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ON SIMPLE EXTENDED LIE ALGEBRAS OVER FIELDS OF CHARACTERISTIC ZERO

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In this paper we shall investigate algebras which generalize Lie algebras, Malcev algebras and binary-Lie algebras (every two elements generate a Lie subalgebra). Such an algebra A is called an *extented Lie algebra* (briefly *el-algebra*) and is defined by

$$xy = -yx \qquad \text{and} \qquad J(x, y, xy) = 0$$

for all x, y in A where $J(x, y, z) = xy \cdot z + yz \cdot x + zx \cdot y$. We prove the following.

THEOREM. **Let A be a simple finite dimensional el-algebra over an algebraically closed field of characteristic zero, then A is a simple Lie algebra or the simple seven dimensional Malcev algebra if and only if the trace form, $(x, y) = \text{trace } R_xR_y$, is a nondegenerate invariant form.**

The identities for right multiplications in the Lie [1] and Malcev [2] algebras mentioned in the theorem yield that the form (x, y) is nondegenerate and invariant i.e. $(xy, z) = (x, yz)$; so this paper is concerned with the converse statement. All algeras considered in this paper are finite dimensional.

2. Precartan subalgebras.

In this section we shall consider subalgebras of an arbitrary anti-commutative algebra analogous to Cartan subalgebras of a Lie algebra.

DEFINITION. A subalgebra N of an anti-commutative algebra A is a *precartan subalgebra* if

1. N is a nilpotent Lie subalgebra of A;
2. the mapping $N \to E_L(A) : n \to R_n$ is a representation of N where R_x or $R(x)$ denotes the mapping $a \to ax$ and $E_L(A)$ denotes the Lie algebra of all linear transformations on A.

Thus N is a Lie subalgebra of A such that there exists an integer k with $(\cdots(n_1n_2)\cdots n_k) = 0$ for all n_i in N and $[R_n, R_m] = R_{nm}$ for all n, m in N.

Now $R(N) = \{R_n : n \in N\}$ is a Lie algebra of linear transformations which is also a nilpotent Lie algebra of linear transformations:

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$$0 = R[(n_1 n_2) \cdots) n_k)] = [R((n_1 n_2) \cdots) n_{k-1}), R(n_k)]$$
$$= \cdots\cdot = [\cdots [R(n_1), R(n_2)], \cdots, R(n_k)] .$$

So if we assume the base field F is algebraically closed we can, analogous to Lie algebra theory, decompose A into a direct sum of weight spaces relative to $R(N)$:

$$A = A(N, 0) \oplus \sum_{\alpha \neq 0} A(N, \alpha)$$

where if $\lambda: R(N) \to F : R_n \to \lambda(R_n) \equiv \lambda(n)$ is a weight of $R(N)$, then the weight space corresponding to λ is

$$A(N, \lambda) = \{x \text{ in } A\text{: all } n \text{ in } N,\ x(R_n - \lambda(n)I)^t = 0$$
$$\text{for some integer } t > 0\} = \bigcap_{n \in N} A(R_n, \lambda(n))$$

where $A(R_n, \lambda(n)) = \{x \in A: x(R_n - \lambda(n)I)^m = 0$ for some integer $m > 0\}$. By Lie's theorem the weights λ are linear functionals on $R(N)$ and can actually be considered as linear functionals on N via $\lambda(n) \equiv \lambda(R_n)$ and noting R_n is a linear transformation. The following facts concerning the above decomposition are known from Lie algebra theory [1].

THEOREM 2.1.
1. *Precartan subalgebras exist.*
2. *The weight spaces $A(N, \lambda)$ are $R(N)$-invariant subspaces.*
3. *λ is the only weight of $R(N)$ in $A(N, \lambda)$.*
4. *If M is a precartan subalgebra of A containing N, then $M \subset A(N, 0)$.*
5. *There exists an element n in N such that $A(N, 0) = A(R_n, 0)$.*
6. *There are finitely many weights.*

DEFINITION. The *normalizer* $Z(B)$ of a subalgebra B of an anticommutative algebra A is the set of all x in A such that $xB \subset B$.

PROPOSITION 2.2. Let N be a precartan subalgebra of an anticommutative algebra A over an algebraically closed field, then $N = Z(N)$ if and only if $N = A(N, 0)$.

Proof. First note $Z(N) \subset A(N, 0)$. For if $x \in Z(N)$, then for all $n \in N$, $xR_n = xn \in N$ and since N is a nilpotent Lie subalgebra of A, there exists an integer k with $xR_n^k = 0$ and so $x \in A(N, 0)$. Thus noting that $N \subset Z(N)$ we always have

$$N \subset Z(N) \subset A(N, 0) .$$

In particular if $N \neq Z(N)$, then $N \neq A(N, 0)$.

Next assume $N \neq A(N, 0)$, then we shall show $N \neq Z(N)$. $A(N, 0)$ and N are $R(N)$-invariant subspaces and $R(N)$ restricted to $A(N, 0)$ is a nilpotent Lie algebra of linear transformations. Hence we obtain a nilpotent Lie algebra of linear transformations $\overline{R(N)}$ acting in the nonzero space $A(N, 0)/N$. Now by Engel's theorem there exists an $\bar{x} = x + N \neq \bar{0}$ in $A(N, 0)/N$ such that $\overline{xR(N)} = \bar{0}$, But this means $\bar{0} = \bar{x}\bar{R}_n = \overline{xR_n}$ for all n in N and so $xN \subset N$. But by definition of $Z(N)$, $x \in Z(N)$. However $\bar{x} \neq 0$ means x is not in N and so $N \neq Z(N)$.

