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INFINITE SUMS IN ALGEBRAIC STRUCTURES

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The purpose of this note is an outline of an algebraic theory of summability in algebraic structures like abelian groups, ordered abelian groups, modules, and rings. "Infinite sums" of elements of these structures will be defined by means of homomorphisms satisfying some weak requirements of permanency which hold in all usual linear summability methods. It will turn out that several elementary well known theorems from the theory of infinite series, proved ordinarily by methods of analysis, (i.e. by use of some concept of a limit) are consequences of algebraic properties.

1. Definitions and existence theorems. Let G be an abelian group with a ring T operating from the left; we assume, without loss of generality, that T contains the integers. Denote by G^ω the strong direct sum of countably many copies of G, i.e., the set of all infinite sequences $s = (g_i)_{i=1}^\infty = (g_1, g_2, \cdots, g_i, \cdots)$ of elements of G, with the natural definitions of addition and of left multiplication by elements of T. Let Γ be the weak direct sum of countably many copies of G, i.e., the subgroup of G^ω consisting of all infinite sequences with at most a finite number of coordinates different from 0 (the neutral element of G). For $s=(g_1, g_2, \cdots, g_i, \cdots) \in G^\omega$, denote by s' the element $(0, g_1, g_2, \cdots, g_{i-1}, g_i, \cdots)$; s' will be called the translate of s.

DEFINITION 1. The T-subgroup S of G^ω will be called admissible if

$$\Gamma \subset S \tag{1}$$

and if

(2) $s \in S$ if and only if $s' \in S$, where s' is the translate of s.

Obviously, both Γ and G^ω are admissible, and any subset K of G^ω can be completed in a unique way to a minimal admissible subgroup containing K.

DEFINITION 2. Let S be admissible, and φ a T-homomorphism $S \to G$ with the following properties:

$$\varphi(g, 0, 0, \cdots) = g , \qquad (g \in G) \tag{3}$$

and

$$\varphi(s) = \varphi(s') , \qquad (s \in S) . \tag{4}$$

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Then φ will be called a *summation method on G with domain S*, and we shall refer to it briefly as the *summation method* $[S, \varphi]$.

Using the fact that S is admissible, and by properties (3) and (4) of the homomorphism φ, it follows immediately that

$$\varphi(g_1, g_2, \cdots, g_n, 0, 0, \cdots) = \sum_{i=1}^{n} g_i$$

for any summation method $[S, \varphi]$. Therefore, the (unique) summation method $[\Gamma, \varphi]$ shall be called *trivial*. Furthermore, by (3), φ is always a T-homomorphism *onto* G.

We ask first the following question:

When does there exist a summation method containing in its domain a given $s \in G^\omega$?

Denote by $s^{(n)}$ for any integer n the nth translate of s, i.e.

$$\begin{aligned} s^{(0)} &= s \\ s^{(n)} &= (s^{(n-1)})' = (\underset{n}{0, \cdots, 0}, g_1, g_2, \cdots) && \text{for } n > 0 \\ s^{(n)} &= (g_{-n+1}, g_{-n+2}, \cdots) && \text{for } n < 0 \,. \end{aligned}$$

Let S be the minimal admissible subgroup of G^ω containing s. The elements of S have the form

$$(5) \qquad \sum_{k=-m}^{n} t_k s^{(k)} + \gamma \,,$$

where m, n are nonnegative integers, the t_k are arbitrary elements of T, and γ is an element of Γ. This representation of an element of S is not necessarily unique, and a T-homomorphism $\varphi\colon S \to G$ has to be independent of it. But, since all summation methods agree on Γ, one has to answer first the question when an expression of type (5) will be in Γ. We may evidently assume $m = 0$ (changing s if necessary); hence we shall study linearly independent expressions of type

$$(6) \qquad l_i = \sum_{k=0}^{i} t_{ik} s^{(k)} = \gamma_i \in \Gamma \qquad (i = 0, 1, 2, \cdots)$$

where $t_{ik} \in T$. For each i, the coefficients t_{ii} appearing in (6) form a left ideal T_i of T, and $T_0 \subset T_1 \subset \cdots$

