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NORMS AND INEQUALITIES FOR CONDITION NUMBERS

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The condition number c_φ of a nonsingular matrix A is defined by $c_\varphi(A) = \varphi(A)\varphi(A^{-1})$ where ordinarily φ is a norm. It was proved by O. Taussky-Todd that (c) $c_\varphi(A) \leqq c_\varphi(AA^*)$ when $\varphi(A) = (tr\, AA^*)^{1/2}$ and when $\varphi(A)$ is the maximum absolute characteristic root of A. It is shown that (c) holds whenever φ is a unitarily invariant norm, i.e., whenever φ satisfies $\varphi(A) > 0$ for $A \neq 0$; $\varphi(\alpha A) = |\alpha|\varphi(A)$ for complex α; $\varphi(A + B) \leqq \varphi(A) + \varphi(B)$; $\varphi(A) = \varphi(AU) = \varphi(AU)$ for all unitary U. If in addition, $\varphi(E_{ij}) = 1$, where E_{ij} is the matrix with one in the (i, j)th place and zeros elsewhere, then $c_\varphi(A) \geqq [c_\varphi(AA^*)]^{1/2}$. Generalizations are obtained by exploiting the relation between unitarily invariant norms and symmetric gauge functions. However, it is shown that (c) is independent of the usual norm axioms.

1. Introduction. The genesis of this study is the proposition that under certain conditions, the matrix AA^* is more "ill-conditioned" than A. More precisely, the condition number $c_\varphi(A)$ is defined for nonsingular matrices A as

$$c_\varphi(A) = \varphi(A)\varphi(A^{-1}) ,$$

where ordinarily φ is a norm. The statement concerning ill-conditioning of AA^* is the inequality

$$(c) \qquad c_\varphi(A) \leqq c_\varphi(AA^*) .$$

Where $\varphi(A)$ is the maximum absolute characteristic root of A and where $\varphi(A) = (tr\, AA^*)^{1/2}$, inequality (c) was proved by O. Taussky-Todd [7]. This raises the question of whether (c) is true for all norms. In this paper, we show that quite the contrary is true; (c) is independent of the usual norm axioms. However, we also prove that (c) does hold for a quite general class of norms.

In the course of proving these results, we obtain some inequalities for symmetric gauge functions, which may be of independent interest.

2. Gauge functions and matrix norms. We call φ a *matrix norm* if

$$(\text{aI}) \qquad \varphi(A) > 0 \qquad \text{when } A \neq 0 ,$$

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(aII) $\varphi(\alpha A) = |\alpha| \varphi(A)$ for complex α ,

(aIII) $\varphi(A + B) \leqq \varphi(A) + \varphi(B)$.

In addition to these basic axioms, various other conditions are sometimes imposed:

(aIV) $\varphi(E_{ij}) = 1$,

where E_{ij} is the matrix with one in the (i, j)th position and zero elsewhere,

(aV) $\varphi(AB) \leqq \varphi(A)\varphi(B)$,

(aVI) $\varphi(A) = \varphi(UA) = \varphi(AU)$ for all unitary matrices U.

If φ satisfies aI, aII, aIII, and aVI, φ is called a *unitarily invariant norm*.

There is an important connection between unitarily invariant norms and symmetric gauge functions. A function Φ on a complex vector space is called a *gauge function* if

(bI) $\Phi(u) > 0$ when $u \neq 0$,

(bII) $\Phi(\alpha u) = |\alpha| \Phi(u)$ for complex α ,

(bIII) $\Phi(u + v) \leqq \Phi(u) + \Phi(v)$.

Often it is convenient to assume, in addition, that

(bIV) $\Phi(e_i) = 1$,

where e_i is the vector with one in the ith place and zero elsewhere. If, in addition to bI, bII, and bIII,

(bV) $\Phi(u_1, \cdots, u_n) = \Phi(\varepsilon_1 u_{i_1}, \cdots, \varepsilon_n u_{i_n})$

whenever $\varepsilon_j = \pm 1$ and $(i_1, \cdots, i_n)$ is a permutation of $(1, \cdots, n)$, then Φ is called a *symmetric gauge function.*

It was noted by Von Neumann [8] that a norm φ is unitarily invariant if and only if there exists a symmetric gauge function Φ such that $\varphi(A) = \Phi(\alpha)$ for all A, where $\alpha_1^2, \cdots, \alpha_n^2$ are the eigenvalues of AA^*.