DEFINITION. A precartan subalgebra N of an anti-commutative algebra A is a *Cartan subalgebra* of A if $N = Z(N)$.

Cartan subalgebras are difficult to find, however if there exists an element u in A such that $N = A(R_u, 0)$ is a precartan subalgebra of A over an algebraically closed field, then N is a Cartan subalgebra. For decompose A relative to $R(N)$, then $N \subset A(N, 0) = \cap_{n \in N} A(R_n, 0) \subset A(R_u, 0) = N$ and the results follow from Proposition 2.2.

The following notation will be used: if $h(x_1, \cdots, x_n)$ is a function of n indeterminates such that for any n subsets B_i of A the elements $h(b_1, \cdots, b_n)$, for $b_i \in B_i$, are in A, then $h(B_1, \cdots, B_n)$ denotes the linear subspace spanned by all the elements $h(b_1, \cdots, b_n)$ for $b_i \in B_i$. Also we shall identity the element b and the set $\{b\}$.

3. **Identities.** We shall now restrict ourselves to el-algebras, that is, we assume the algebra A satisfies

$$xy = -yx \quad \text{and} \quad J(x, y, xy) = 0 \tag{3.1}$$

for all x, y in A. First we note for any anti-commutative algebra A that a straightforward calculation yields

$$\begin{aligned} wJ(x, y, z) - xJ(y, z, w) + yJ(z, w, x) - zJ(w, x, y) &= J(wx, y, z) + J(yz, w, x) \\ &+ J(wy, z, x) + J(zx, w, y) \\ &+ J(wz, x, y) + J(xy, w, z) \end{aligned} \tag{3.2}$$

for all w, x, y, z in A.

Next a linearization of (3.1) yields

$$J(xy, y, z) + J(zy, y, x) = 0 \tag{3.3}$$

for all x, y, z in A and a further linearization of (3.3) yields

$$\begin{aligned} J(wx, y, z) + J(yz, w, x) &= J(wy, z, x) + J(zx, w, y) \\ &= J(wz, x, y) + J(xy, w, z) \end{aligned} \tag{3.4}$$

for all w, x, y, z in A. Combining (3.2) and (3.4) we have

$$\begin{aligned} wJ(x, y, z) - xJ(y, z, w) + yJ(z, w, x) - zJ(w, x, y) \\ = 3[J(wx, y, z) + J(yz, w, x)] \end{aligned} \tag{3.5}$$

for all w, x, y, z in A.

For the remainder of this paper we shall assume A is a simple, finite dimensional el-algebra over an algebraically closed field of characteristic zero; however it should be clear when these various conditions can be relaxed. Let N be a precartan subalgebra of A and decompose

$$A = A(N, 0) \oplus \sum_{\lambda \neq 0} A(N, \lambda)$$

relative to $R(N)$. Next let $m, n \in N$ and $w, z \in A$, then from (3.4) we have

$$\begin{aligned} J(wn, m, z) + J(mz, w, n) &= J(wm, z, n) + J(zn, w, m) \\ &= J(wz, n, m) + J(nm, w, z) \\ &= J(nm, w, z) \end{aligned}$$

since $m, n \in N$ and $J(A, n, m) = 0$. Now set $w_1 = wR_n$ and $w_k = w_{k-1}R_n = wR_n^k$ and set $z_1 = zR_m, z_k = z_{k-1}R_n + zR(mR_n^{k-1})$ for $k = 2, 3, \cdots$. From the above set of equations we have

$$J(wR_n, m, z) = J(w, mR_n, z) + J(w, n, zR_m) , \tag{3.6}$$

that is,

$$J(w_1, m, z) = J(w, mR_n, z) + J(w, n, z_1) .$$

Now assume

$$J(w_k, m, z) = J(w, mR_n^k, z) + J(w, n, z_k) ,$$

then for $k + 1$ we have

$$\begin{aligned} J(w_{k+1}, m, z) &= J(vR_n^k, m, z) , \quad \text{where } v = wR_n \\ &= J(v_k, m, z) , \quad \text{where } v_k = vR_n^k \\ &= J(v, mR_n^k, z) + J(v, n, z_k) , \text{ using induction hypothesis} \\ &= J(wR_n, mR_n^k, z) + J(wR_n, n, z_k) \\ &= J(w, (mR_n^k)R_n, z) + J(w, n, zR(mR_n^k)) \\ &\quad + J(w, nR_n, z_k) + J(w, n, z_kR_n) , \quad \text{using (3.6) twice} \\ &= J(w, mR_n^{k+1}, z) + J(w, n, z_{k+1}) . \end{aligned}$$

Thus for every $m, n \in N$ and $w, z \in A$,

$$J(wR_n^k, m, z) = J(w, mR_n^k, z) + J(w, n, z_k) \tag{3.7}$$

where $z_1 = zR_n$ and $z_k = z_{k-1}R_n + zR(mR_n^k)\ k = 2, 3, \cdots$.

Let l be the dimension of $A(N, 0)$, then choosing k large enough e.g. $k \geqq l + 2$ and using $N \subset A(N, 0)$ we have $mR_n^k = 0$ and

$$J(w_k, m, z) = J(w, n, z_k) \tag{3.8}$$

where $k \geqq l + 2$, $w_k = wR_n^k$ and now $z_k = z_{k-1}R_n$.