We now *assume that T satisfies an ascending chain condition*, so that each T_i is finitely generated, and that there exists an index m such that $T_m = T_{m+1} = \cdots$. Let t_{ii}^j $(j = 1, \cdots, n_i)$ be a system of generators of T_i $(i = 1, \cdots, m)$. Then a finite system of equations

$$l_i^j = \sum_{k=0}^{i} t_{ik}^j s^{(k)} = \gamma_i^j \in \Gamma , \qquad j = 1, \cdots, n_i; \quad i = 1, \cdots, m \tag{6'}$$

implies all equations of (6) in the sense that each l_i in (6) is a linear combination over T of the l_i and their translates.

A summation method φ on G with domain S exists if and only if $\varphi(s)$ satisfies all the equations

$$\left(\sum_{k=0}^{i} t_{ik}^j\right)\varphi(s) = \varphi(\gamma_i^j) , \qquad j = 1, \cdots, n_i; \quad i = 1, \cdots, m \tag{6''}$$

where the right side is independent of φ, since on Γ the homomorphism φ is the ordinary sum of finitely many elements of G. Once $\varphi(s)$ is determined it extends by linearity (over T) to all of S.

This may be generalized easily for any finite number of elements $s_1, s_2, \cdots, s_r$. Assume a summation method φ defined for the minimal admissible subgroup Γ_1 containing s_1. We can now obtain a finite system of relations of the type (6′), with s replaced by s_2, and Γ by Γ_1. This leads to a system of necessary and sufficient conditions for $\varphi(s_2)$ compatible with $\varphi(s_1)$, which is analogous to (6″) (the right side there being already defined by the previous step). Proceed by induction.

As a consequence we can prove the following existence theorem:

THEOREM 1. *For any abelian group $G \neq \{0\}$ with ring of operators T satisfying an ascending chain condition, there exists a nontrivial summation method.*

Proof. Let $g \in G$ be $\neq 0$. Define $s = (g_n)_{n=1}^{\infty}$ by

$$g_n = \begin{cases} g \text{ if } n = 2^k \\ 0 \text{ otherwise} . \end{cases}$$

Let S be the minimal admissible subgroup containing s, and $\bar{g}$ any element of G such that $tg = 0$ implies $t\bar{g} = 0$ for all $t \in T$ (for example $\bar{g} = g$). Then obviously the only relations of type (6) are of the form $ts = 0$ (because $tg = 0$), so that (6″) reduces to $t\varphi(s) = 0$ whenever $tg = 0$. These conditions are satisfied by setting $\varphi(s) = \bar{g}$.

REMARK 1. From the $2^{\aleph_0}$ sequences in G^{ω} whose elements are g or 0 one can pick a subset R, of power $2^{\aleph_0}$ so that any relation $\sum_{j=0}^{m}\sum_{i=1}^{n} t_{ij} r_i^{(j)} \in \Gamma$ for elements t_{ij} of T and $r_i \in R$ implies $t_{ij}g = 0$ for all t_{ij}. Thus we can define $2^{2^{\aleph_0}}$ different summation methods for the least admissible S which contains R by setting $\varphi(r)$ to be 0 or g arbitrarily for each $r \in R$, and then extending φ to all of S by linearity (over T).

On the other hand, in a nontrivial group no summation method can assign a sum to all the sequences of elements of the group.

THEOREM 2. *Let $G \neq 0$ be a T-group and $g_i \in G$, $(i = 1, \cdots, n)$ such that $\sum_{i=1}^{n} g_i \neq 0$. Then there exists no summation method defined for*

$$s = (g_1, g_2, \cdots, g_n, g_1, g_2, \cdots, g_n, g_1, \cdots) .$$

Proof. $s^{(n+1)} - s = (g_1, g_2, \cdots, g_n, 0, 0, \cdots)$ would lead to

$$\varphi(s^{(n+1)} - s) = \varphi(s^{(n+1)}) - \varphi(s) = \varphi(s) - \varphi(s) = 0 = g_1 + g_2 + \cdots + g_n ,$$

a contradiction.