If Φ is a symmetric gauge function and u, v satisfy $u_i \leqq v_i$, $i = 1, \cdots, n$, then it follows [6, p. 85] that

(2.1) $$\Phi(u_1, \cdots, u_n) \leqq \Phi(v_1, \cdots, v_n) .$$

If Φ is a symmetric gauge function satisfying bIV, then [6, p. 86]

(2.2) $$\max_i |u_i| \leqq \Phi(u_1, \cdots, u_n) \leqq \sum_{i=1}^n |u_i| .$$

If φ is the unitarily invariant matrix norm determined by Φ as above, then it follows that

$$\frac{\varphi(AB)}{\varphi(A)\varphi(B)} \leqq \frac{\sum_{i=1}^{n} \lambda_i(ABB^*A^*)}{[\max_i \lambda_i (AA^*)][\max_j \lambda_j (BB^*)]} \leqq \frac{n \max_i \lambda_i (BB^*A^*A)}{[\max_i \lambda_i (AA^*)][\max_j \lambda_j (BB^*)]} \leqq n ,$$

where $\lambda_i(M)$ are the eigenvalues of M. Thus, for any $k \geqq n$, $k\varphi$ is a unitarily invariant matrix norm also satisfying aV. Of course, φ itself satisfies aIV (since Φ satisfies bIV), and this property is destroyed by the renormalization.

3. The condition number inequality.

THEOREM 3.1. *If φ is a unitarily invariant norm, then*

$$c_\varphi(A) \leqq c_\varphi(AA^*) . \tag{c}$$

If Φ is a symmetric gauge function which determines φ, then we may rewrite (c) in the form

$$\Phi(\alpha_1, \cdots, \alpha_n)\Phi(\alpha_1^{-1}, \cdots, \alpha_n^{-1}) \leqq \Phi(\alpha_1^2, \cdots, \alpha_n^2)\Phi(\alpha_1^{-2}, \cdots, \alpha_n^{-2}) .$$

Thus, Theorem 3.1 is a very special case of

THEOREM 3.2. *If Φ is a symmetric gauge function, then $\Phi(\alpha_1^r, \cdots, \alpha_n^r)\Phi(\alpha_1^{-r}, \cdots, \alpha_n^{-r})$ is increasing in $r > 0$, where $\alpha_i > 0$.*

The proof of Theorem 3.2 is embodied in the lemmas below.

Following [2] we say $(a_1, \cdots, a_n)$ is *majorized* by $(b_1, \cdots, b_n)$, written (a) $\prec$ (b), if

(i) $a_1 \geqq \cdots \geqq a_n > 0,\ b_1 \geqq \cdots \geqq b_n > 0,$

(ii) $\sum_1^k a_i \leqq \sum_1^k b_i, \qquad k = 1, \cdots, n-1,$

(iii) $\sum_1^n a_i = \sum_1^n b_i .$

LEMMA 3.3. *If* (a) $\prec$ (b), *and Φ is a symmetric gauge function, then*

$$\Phi(a_1, \cdots, a_n) \leqq \Phi(b_1, \cdots, b_n) , \tag{3.1}$$

$$\Phi(a_1^{-1}, \cdots, a_n^{-1}) \leqq \Phi(b_1^{-1}, \cdots, b_n^{-1}) . \tag{3.2}$$

Proof. Proofs of (3.1) have been given by Fan [1] and Ostrowski

[3]; by an argument similar to that of Fan, we prove (3.2).