We shall use (3.8) with the following lemmas to prove

$$A(N, \sigma)A(N, \rho) \subset A(N, \rho + \sigma) \text{ if } \rho \neq \sigma . \tag{3.9}$$

LEMMA 3.10. *Let ρ be any weight of N and $n \in N$, then R_n is nonsingular on $A(N, \rho)$ if and only if $\rho(n) \neq 0$.*

LEMMA 3.11. *Let ρ, σ be any nonzero weights of N in A with $\rho \neq \sigma$, then there exists $u \in N$ such that $\rho(u) \neq \sigma(u)$ and R_u is nonsingular on $A(N, \rho)$.*

Proof. Since $\rho \neq \sigma$, there exists $h \in N$ with $\rho(h) \neq \sigma(h)$. If R_h is nonsingular on $A(N, \rho)$, we are finished. Otherwise R_h is singular on $A(N, \rho)$ and by Lemma 3.10, $0 = \rho(h) \neq \sigma(h)$. Now there exists $k \in N$ so that R_k is nonsingular on $A(N, \rho)$, since $\rho \neq 0$ and so $\rho(k) \neq 0$. If $\sigma(k) \neq \rho(k)$, then we are finished. Otherwise if $\sigma(k) = \rho(k)$, set $u = h + k$ and note $\sigma(u) = \sigma(h) + \sigma(k) = \sigma(h) + \rho(k) \neq \rho(k) = \rho(k) + \rho(h) = \rho(h + k) = \rho(u)$ and also $\rho(u) \neq 0$ so that R_u is nonsingular on $A(N, \rho)$.

For the proof of (3.9) first consider the case $\sigma = 0$ and $\rho \neq 0$. Let $z \in A(N, 0)$, $w \in A(N, \rho)$ and $m, n \in N$, then we see by definition that $z_k \in A(N, 0)$ for all k. But for $k \geq l + 2$ (where $l =$ dimension $A(N, 0)$) we see $z_k = z_{k-1}R_n$ and this implies $z_{l+2+k} = z_{k-1}R_n^{l+3}$ and so for large enough M, $z_k = 0$ for all $k \geqq M$. This and (3.8) imply $J(w_k, m, z) = 0$ for all $k \geqq M$. But since $w_k = wR_n^k$ where $w \in A(N, \rho)$ with $\rho \neq 0$, there exists $n \in N$ with R_n nonsingular on $A(N, \rho)$ and therefore $A(N, \rho) = A(N, \rho)R_n^k$. Thus any $x \in A(N, \rho)$ is of the form $x = wR_n^k$ for some $w \in A(N, \rho)$ and therefore $J(x, m, z) = 0$. But m and z are arbitrary in N and $A(N, 0)$ respectively which implies

$$J(A(N, \rho), A(N, 0), N) = 0 . \tag{3.12}$$

Next consider the case $0 \neq \sigma \neq \rho \neq 0$ and let $w \in A(N, \rho)$, $z \in A(N, \sigma)$ and $m, n \in N$. Then from (3.8) we have for k large enough,

$$\begin{aligned} J(wR_n^k, m, z) &= J(w_k, m, z) \\ &= J(w, n, z_k), \text{ where } z_k = z_{k-1}R_n \\ &= J(w, n, z_{k-1}n) \\ &= -J(z_{k-1}n, n, w) \\ &= J(wn, n, z_{k-1}), \text{ using (3.3)} . \end{aligned} \tag{3.13}$$

Therefore for $x = z_{k-1} \in A(N, \sigma)$ we have from (3.13)

$$J(w, n, x(R_n - \sigma(n)I)) = J(w(R_n - \sigma(n)I), n, x)$$

and by induction

$$J(w, n, x(R_n - \sigma(n)I)^t) = J(w(R_n - \sigma(n)I)^t, n, x) .$$

But since $x \in A(N, \sigma)$, the left side of the above equation is zero for large enough t and so $J(w(R_n - \sigma(n)I)^t, n, x) = 0$ for all $n \in N$. Now choose $n = u$ of Lemma 3.11, then since $\sigma(u) \neq \rho(u)$, $R_u - \sigma(u)I$ is nonsingular on $A(N, \rho)$ and therefore for any integer $t > 0$, $A(N, \rho) = A(N, \rho)(R_u - \sigma(u)I)^t$. Thus any $v \in A(N, \rho)$ is of the form $v = w(R_u - \sigma(u)I)^t$ and therefore $J(v, u, x) = 0$ where x and u are defined above. Thus using the preceding notaion and (3.13) we have

$$\begin{aligned} 0 &= J(vu, u, x), \text{ since } vu \in A(N, \rho) \\ &= J(vu, u, z_{k-1}) \\ &= J(v, u, z_{k-1}u) \\ &= J(vR_u^k, m, z) . \end{aligned}$$

But also from Lemma 3.11, R_u is nonsingular on $A(N, \rho)$ so that $A(N, \rho) = A(N, \rho)R_u^k$. Thus by the choise of z, m, v we have

$$(3.14) \qquad J(A(N, \rho), A(N, \sigma), N) = 0 \text{ if } 0 \neq \sigma \neq \rho \neq 0 .$$

Using the usual Lie algebra arguments we combine (3.12) and (3.14) to obtain

$$(3.9) \qquad A(N, \rho)A(N, \sigma) \subset A(N, \rho + \sigma) \text{ if } \rho \neq \sigma .$$

Next we prove

$$(3.15) \qquad J(A(N, \rho), A(N, \sigma), A(N, \tau)) = 0 \text{ if } \rho \neq \sigma \neq \tau \neq \rho .$$

For let $n \in N$, $x \in A(N, \rho)$, $y \in A(N, \sigma)$ and $z \in A(N, \tau)$, then assuming first that $\rho \neq \sigma + \tau$ we have, using (3.5) and (3.12) or (3.14),

$$\begin{aligned} nJ(x, y, z) &= 3[J(nx, y, z) + J(yz, n, x)] \\ &= 3J(nx, y, z) . \end{aligned}$$