THEOREM 3. *If $\varphi_1, \varphi_2, \cdots, \varphi_n$ are summation methods on G with domain S, and $e_1, e_2, \cdots, e_n$ are T-endomorphisms of G so that $e_1 + e_2 + \cdots + e_n = 1$, then $e_1\varphi_1 + e_2\varphi_2 + \cdots + e_n\varphi_n$ is a summation method on G with domain S.*

Proof. Let $\varphi = e_1\varphi_1 + e_2\varphi_2 + \cdots + e_n\varphi_n$. Then φ is obviously a T-homomorphism $S \to G$. Since $\varphi_i(s') = \varphi_i(s)$, the same is true for φ, and for a $g \in G$ we have $\varphi(g, 0, 0, \cdots) = g$.

THEOREM 4. *Let $[S_1, \varphi_1]$, $[S_2, \varphi_2]$ be two summation methods on G which agree on $D_0 = S_1 \cap S_2$. Then there is a summation method φ on G with domain $S = S_1 + S_2$, such that $\varphi \mid S_i = \varphi_i$ for $i = 1, 2$.*

Proof. The group S is evidently admissible. Denote $D_i = (S_i \backslash D_0) \cup \{0\}$, $i = 1, 2$. Then any $s \in S$ can be written (not necessarily uniquely)

$$s = d_0 + d_1 + d_2, \ d_i \in D_i, \ i = 0, 1, 2 . \tag{7}$$

Define φ by

$$\varphi(s) = \varphi_1(d_0) + \varphi_1(d_1) + \varphi_2(d_2) .$$

This definition is independent of the representation (7), since if $s = \bar{d}_0 + \bar{d}_1 + \bar{d}_2$ with $\bar{d}_i \in D_i$, then $A = \varphi(d_0 + d_1 + d_2) - \varphi(\bar{d}_0 + \bar{d}_1 + \bar{d}_2) = \varphi_1(d_0 - \bar{d}_0) + \varphi_1(d_1 - \bar{d}_1) + \varphi_2(d_2 - \bar{d}_2)$. The element $d_2 - \bar{d}_2$ is in S_2, but since $d_2 - \bar{d}_2 = \bar{d}_0 - d_0 + \bar{d}_1 - d_1$, it is in D_0, and therefore $\varphi_2(d_2 - \bar{d}_2) = \varphi_2(\bar{d}_0 - d_0 + \bar{d}_1 - d_1)$. Hence $A = \varphi_1(d_0 - \bar{d}_0) + \varphi_1(d_1 - \bar{d}_1) + \varphi_1(\bar{d}_0 - d_0 + \bar{d}_1 - d_1) = 0$. A similar reasoning is needed in order to show that $\varphi(s + \bar{s}) = \varphi(s) + \varphi(\bar{s})$ for s, $\bar{s} \in S$, since the sum of two representations of type (7) is generally not of the same type. Property (3) of φ is obvious, since $\Gamma \subset D_0$, and (4) follows easily, since (7) implies $s' = d'_0 + d'_1 + d'_2$, where $d'_i \in D_i$, $i = 0, 1, 2$. Since the decomposition (7) can be extended to ts, φ is a T-homomorphism, which finishes the proof.

REMARK 2. On the other hand, if $[S_1, \varphi_1]$ and $[S_2, \varphi_2]$ are summation methods which do not agree on $S_1 \cap S_2$, then there need not exist a summation method for the admissible subgroup $S_1 + S_2$. Take S_1 and φ_1 as S and φ in Theorem 1, and define $s_2 = (g_n^*)_{n=1}^{\infty}$ by

$$g_n^* = \begin{cases} 0 \text{ if } n = 2^k \\ g \text{ otherwise .} \end{cases}$$

Again, if $\bar{\bar{g}}$ is any element of G such that $tg = 0$ implies $t\bar{\bar{g}} = 0$ for any $t \in T$, then $\varphi_2(s_2) = \bar{\bar{g}}$ is a valid definition that can be extended to a summation method on the minimal admissible subgroup S_2 containing s_2. But $S_1 + S_2$ can not be the domain of any summation method, since it contains the element $(g, g, g, \cdots)$, in contradiction to the construction in Theorem 2.