First, note that we can assume for h and j fixed, $h < j$,

$$(3.3)\qquad a_h = \alpha b_h + (1-\alpha)b_j,\ a_j = (1-\alpha)b_h + \alpha b_j,\ a_i = b_i,\ i \neq h, j\ .$$

That this is true follows from the fact that if (a) $\prec$ (b), then a can be derived from b by successive applications of a finite number of transformations of the form (3.3) (see [2, p. 47]).

Let $\tilde{b} = (b_1, \cdots, b_{h-1}, b_j, b_{h+1}, \cdots, b_{j-1}, b_h, b_{j+1}, \cdots, b_n)$, so that $\Phi(b_1, \cdots, b_n) = \Phi(\tilde{b}_1, \cdots, \tilde{b}_n)$. By convexity,

$$(\alpha b_i + (1-\alpha)\tilde{b}_i)^{-1} \leqq \alpha b_i^{-1} + (1-\alpha)\tilde{b}_i^{-1}\ .$$

Then using (2.1) and the convexity of Φ, it follows that

$$\begin{aligned}\Phi(a_1^{-1}, \cdots, a_n^{-1}) &= \Phi[(\alpha b_1 + (1-\alpha)\tilde{b}_1)^{-1}, \cdots, (\alpha b_n + (1-\alpha)\tilde{b}_n)^{-1}] \\ &\leqq \Phi(\alpha b_1^{-1} + (1-\alpha)\tilde{b}_1^{-1}, \cdots, \alpha b_n^{-1} + (1-\alpha)\tilde{b}_n^{-1}) \\ &\leqq \alpha\Phi(b_1^{-1}, \cdots, b_n^{-1}) + (1-\alpha)\Phi(\tilde{b}_1^{-1}, \cdots, \tilde{b}_n^{-1})\ . \quad ||\end{aligned}$$

As a consequence of Lemma 3.3., we have that if (a) $\prec$ (b) then

$$\Phi(a_1, \cdots, a_n)\Phi(a_1^{-1}, \cdots, a_n^{-1}) \leqq \Phi(b_1, \cdots, b_n)\Phi(b_1^{-1}, \cdots, b_n^{-1})\ .$$

The proof of Theorem 3.2 is completed by the following

LEMMA 3.4. *If* $\alpha_1 \geqq \cdots \geqq \alpha_n > 0$ *and* $a_i = \alpha_i^r/\Sigma\alpha_j^r$, $b_i = \alpha_i^s/\Sigma\alpha_j^s$, $0 < r < s$, *then* (a) $\prec$ (b).

Proof. We must show that for all k,

$$\frac{\sum_1^k \alpha_i^r}{\sum_1^n \alpha_i^r} \leqq \frac{\sum_1^k \alpha_i^s}{\sum_1^n \alpha_i^s}\ , \qquad r < s\ ,$$

which is true if and only if

$$\sum_1^k \alpha_i^s \sum_{k+1}^n \alpha_j^r - \sum_1^k \alpha_i^r \sum_{k+1}^n \alpha_j^s = \sum_{i=1}^k \alpha_i^r \sum_{j=k+1}^n \alpha_j^r(\alpha_i^{s-r} - \alpha_j^{s-r}) \geqq 0\ .$$

The latter follows from $\alpha_i \geqq \alpha_j$, $i < j$. ||

Observe that by (3.1) and Lemma 3.4, we have

$$\frac{\Phi(\alpha_1^r, \cdots, \alpha_n^r)}{\Phi(\alpha_1^s, \cdots, \alpha_n^s)} \leqq \frac{\Sigma\alpha_i^r}{\Sigma\alpha_i^s}\ .$$

In view of (2.2), it is perhaps natural to expect that

$$(3.4)\qquad \frac{\alpha_1^r}{\alpha_1^s} \leqq \frac{\Phi(\alpha_1^r, \cdots, \alpha_n^r)}{\Phi(\alpha_1^s, \cdots, \alpha_n^s)} \leqq \frac{\Sigma\alpha_i^r}{\Sigma\alpha_i^s}\ ,\ 0 < r < s,\ \alpha_1 \geqq \cdots \geqq \alpha_n > 0\ ,$$

for any symmetric gauge function Φ. To see this we need only prove the left hand inequality, which may be written in the form

$$\Phi\left(\left[\frac{\alpha_1}{\alpha_1}\right]^s, \cdots, \left[\frac{\alpha_n}{\alpha_1}\right]^s\right) \leqq \Phi\left(\left[\frac{\alpha_1}{\alpha_1}\right]^r, \cdots, \left[\frac{\alpha_n}{\alpha_1}\right]^r\right), \tag{3.5}$$

and which is a consequence of (2.1).