Therefore $J(x, y, z)(R_n - 3\rho(n)I) = 3J(x(R_n - \rho(n)I), y, z)$ and we proceed by induction to conclude

$$J(x, y, z)(R_n - 3\rho(n)I)^t = 0 \text{ for large enough } t .$$

Thus with $\rho \neq \sigma \neq \tau \neq \rho$ we conclude

$$J(A(N, \rho), A(N, \sigma), A(N, \tau)) \subset A(N, 3\rho) \text{ if } \rho \neq \sigma + \tau .$$

By the symmetry of the ρ, σ and τ we also conclude

$$\begin{aligned} &J(A(N, \rho), A(N, \sigma), A(N, \tau)) \subset A(N, 3\sigma) \text{ if } \sigma \neq \rho + \tau \text{ and} \\ &J(A(N, \rho), A(N, \sigma), A(N, \tau)) \subset A(N, 3\tau) \text{ if } \tau \neq \rho + \sigma . \end{aligned}$$

Now suppose $\rho = \sigma + \tau$. If $\sigma = \rho + \tau$, then $\tau = 0$ and therefore $\sigma = \rho$, a contradiction; thus $\sigma \neq \rho + \tau$. Similarly we have $\tau \neq \rho + \sigma$ and from these we may conclude

$$J(A(N, \rho), A(N, \sigma), A(N, \tau)) \subset A(N, 3\sigma) \cap A(N, 3\tau) = 0 .$$

Next suppose $\rho \neq \sigma + \tau$. If both $\sigma = \rho + \tau$ and $\tau = \rho + \sigma$ we obtain $\sigma = \tau$, a contradiction; thus $\sigma \neq \rho + \tau$ or $\tau \neq \rho + \sigma$ and using the equations involving $J(A(N, \rho), A(N, \sigma), A(N, \tau))$ we see that this expression is also zero in case $\rho \neq \sigma + \tau$. This completes the proof.

4. **Some assumptions.** The el-algebra identities do not appear strong enough to determine $A(N, \alpha)^2$ in any reasonable manner, so we assume N is a precartan subalgebra such that

$$A(N, \alpha)^2 \subset \sum_{\beta \neq 0} A(N, \beta) \quad \text{if } \alpha \neq 0 . \tag{4.1}$$

That is, if $x, y \in A(N, \alpha)$ and $xy = \sum_\beta z_\beta$, then no z_β is in $A(N, 0)$ except $z_\beta = 0$. By considering Lie and Malcev algebras, this is a natural assumption. Now let

$$B = \sum_{\beta \neq 0} A(N, \beta)A(N, -\beta) \oplus \sum_{\alpha \neq 0} A(N, \alpha)$$

where we use the usual convention that if $-\beta$ is not a weight of N, then $A(N, -\beta) = 0$. By (3.9), $\sum_{\beta \neq 0} A(N, \beta)A(N, -\beta) \subset A(N, 0)$ and for any weight $\gamma \neq 0$ of N we have

$$\begin{aligned} BA(N, \gamma) &\subset A(N, 0)A(N, \gamma) + \sum_{\alpha \neq 0} A(N, \alpha)A(N, \gamma) \\ &\subset A(N, \gamma) + A(N, \gamma)A(N, \gamma) + A(N, \gamma)A(N, -\gamma) \\ &\qquad + \sum_{\alpha \neq 0, \pm\gamma} A(N, \alpha + \gamma) \\ &\subset B, \text{ using (4.1)} . \end{aligned}$$

Next from (3.15) we have for any $\beta \neq 0$,

$$J(A(N, \beta), A(N, -\beta), A(N, 0)) = 0 .$$

From this it follows that

$$\begin{aligned} [A(N, \beta)A(N, -\beta)]A(N, 0) &\subset [A(N, \beta)A(N, 0)]A(N, -\beta) \\ &\quad + [A(N, -\beta)A(N, 0)]A(N, \beta) \\ &\subset A(N, \beta)A(N, -\beta) \end{aligned}$$

and so

$$BA(N, 0) \subset \sum_{\beta \neq 0} [A(N, \beta)A(N, -\beta)]A(N, 0) + \sum_{\alpha \neq 0} A(N, \alpha)A(N, 0) \subset B .$$

Thus B is an ideal of A and since A is simple $B = 0$ or $B = A$. This proves

THEOREM 4.2. *If A is a simple el-algebra and N a precartan subalgebra such that* (4.1) *holds, then*

1. $$A = \sum_{\beta\neq 0} A(N, \beta)A(N, -B) \oplus \sum_{\alpha\neq 0} A(N, \alpha) \text{ and}$$
$$A(N, 0) = \sum_{\beta\neq 0} A(N, \beta)A(N, -\beta); \text{ or}$$

2. *the only weight of N is* 0 *and* $A = A(N, 0)$.

COROLLARY 4.3. *If A is an el-algebra as in Theorem* 4.2, *then $A(N, 0)$ is a subalgebra. Furthermore if conclusion* (1) *of the theorem holds, then $A(N, 0)$ is a Lie subalgebra.*

Proof. If conclusion (2) holds, the result is trivial. So assume conclusion (1) holds, then $A(N, 0) = \sum_{\beta\neq 0} A(N, \beta)A(N, -\beta)$ and as before

$$A(N, 0)A(N, 0) = \sum_{\beta\neq 0} [A(N, \beta)A(N, -\beta)]A(N, 0) \subset A(N, 0) .$$

Next for any $\beta \neq 0$ we shall show

$$J(A(N, \beta)A(N, -\beta), A(N, 0), A(N, 0)) = 0$$

and therefore $A(N, 0)$ will be a Lie subalgebra. Let $x \in A(N, \beta)$, $y \in A(N, -\beta)$ and $s, t \in A(N, 0)$, then using $A(N, 0)^2 \subset A(N, 0)$ and (3.15) we have

$$\begin{aligned} J(xy, s, t) &= J(xy, s, t) + J(st, x, y) \\ &= J(xs, t, y) + J(ty, x, s), \text{ using (3.4)} \\ &= 0, \text{ using (3.15)} . \end{aligned}$$

COROLLARY 4.4. *If A is an el-algebra as in Theorem* 4.2 *which satisfies conclusion* (1), *then the mapping* $A(N, 0) \to R(A(N, 0)): m \to R_m$ *is a representation of the Lie algebra* $A(N, 0)$.