REMARK 3. Let $(G_\alpha)_{\alpha \in A}$, where A is a set of indices, be a family of abelian groups with operators T; assume that S_α is an admissible subgroup of G_α^ω and that φ_α is a summation method on G_α with domain S_α for each $\alpha \in A$. Consider the (weak or strong) direct sum $G = \bigoplus_{\alpha \in A} G_\alpha$. Then it is easily shown that $S = \bigoplus_{\alpha \in A} S_\alpha$ is admissible for G, and that $\varphi = (\varphi_\alpha)_{\alpha \in A}$ is a summation method with domain S on G. It is clear that $[S, \varphi]$ is nontrivial if and only if at least one of the summation methods $[S_\alpha, \varphi_\alpha]$ is nontrivial.

2. Subgroups and ideals. To each subgroup H of G we associate the (left) annihilator ideal T_H of T consisting of all $t \in T$ such that $tH = 0$. If H is a T-subgroup of G, then T_H is a two-sided ideal, since $0 = t_H(tH) = (t_H t)H$ for every $t_H \in T_H$ and $t \in T$. Clearly $T_{H^\omega} = T_H$.

Let $[S, \varphi]$ be a summation method on G, and let H be a T-subgroup of G. Then $\varphi(S \cap H^\omega) = H_1$ is a T-subgroup of G which contains H. We call this group the *$[S, \varphi]$-extension of H*. It is easy to see that *if H_1 is an $[S, \varphi]$-extension of H, then $T_{H_1} = T_H$*; since $H_1 \supset H$, we obviously have $T_{H_1} \subset T_H$. On the other hand, $T_{H_1} \supset T_{H^\omega} = T_H$. From this, it follows:

THEOREM 5. *If H is a maximal T-subgroup for the annihilator ideal T_H, then H has no proper $[S, \varphi]$-extensions.*

THEOREM 6. *Let H_1 be a denumerable T-subgroup of G, and*

$H_2 > H_1$ a T-subgroup of G of cardinality not greater than $2^{\aleph_0}$ such that $T_{H_1} = T_{H_2}$. Then there is a summation method $[S, \varphi]$ on G so that H_2 is the $[S, \varphi]$-extension of H_1.

Proof. Let $\{h_1, h_2, \cdots\}$ be an enumeration of H_1, and let M be an increasing sequence of integers. Define sequences $s_{M,i} = (g_{n,i})_{n=1}^{\infty}$ by

$$g_{n,i} = \begin{cases} h_i \text{ if } n = 2^{p_i^m},\ m \in M,\ p_i = i\text{th prime} \\ 0 \text{ otherwise}\ . \end{cases}$$

It is easy to find (see Remark 1) a set $\mathfrak{M}$ of $2^{\aleph_0}$ sequences M such that any relation of the form $\sum_{r,j} t_{rj} s_{M}^{(r)}{}_j \in \Gamma$ implies $t_{ij} s_{M,j}^{(r)} = 0$ for all r and j, which in turn implies that $t_{rj} \in T_{H_1}$. Now, let $\{h_\alpha^{(2)}\}_{\alpha \in A}$ be a minimal system of generators of H_2, that is $\sum_\alpha t_\alpha h_\alpha^{(2)} = 0$ (finite sum) if and only if $t_\alpha h_\alpha = 0$ for all α. For any choice of the subsystem M_α of $\mathfrak{M}$ the definition $(s_{M,\alpha}) = h_\alpha^{(2)}$ for $\alpha \in A$ yields a summation method on the minimal admissible subgroup S of G^ω containing all the s_{M_α}.