An interesting counterpart to Theorem 3.2 can be obtained from (3.4).

THEOREM 3.5. *If Φ is a symmetric gauge function satisfying* bIV, *then $[\Phi(\alpha_1^r, \cdots, \alpha_n^r)]^{1/r}$ is decreasing in $r > 0$ whenever $\alpha_i > 0$, $i = 1, 2, \cdots, n$. Thus $[\Phi(\alpha_1^r, \cdots, \alpha_n^r)\Phi(\alpha_1^{-r}, \cdots, \alpha_n^{-r})]^{1/r}$ is decreasing in $r > 0$.*

Proof. We have that

$$1 \leqq \Phi\left(\left[\frac{\alpha_1}{\alpha_1}\right]^s, \cdots, \left[\frac{\alpha_n}{\alpha_1}\right]^s\right) \leqq \Phi\left(\left[\frac{\alpha_1}{\alpha_1}\right]^r, \cdots, \left[\frac{\alpha_n}{\alpha_1}\right]^r\right),$$

the first inequality by bIV and (2.1). The second inequality is (3.5). Thus

$$\begin{aligned}\left\{\Phi\left(\left[\frac{\alpha_1}{\alpha_1}\right]^s, \cdots, \left[\frac{\alpha_n}{\alpha_1}\right]^s\right)\right\}^r &\leqq \left\{\Phi\left(\left[\frac{\alpha_1}{\alpha_1}\right]^r, \cdots, \left[\frac{\alpha_n}{\alpha_1}\right]^r\right)\right\}^r \\ &\leqq \left\{\Phi\left(\left[\frac{\alpha_1}{\alpha_1}\right]^r, \cdots, \left[\frac{\alpha_n}{\alpha_1}\right]^r\right)\right\}^s,\end{aligned}$$

so that

$$\left\{\Phi\left(\left[\frac{\alpha_1}{\alpha_1}\right]^s, \cdots, \left[\frac{\alpha_n}{\alpha_1}\right]^s\right)\right\}^{1/s} \leqq \left\{\Phi\left(\left[\frac{\alpha_1}{\alpha_1}\right]^r, \cdots, \left[\frac{\alpha_n}{\alpha_1}\right]^r\right)\right\}^{1/r}.$$

The theorem now follows from bII. ||

Theorem 3.5 can, of course, be specialized to yield a kind of converse to (c).

THEOREM 3.6. *If φ is a unitarily invariant norm satisfying* aIV, *then*

$$[c_\varphi(AA^*)]^{1/2} \leqq c_\varphi(A). \tag{c*}$$

Condition (c*) can also be obtained under somewhat different hypotheses. In particular, if φ satisfies aV, then

$$\begin{aligned}c_\varphi(AA^*) &= \varphi(AA^*)\varphi((AA^*)^{-1}) \\ &\leqq \varphi(A)\varphi(A^{-1})\varphi(A^*)\varphi(A^{*-1}) = c_\varphi(A)c_\varphi(A^*).\end{aligned}$$

If also $\varphi(A) = \varphi(A^*)$, then (c*) follows. Of course, $\varphi(A) = \varphi(A^*)$ if φ is unitarily invariant.

4. Independence of the norm axioms and (c). It is our purpose here to show that the condition number inequality (c) does not follow from the usual norm axioms aI – aV. In fact, aII, aIII, aIV, aV and (c) are independent.

REMARK. It has been shown by Ostrowski [4] that aI is implied by aII, aIII, aV, together with $\varphi(A) \not\equiv 0$, so that aI is not included in the list of independent properties. Rella [5] has shown that aII, aIII, aIV and aV are independent, and we add (c) to this list.