Proof. From Corollary 4.3 it suffices to prove

$$J(A(N, \beta), A(N, 0), A(N, 0)) = 0 \text{ for any } \beta \neq 0 .$$

Let $w \in A(N, \beta)$ and $y, z \in A(N, 0)$, then for any $n \in N$ we have

$$\begin{aligned} J(wn, y, z) &= J(wn, y, z) + J(yz, w, n), \text{ using (3.12)} \\ &= J(wy, z, n) + J(zn, w, y) \\ &= J(zn, w, y), \text{ using (3.12)} \end{aligned}$$

and by induction we have

$$J(wR_n^t, y, z) = J(zR_n^t, w, y) .$$

But for large enough $t, zR_n^t = 0$ and so for any $n \in N$ and t large enough $J(wR_n^t, y, z) = 0$. However there exists in $n \in N$ so that R_n and therefore R_n^t is nonsingular on $A(N, \beta)$ and the usual argument proves the corollary.

COROLLARY 4.5. *If A is an el-algebra as in Theorem 4.2 which satisfies conclusion (1) for some precartan subalgebra N, then $A(N, 0)$ is a Cartan subalgebra provided $A(N, 0)$ is a nilpotent Lie subalgebra.*

Proof. Let $M = A(N, 0)$, then from Corollary 4.4 and the hypothesis we see that M is a precartan subalgebra and from § 2

$$N \subset M = \bigcap_{n \in N} A(R_n, 0) .$$

Thus decompose $A = A(M, 0) \oplus \sum_{\beta \neq 0} A(M, \beta)$ relative to $R(M)$ and we must show $A(M, 0) \subset M$. Well,

$$N \subset A(N, 0) = M \subset A(M, 0) = \bigcap_{m \in M} A(R_m, 0)$$

and since $N \subset M$ we have $\bigcap_{m \in M} A(R_m, 0) \subset \bigcap_{n \in N} A(R_n, 0) = M$. Thus $A(M, 0) = M$.

As in Lie algebras we make the following

DEFINITION. An element u in an el-algebra A is *regular* if the dimension of $A(R_u, 0)$ is minimal.

COROLLARY 4.6. *If A is an el-algebra as in Theorem 4.2 which satisfies conclusion (1) for the precartan subalgebra $N = Fu$ where u is regular, then $A(N, 0)$ is a Cartan subalgebra.*

Proof. From Corollary 4.5 it suffices to prove $A(N, 0)$ is a nilpotent Lie subalgebra and the proof in [1, Th. 3.1] can easily be modified to prove this fact.

Next we investigate the symmetric bilinear form $(x, y) = \operatorname{trace} R_x R_y$ and impose a condition on it which is satisfied by Lie and Malcev algebras and implies assumption (4.1). Some notation: for $x \in A$ or $S, T \subset A$, (x, S) denotes $\{(x, S) : s \in S\}$ and (S, T) denotes $\{(s, t) : s \in S$ and $t \in T\}$. (x, y) is an *invariant form* if $(xy, z) = (x, yz)$ for all $x, y, z \in A$.

THEOREM 4.7. *Let N be* any *precartan subalgebra of A and decom-*

pose $A = A(N, 0) \oplus \sum_{\alpha \neq 0} A(N, \alpha)$, *then*
1. *if* (x, y) *is an invariant form, then*

$$(A(N, \alpha), A(N, \beta)) = 0 \quad \text{if } \alpha+\beta \neq 0 \; ;$$

2. *if* (x, y) *is a nondegenerate invariant form, then*

$$A(N, \alpha)^2 \subset \sum_{\beta \neq 0} A(N, \beta) \quad \text{if } \alpha \neq 0 \; ;$$

furthermore there exists a precartan subalgebra N_0 *such that* $N_0 = Fu$ *where* u *is regular and* $A \neq A(N_0, 0)$.

Proof. (1) Suppose (x, y) is an invariant form and let $x \in A(N, \alpha)$, $y \in A(N, \beta)$ and $n \in N$. Then by induction we have

$$(x(R_n - \alpha(n)I)^k, y) = (-1)^k(x, y(R_n + \alpha(n)I)^k)$$

for $k = 1, 2, \cdots$. But for large enough k, $(x(R_n - \alpha(n)I)^k, y) = 0$, since $x \in A(N, \alpha)$ and so $(x, y(R_n + \alpha(n)I)^k) = 0$. But since $\beta \neq -\alpha$, there exists $n \in N$ with $(R_n + \alpha(n)I)^k$ nonsingular on $A(N, \beta)$ and therefore $A(N, \beta) = A(N, \beta)(R_n + \alpha(n)I)^k$. This proves (1).
(2) Suppose (x, y) is a nondegenerate invariant form, then there exists $n \in A$ such that $(n, n) = \text{trace } R_n^2 \neq 0$; otherwise by linearization, $(x, y) = 0$ for all $x, y \in A$. Thus R_n is not nilpotent and we can choose an element $u \in A$ such that $A \neq A(R_u, 0)$ and u regular. Thus $N_0 = Fu$ is a precartan subalgebra such that $A \neq A(N_0, 0)$. Next let $\alpha \neq 0$ and $x, y \in A(N, \alpha)$ and $x, y \in A(N, \alpha)$ and $xy = z_0 + \sum_{\beta \neq 0} z_\beta$ where N is *any* precartan subalgebra. Now for $\gamma \neq 0$ we have