REMARK 4. The restrictions on the cardinalities of H_1 and H_2 can be removed if we allow summation methods using, instead of G^ω, the strong direct sum G^ξ, where ξ is an arbitrary infinite ordinal.

EXAMPLE 1. Let G be a finite abelian group, and T the ring of integers modulo the minimal annihilator N of G. To each subgroup H of G corresponds the ideal generated by its minimal annihilator. Clearly, to every divisor D of N, there corresponds a unique maximal subgroup H_D of G with minimal annihilator D. Each subgroup of G can be $[S, \varphi]$-extended to exactly one H_D.

EXAMPLE 2. If G is the additive group of a ring R considered as the ring of operators T on G, then T-subgroups of G are the left ideals of R. Given now a subset $M \subset R$, it determines a left annihilator ideal T_M of M. Any finitely generated left ideal containing M whose annihilator is T_M can be represented as an $[S, \varphi]$-extension of the left ideal generated by M.

3. Ordered groups. Let G be an abelian group with a partial ordering relation $\geqq$ satisfying: (1) there is a semigroup $H \subset G$ containing the zero element and at least one element $\neq 0$, in which the binary reflexive and transitive relation $\geqq$ is defined; (2) if $h, h_1 \in H$ and $h > 0$, then $h_1 + h > h_1$; (3) the archimedean axiom: if $h_1, h_2 \in H$, $h_1 > 0$ and $h_2 > 0$, then there is a positive integer n such that $nh_1 > h_2$.

DEFINITION 3. Let G be a partially ordered abelian group. $s = (g_1, g_2, \cdots, g_n, \cdots) \in G^\omega$ will be called *positive* if $g_n \in H$ and $g_n \geqq 0$ for

$n = 1, 2, \cdots$, and if $g_{n_0} > 0$ for at least one index n_0. A summation method $[S, \varphi]$ will be called *positive* if $s \in S$ and s positive imply $\varphi(s) > 0$.

The positive elements of G^ω or of S evidently form a semigroup. Furthermore, if s is positive, so is its translate s'.

THEOREM 7. *Let G be a partially ordered abelian group, and $[S, \varphi]$ a positive summation method on G. If $s = (g_1, g_2, \cdots, g_n, \cdots) \in G$ is such that $g_{k_n} \geqq g > 0$ for infinitely many indices k_n, then $s \notin S$.*

Proof. The hypothesis implies that s is a positive element. Assume $s \in S$ and $\varphi(s) = \gamma$, then $0 > \gamma = \varphi(s) = \varphi(g_1, g_2, \cdots, g_{k_n}, 0, 0, \cdots) + \varphi(0, \cdots, 0, g_{k_n}, g_{k_n+1}, \cdots) = \sum_{i=1}^{k_n} g_i + \varphi(0, \cdots, 0, g_{k_n}, g_{k_n+1}, \cdots) > ng$ for each positive integer n. This contradicts the archimedean axiom.

COROLLARY 7.1. *There is no positive nontrivial summation method for the group of integers with their natural ordering.*

COROLLARY 7.2. *Let G be an abelian group with a linear ordering, and $[S, \varphi]$ a positive summation method on G. If $s = (g_1, g_2, \cdots, g_n, \cdots) \in S$ is positive, then* g.l.b $g_n = 0$ *and* $\varphi(s) \geqq \text{l.u.b.}_{1 \leqq n < \infty} \sum_{i=1}^{n} g_i$.

From the last part of Corollary 7.2 it follows that if the partial sums of a "series" with positive terms are unbounded, then the "series" does not belong to the domain of any positive summation method.

