \ \cdots, j(g(2), c_{r(2)} - 1) \\ \quad \vdots \qquad\qquad\quad \vdots \end{array}$$

reading from the left to the right in each row and from the top row down; it is understood that every row which starts with $j(g(k), 0)$ for some k with $c_{r(k)} = 0$ is to be deleted. From the definitions of ρ_k and $r(k)$ we see that

$$k \in (2^0, 2^1, 2^2, \cdots) \Longrightarrow r(k) = 1 \Longrightarrow c_{r(k)} = c_1 > 0 \, .$$

The function $g(n) = t'(n)$ is regressive by Lemma 2. Taking into account that c_i is a recursive function, it readily follows that w_n is a regressive function. In view of (6) we have $\rho w_n \in F(T)$ it therefore suffices to prove that $\rho w_{f(n)} \in T$. By Lemma 1

$$f(n) = \sum_{x=0}^{2^n-1} c_{r(x)} \, ,$$

hence

$$f(0) = c_{r(0)} \, , \quad f(1) = c_{r(0)} + c_{r(1)} \, , \quad f(2) = c_{r(0)} + c_{r(1)} + c_{r(2)} + c_{r(3)}, \cdots$$

and

$$w_{f(0)} = j(g(1), 0) \, , \quad w_{f(1)} = j(g(2), 0), \cdots , \quad w_{f(n)} = j(g(2^n), 0), \cdots \, .$$

We conclude that $w_{f(n)} \cong g(2^n)$. However, by Lemma 2

$$g(2^n) = t'(2^n) \cong t(n) \, .$$

Thus $w_{f(n)} \cong t_n$ and $\rho w_{f(n)} \in T$. This completes the proof.

References

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