$$(z_0, A(N, \gamma)) = 0, \quad \text{using part (1)} \; .$$

For $\gamma = 0$ we have

$$\begin{aligned}(z_0, A(N, 0) &= (xy - \sum_{\beta \neq 0} z_\beta, A(N, 0)) \\ &\quad (x, yA(N, 0)) - \sum_{\beta \neq 0} (z_\beta, A(N, 0)) \\ &= 0 \; , \quad \text{using part (1)} \; .\end{aligned}$$

Thus $(z_0, A) = 0$ and since (x, y) is nondegenerate $z_0 = 0$.

For the remainder of this paper we shall assume (x, y) is a nondegenerate invariant form on A. With this assumption we may conclude that A has a precartan subalgebra N_0 such that (4.1) actually holds, $A \neq A(N_0, 0)$ and $A(N_0, 0)$ is a Cartan subalgebra. Thus we may assume the existence of a Cartan subalgebra $N = A(N, 0) \neq A$ satisfying 4.1, and call A "the usual el-algebra." We shall work with a fixed Cartan subalgebra described above and use the notation $A(\alpha)$ or A_α for $A(N, \alpha)$ and also N for $A(N, 0)$. The proof of the following is similar to the proof for Lie algebras [1, Section 4.1].

THEOREM 4.8. *Let A be the usual el-algebra, then*
1. *(x, y) is nondegenerate on N and for $x, y \in N$, $(x, y) = \sum_\rho N_\rho \rho(x)\rho(y)$ where the sum is over all weights ρ and $N_\rho =$ dimension $A(\rho)$.*
2. *$N^2 = 0$.*
3. *$0 \neq \rho$ is a weight if and only if $0 \neq -\rho$ is a weight; furthermore $A(\rho)$ and $A(-\rho)$ are dual relative to (x, y) so that dimension $A(\rho) =$ dimension $A(-\rho)$.*
4. *Let N^* be the dual space of N. If $g \in N^*$, then there exists a unique element $n_g \in N$ such that $g(n) = (n, n_g)$ for all $n \in N$. Furthermore the mapping $N^* \to N: g \to n_g$ is a bijection.*
5. *If $l =$ dimension N over F, then there are l linearly independent weights of N and these form a basis of N^*. Furthermore if $\rho, \sigma \in N^*$ and n_ρ, n_σ are determined as in (4), then the symmetric bilinear form $< \rho, \sigma > = (n_\rho, n_\sigma)$ is nondegenerate on N^*.*
6. *If ρ is a nonzero weight of N and $x_\rho \in A(\rho)$ is such that*

$$x_\rho R_n = \rho(n) x_\rho \text{ for all } n \in N$$

and if $x_{-\rho}$ is any element in $A(-\rho)$, then

$$x_{-\rho} x_\rho = (x_\rho, x_{-\rho}) n_\rho$$

where n_ρ is determined in (4).
7. *$\rho(n_\rho) \neq 0$ if $\rho \neq 0$.*

We shall prove (7) since its proof is different from the proof of the corresponding statement in [1]. First we need

LEMMA 4.9 *Let $A = N \oplus \sum_{\alpha \neq 0} A(\alpha)$ be the usual el-algebra with Cartan subalgebra $N = \sum_{\alpha \neq 0} A(\alpha)A(-\alpha)$. Let ϕ be any weight of N and ρ any nonzero weight of N and let $h \in \sum_{i=1}^{\infty} A(i\rho)A(-i\rho)$. Then $\phi(h) = r\rho(h)$ where r is a rational number.*

Proof. If $\phi = r\rho$ for some rational number, then we are finished. Otherwise $\phi \neq r\rho$ for any rational number r and we consider

$$M = \sum_k A(\phi + k\rho) \qquad k = 0, \pm 1, \pm 2, \cdots.$$

Now since there are finitely many weights we may write $h = \sum_{i=1}^{m} h_i$ where $h_i \in A(i\rho)A(-i\rho)$ and therefore $h_i = \sum_j x^j(i\rho)x^j(-i\rho)$ where $x^j(k\rho) \in A(k\rho)$, $k = \pm i$. Now since $h_i \in N$, M is $R(h_i)$-invariant and so $R(h)$-invariant. But since $\phi \neq r\rho$ for any rational number r, we use (3.9) to see that M is also $R(x^j(i\rho))$- and $R(x^j(-i\rho))$-invariant for $i = 1, \cdots, m$.

Next for $z = \sum_k z_k \in M$ where $z_k \in A(\phi + k\rho)$ we have, using (3.15),

$$J(z, x^j(i\rho), x^j(-i\rho)) = \sum_k J(z_k, x^j(i\rho), x^j(-i\rho)) = 0 .$$

Therefore on M we have

$$\begin{aligned} R(h_i) &= \textstyle\sum_j R(x^j(i\rho)x^j(-i\rho)) \\ &= \textstyle\sum_j [R(x^j(i\rho)), R(x^j(-i\rho))] \end{aligned}$$

so that on M

$$R(h) = \textstyle\sum_i R(h_i) = \sum_{ij}[R(x^j(i\rho)), R(x^j(-i\rho))] .$$

Therefore $\operatorname{trace}_M R(h) = 0$. But since $h \in N$ we can calculate the trace of $R(h)$ on M from its matrix to see that

$$\begin{aligned} \operatorname{trace}_M R(h) &= \textstyle\sum_k N_{\phi+k\rho}(\phi + k\rho)(h) \\ &= (\textstyle\sum_k N_{\phi+k\rho})\phi(h) + (\sum_k k N_{\phi+k\rho})\rho(h) , \end{aligned}$$

where $N_{\phi+k\rho} =$ dimension $A(\phi + k\rho)$. The result now follows.