THEOREM. 8. *Let G be a linearly ordered abelian group. Then there is a nontrivial positive summation method on G if and only if G contains an infinite sequence $g_1, g_2, \cdots$, of positive elements and an element g, such that $g_1 + \cdots + g_n \leqq g$ for all n.*

Proof. The necessity follows immediately from Corollary 7.2. To prove sufficiency, set $s = (s_n)_{n=1}^{\infty}$ and define

$$s_n = \begin{cases} g_k \text{ for } n = 2^k \\ 0 \text{ otherwise .} \end{cases}$$

Then the least admissible S which contains s has elements which can be expressed uniquely in the form

$$t = \gamma + \sum_{i=0}^{n} a_i s^{(i)}$$

where the a_i are integers and $\gamma = (\gamma_1, \gamma_2, \cdots, \gamma_m, 0, 0, \cdots) \in \Gamma$. $t \geqq 0$ implies $a_i \geqq 0$ and $\sum_{j=1}^m \gamma_j > -(\sum_{i=0}^n a_i)g$. Thus if we define $\varphi(s) = g$ we obtain

$$\varphi(t) = \sum_{j=1}^m \gamma_i + \left(\sum_{i=0}^n a_i\right)g$$

where $\varphi(t) \geqq 0$ whenever $t \geqq 0$, and φ can be extended in an obvious way to a summation method, and is nontrivial.

THEOREM 9. *Let G be a linearly ordered abelian group and $[S, \varphi]$ a positive summation mothod such that S contains all the positive elements $s = (g_i)_{i=1}^\infty \in G$ for which the "partial sums" $\sum_{i=1}^n g_i$ are bounded for all n. Then $\varphi(s) = \text{l.u.b.}_{1 \leqq n < \infty} \sum_{i=1}^n g_i$ for any positive $s \in S$.*

Proof. By Corollary 7.2 we know that $\varphi(s) \geqq \bar{g} = \text{l.u.b.}_n \sum_{i=1}^n g_i$. Assume $\varphi(s) > \bar{g}$. Then $\varphi(0, \cdots, 0, g_N, g_{N+1}, \cdots) \geqq \varphi(s) - \bar{g} > 0$ for any N, and $g_N + g_{N+1} + \cdots + g_{N+k} < \bar{g}$ for all k. It follows that there is a greatest positive integer n_1 such that $(2n_1)(g_1 + \cdots + g_k) < \bar{g}$ for all k. Determine n_2 as greatest positive integer such that $(2n_2)(g_2 + \cdots + g_k) < \bar{g} - n_1 g$ for all k, etc. This defines a nondecreasing sequence of positive integers n_j with $n_j \to \infty$. Consider the element $\bar{s} = (n_j g_j)_{j=1}^\infty \in G^\omega$. It is obviously in S, since the partial sums $\sum_{j=1}^r n_j g_j$ are bounded for all r. On the other hand

$$\varphi(\bar{s}) > n_j(\varphi(s) - \bar{g})$$

for all j, which is in contradiction with the archimedean property of the order in G.

4. Limits.

DEFINITION 4. Let $[S, \varphi]$ be a summation method on the abelian group G. The sequence $\{g_1, g_2, \cdots, g_n, \cdots\}$ of elements of G will be called *$[S, \varphi]$-convergent to g* (notation: $g = \lim_{[S\ \varphi]} g_n$, or $g_n \xrightarrow[{[S,\varphi]}]{} g$) if (1) $s = (g_n - g_{n-1})_{n=1}^\infty \in S$, and (2) $\varphi(s) = g$. (Here $g_0 = 0$.)

The following properties are immediate:

THEOREM 10. (1) *The sequence $\{g, g, g, \cdots\}$ is $[S, \varphi]$-convergent to g for any $[S, \varphi]$.* (2) *If $g_n \xrightarrow[{[S,\varphi]}]{} g$ and $\bar{g}_n \xrightarrow[{[S,\varphi]}]{} \bar{g}$ then $g_n + \bar{g}_n \xrightarrow[{[S,\varphi]}]{} g + \bar{g}$.* (3) $\lim_{[S\ \varphi]}(-g_n) = -\lim_{[S\ \varphi]} g_n$. (4) *If $g = \lim_{[S,\varphi]} g_n$ and $h_1, h_2, \cdots, h_k$ are arbitrary elements of G, then the sequence $\{h_1, h_2, \cdots, h_k, g_1, g_2, \cdots, g_n, \cdots\}$ is $[S, \varphi]$-convergent to g.*

The last part of Theorem 10 implies that if $\lim_{[S,\varphi]} g_n = g$, then

$\{g_k, g_{k+1}, \cdots\}$ is $[S, \varphi]$-convergent to g, too.