The results which prove the independence of aII – aV and (c) are summarized in the following table, where +(–) indicates that a property is true (false).

$\varphi(A)$	aII	aIII	aIV	aV	(c)
1	–	+	+	+	+
$(\text{rank } A)(\text{tr } AA^*)^{1/2}$	+	–	+	+	+
$n \max \mid a_{ij} \mid$	+	+	–	+	+
$\max \mid a_{ij} \mid$	+	+	+	–	+
$\Sigma \mid a_{ij} \mid$	+	+	+	+	–

An example which serves in the last line of the table just as well as $\Sigma \mid a_{ij} \mid$ is the norm $\max_i \sum_j \mid a_{ij} \mid = \sup_x \Phi(xA)/\Phi(x)$, where $\Phi(x) = \sum_i \mid x_i \mid$. Norms of this form are called "subordinate" or "lub" norms, and in this case Φ is a symmetric guage function.

The remainder of this paper is devoted to proving the propositions indicated in the table.

The results for $\varphi(A) \equiv 1$ are obvious, so we begin by considering $\varphi(A) = (\text{rank } A)(\text{tr } AA^*)^{1/2}$. In this case, aII and aIV are obvious, and (c) follows from Theorem 3.1, since $(\text{tr } AA^*)^{1/2}$ is unitarily invariant. As is well known, $(\text{tr } AA^*)^{1/2}$ satisfies aV; this together with rank $AB \leqq (\text{rank } A)(\text{rank } B)$ yields aV for $\varphi(A) = (\text{rank } A)(\text{tr } AA^*)^{1/2}$. That aIII is violated may be seen by taking $A = I$ and B the matrix with a unit in the (1, 1)th place and zeros elsewhere.

For $\varphi(A) = n \max_{i,j} \mid a_{ij} \mid$ and $\max_{i,j} \mid a_{ij} \mid$ the first four columns of the table are well known, and we need only prove (c). Let e_i be the row vector with one in the ith position and zero elsewhere. Denote $M^{-1} = (m^{ij})$ where $M = (m_{ij})$, and let $U = AA^*$. By Cauchy's inequality,

$$|a_{ij}||a^{\alpha\beta}| = |e_i A e_j^*||e_\alpha A^{-1} e_\beta^*| \leqq [(e_i U e_i^*)(e_j e_j^*)(e_\alpha e_\alpha^*)(e_\beta U^{-1} e_\beta^*)]^{1/2}$$
$$= (u_{ii} u^{\beta\beta})^{1/2} .$$

Hence,

$$\max_{i\ j} |a_{ij}| \max_{\alpha\ \beta} |a^{\alpha\beta}| \leqq (\max_i |u_{ii}| \max_\alpha |u^{\alpha\alpha}|)^{1/2} ,$$

or

$$c_\varphi(A) \leqq [c_\varphi(AA^*)]^{1/2} .$$

Since $U = AA^*$ is positive semi-definite,

$$u_{ii}u^{ii} = (e_i U e_i^*)(e_i U^{-1} e_i^*) \geqq (e_i e_i^*)^2 = 1 ,$$

and it follows that $c_\varphi(AA^*) \geqq 1$. Thus, we have that

$$c_\varphi(A) \leqq [c_\varphi(AA^*)]^{1/2} \leqq c_\varphi(AA^*) , \tag{4.1}$$

which gives (c).

Note that the left inequality of (4.1) is a reversal of inequality (c*). That (4.1) also holds if $\varphi(A)$ is the maximum of the absolute values of the characteristic values of A was proved by O. Taussky-Todd [6].

Since the first four columns of the table are well known for $\varphi(A) = \Sigma |a_{ij}|$, we again need consider only (c). If $A = \begin{pmatrix} B & 0 \\ 0 & 2I \end{pmatrix}$, where $B = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$. Then (c) is violated. This same example shows that (c) is violated for $\varphi(A) = \max_i \sum_j |a_{ij}|$.

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