For the proof of 4.8.7 suppose ρ is a nonzero weight of N such that $\rho(n_\rho) = 0$. Then for any $n \in N$ we have

$$\rho(n) = (n, n_\rho) = \textstyle\sum_\alpha N_\alpha \alpha(n_\rho)\alpha(n) ,$$

summed over all weights α. Now we can find an element $x_{-\rho} \in A(-\rho)$ such that $(x_\rho, x_{-\rho}) = 1$ and therefore from 4.8.6, $n_\rho = x_{-\rho}x_\rho$ and so using Lemma 4.9 we have $\alpha(n_\rho) = r\rho(n_\rho) = 0$ where r is a rational number depending on α and ρ. Therefore $\rho(n) = 0$ i.e. $\rho = 0$, a contradiction.

Finally we use the nondegenerate invariant form (x, y) to obtain more information on $A(\alpha)^2$.

THEOREM 4.10. *Let A be the usual el-algebra, then*

$$A(\alpha)^2 \subset A(-\alpha) + A(2\alpha) .$$

Proof. For $\alpha = 0$ we know the result. Suppose $\alpha \neq 0$; if $A(\alpha)^2 = 0$, then we are finished. So assume $A(\alpha)^2 \neq 0$, then since (x, y) is nondegenerate there exists a weight β such that

$$(A(\alpha)^2, A(\beta)) \neq 0 .$$

Case 1. If $\beta \neq \alpha$, then

$$0 \neq (A(\alpha)^2, A(\beta)) = (A(\alpha), A(\alpha)A(\beta)) \subset (A(\alpha), A(\alpha + \beta)) .$$

Thus $\alpha + \beta$ must be a weight and so by Lemma 4.7.1, $\alpha + (\alpha + \beta) = 0$, that is, $\beta = -2\alpha$.

Case 2. $\beta = \alpha$.

Thus we have $(A(\alpha)^2, A(-2\alpha) + A(\alpha)) \neq 0$. Now let $x, y \in A(\alpha)$ and $xy = \sum_\sigma z_\sigma$. Suppose there exists a component z_γ with $\gamma \neq -\alpha$ or 2α, then we have

$$\begin{aligned}(xy, A(-\gamma)) &= (x, yA(-\gamma)) \\ &\subset (A(\alpha), A(\alpha-\gamma)), \text{ since } \gamma \neq -\alpha \\ &= 0, \text{ since } \alpha + (\alpha - \gamma) \neq 0 .\end{aligned}$$

Thus

$$\begin{aligned}0 &= (xy, A(-\gamma)) \\ &= \textstyle\sum_\sigma (z_\sigma, A(-\gamma)) \\ &= (z_\gamma, A(-\gamma)), \text{ using Lemma 4.7.1.}\end{aligned}$$

Therefore by Theorem 4.8.3, $z_\gamma = 0$ and so $xy = z_{-\alpha} + z_{2\alpha}$.

5. **Weight space subalgebras.** Let $A = N \oplus \sum_{\rho \neq 0} A(\rho)$ be the usual simple el-algebra discussed in § 4. Since we know $-\rho$ is a weight if and only if ρ is a weight, we may eliminate superfluous weight spaces and write

$$A = \textstyle\sum_\rho A(-\rho) \oplus N \oplus \sum_\rho A(\rho)$$

where ρ is a nonzero weight. Now for a fixed nonzero weight ρ we shall consider the *weight space subalgebra*

$$B(\rho) = \sum_{k=1}^{\infty} A(-k\rho) \oplus \sum_{k=1}^{\infty} A(-k\rho)A(k\rho) \oplus \sum_{k=1}^{\infty} A(k\rho)$$

and show it actually equals

$$A(-\rho) \oplus n_\rho F \oplus A(\rho)$$

where n_ρ determined in Theorem 4.8.4. In the next section we shall show $B(\rho)$ is the split 3-dimensional simple Lie algebra or the simple 7-dimensional Malcev algebra obtained from the split Cayley-Dickson algebra.

PROPOSITION 5.1. Let $B(\rho)$ be a weight space subalgebra, then $\sum_{k=1}^{\infty} A(-k\rho)A(k\rho) = n_\rho F$.

Proof. Let $B = \sum_k A(-k\rho)A(k\rho)$, then it suffices to show dimension $B = 1$; for $0 \neq n_\rho \in B$ and we would have $B = n_\rho F$. To show this let B^* denote the dual space of B and prove dimension $B^* = 1$. Since dimension $B =$ dimension B^* we have the results. Let $x \in A$ be such that for any $a \in B(\rho)$, $ax \in B(\rho)$ and set

$$\bar{R}_x : B(\rho) \to B(\rho) : a \to ax .$$

Let $x, y \in B(\rho)$ and set

$$f(x, y) = \text{trace}\, \bar{R}_x \bar{R}_y (= \text{trace}_{B(\rho)} R_x R_y) .$$

$f(x, y)$ is a bilinear form on $B(\rho)$ and the fact that dimension $B^* = 1$ is a consequence of the following lemma.