An arbitrary subsequence of an $[S, \varphi]$-convergent sequence will not always be $[S, \varphi]$-convergent to the same limit, even if it is $[S, \varphi]$-convergent.

EXAMPLE 3. Let G be an abelian group with an element g of order > 2. Define S to be the minimal admissible subgroup of G^ω containing the element

$$s = (2g, -2g, 2g, -2g, \cdots) .$$

Since $s' + s = (2g, 0, 0, \cdots)$ we may define $\varphi(s) = g$. Then the sequence $\{2g, 0, 2g, 0, \cdots\}$ is $[S, \varphi]$-convergent to g, but the subsequence $\{2g, 2g, \cdots\}$ is $[S, \varphi]$-convergent to $2g$.

This example shows that it is not always possible to define a topology in G by means of $[S, \varphi]$-convergent sequences.

THEOREM 11. *Let G be an abelian group. A non-trivial summation method $[S, \varphi]$ on G, with the property that every subsequence of any $[S, \varphi]$-convergent sequence is $[S, \varphi]$-convergent to the same limit, exists if and only if G is infinite.*

Proof. Let G be finite. If a sequence of elements of G is not eventually constant, then two different elements must occur infinitely often. Hence no summation method $[S, \varphi]$ with the required property is possible.

Assume G infinite, and distinguish among the following cases:

(a) *G contains an element g of infinite order.* Let S be the minimal admissible subgroup of G^ω containing all the sequences $(n_i g)_{i=1}^\infty$ such that $\sum n_i$ converges p-adically to a rational integer n. Define then

$$\varphi((n_i g)_{i=1}^\infty) = ng .$$

(b) *There exists an element $g \neq 0$ of G of finite order divisible by arbitrarily high powers of some prime p.* Let M be the subgroup of the additive group of rationals, containing all the sequences $(p^{-k/n}a_n)_{n=1}^\infty$ where a_n and k_n are integers, such that $\sum_{n=1}^\infty p^{-k/n}a_n$ converges to a number of the form $p^{-k}a$, a and k integers. Let S be the minimal admissible subgroup of G^ω that contains the sequence $(p^{-k_n}a_n g)_{n=1}^\infty$, and define $\varphi((p^{-k_n}a_n g)_{n=1}^\infty) = p^{-k}ag$.

(c) *All elements of G are finite but not of bounded order, and no element of G is infinitely divisible* (by powers of some prime).

Define $G_n = n!G$; let S be the minimal admissible subgroup of G^ω consisting of the sequences $(g_n)_{n=1}^\infty$ so that there exists a g in G with $g - g_1 - g_2 - \cdots - g_n \in G_n$ for $n = 1, 2, \cdots$. Define $\varphi((g_n)_{n=1}^\infty) = g$.

(d) *All elements of G have bounded order $\leqq m$.* Then G must contain an infinite subgroup, all of whose elements have order p for some fixed prime p. Otherwise there would be a least divisor d of m for which there is an infinite subgroup G_1 of G such that $dG_1 = 0$. If d is composite, then for every prime divisor q of d the group qG_1 is finite, and hence the kernel of the homomorphism $G_1 \rightarrow qG_1$ is an infinite group G_2 with $qG_2 = 0$, contrary to the hypothesis.

Now, an infinite abelian group all of whose elements are of order p is the direct sum of infinitely many cyclic groups of order p, say $Z_1^{(p)} \oplus Z_2^{(p)} \oplus \cdots$. Let S be the minimal admissible subgroup of G^ω containing the sequences $(g_n)_{n=1}^\infty$ for which there exists a $g \in G$ such that $g - g_1 - \cdots - g_n \in Z_{n+1}^{(p)} \oplus Z_{n+2}^{(p)} \oplus \cdots$, and define $\varphi((g_n)_{n=1}^\infty) = g$.

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