LEMMA 5.2. 1. *$f(x, y)$ is nondegenerate on B and*
2. *dimension $B^* = 1$.*

Proof. Since there are finitely many weights we can apply Theorem 4.8 to show that for any $h, k \in B$, $f(h, k) = \text{trace}\, \bar{R}(h)\bar{R}(k) =$

$$\text{trace} \begin{pmatrix} H_m & & & & & & 0 \\ & \cdot & & & & & \\ & & \cdot & & & & \\ & & & H_0 & & & \\ & & & & \cdot & & \\ & & & & & \cdot & \\ 0 & & & & & & H_{-m} \end{pmatrix} \begin{pmatrix} K_m & & & & & & 0 \\ & \cdot & & & & & \\ & & \cdot & & & & \\ & & & K_0 & & & \\ & & & & \cdot & & \\ & & & & & \cdot & \\ 0 & & & & & & K_{-m} \end{pmatrix}$$
$$= (\textstyle\sum_j 2j^2 N_{j\rho})\rho(h)\rho(k) ,$$

where

$$H_j = \begin{pmatrix} j\rho(h) & & 0 \\ & \cdot & \\ * & & j\rho(h) \end{pmatrix} \quad \text{and} \quad K_j = \begin{pmatrix} j\rho(k) & & 0 \\ & \cdot & \\ * & & j\rho(k) \end{pmatrix}$$

Now suppose there exists $b \in B$ such that for all $k \in B$

$$0 = f(b, k) = (\textstyle\sum_j 2j^2 N_{j\rho})\rho(b)\rho(k) .$$

Set $k = b$ to conclude $\rho(b) = 0$. Now since $b \in B = \sum_j A(j\rho)A(-j\rho)$ we have form Lemma 4.9 that for any weight ϕ of N in A, $\phi(b) = r\rho(b) = 0$. Now for any $x \in N$ we can apply Theorem 4.8.1 to conclude $(x, b) = 0$ and therefore $b = 0$. Thus $f(x, y)$ is nondegenerate on B.

Next let $g \in B^*$, then since $f(x, y)$ is nondegenerate on B, there exists a unique $b \in B$ so that for all

$$h \in Bg(h) = f(h, b) = (\textstyle\sum_j 2j^2 N_{j\rho})\rho(b)\rho(h) = c\rho(h)$$

where $c \in F$. Thus $B^* = \rho F$ is of dimension one.

COROLLARY 5.3. $B(\rho) = \sum_k A(-k\rho) \oplus n_\rho F \oplus \sum_k A(k\rho)$.

Put $R(n_\rho)$ into its Jordan canonical form on $A(-\rho)$. Thus we can write

$$A(-\rho) = U(1, -\rho) \oplus \cdots \oplus U(k, -\rho)$$

where each $U(j, -\rho)$ has a basis $\{z_{j1}, \cdots, z_{jm_j}\}$ such that, if $\rho \equiv \rho(n_\rho)$,

$$z_{j1}R(n_\rho) = -\rho z_{j1} \quad \text{and} \tag{5.4}$$
$$z_{ji}R(n_\rho) = -\rho z_{ji} + z_{ji-1}$$

$i = 2, 3, \cdots, m_j$. We shall consider just one of the $R(n_\rho)$-invariant spaces $U(j, -\rho)$ and denote it by $U \equiv \{z_1, \cdots, z_m\}$. We use symmetry to argue similar results for the other spaces and also for the weight ρ.

LEMMA 5.5. *Let $x \in A(\rho)$ be such that $xR(n_\rho) = \rho x$, then $z_i x = \lambda_i n_\rho$ and $\lambda_i = 0$ for $i = 1, \cdots, m-1$.*

Proof. From Proposition 5.1 $z_i x = \lambda_i n_\rho$ and from (3.15) we have for $i = 2, \cdots, m$ that

$$\begin{aligned} 0 = J(z_i, x, n_\rho) &= z_i x \cdot n_\rho + x n_\rho \cdot z_i + n_\rho z_i \cdot x = \rho x z_i - z_i n_\rho \cdot x \\ &= \rho x z_i - (-\rho z_i + z_{i-1})x = z_{i-1}x , \end{aligned}$$

also using Theorem 4.8.2. Thus $\lambda_i = 0, i = 1, \cdots, m-1$.

PROPOSITION 5.6. $J(z_i, z_j, n_\rho) = 0$ for $i, j = 1, \cdots, m$.

Proof. Using the skew-symmetry of $J(a, b, c)$, it suffices to show $J(z_i, z_j, n_\rho) = 0$ for $i \geqq j$ and $j = m, m-1, \cdots, 1$. We use induction on j. First for $j = m$, $J(z_m, z_m, n_\rho) = 0$. Next for $j = m-1$ we have

$$\begin{aligned} J(z_m, z_{m-1}, n_\rho) &= J(z_m, z_m(R(n_\rho) + \rho I), n_\rho) \\ &= J(z_m, z_m n_\rho, n_\rho) \\ &= 0, \text{ using (3.1)} . \end{aligned}$$

Now assume that for all $j = m, m-1, \cdots, m-q$ we have shown $J(z_j, z_j, n_\rho) = \cdots = J(z_m, z_j, n_\rho) = 0$, then we must show similar results for $j - 1 = m - (q+1)$. We always have $J(z_{j-1}, z_{j-1}, n_\rho) = 0$ and next

$$\begin{aligned} J(z_j, z_{j-1}, n_\rho) &= J(z_j, z_j(R(n_\rho) + \rho I), n_\rho) \\ &= J(z_j, z_j n_\rho, n_\rho) = 0 . \end{aligned}$$

So we now assume

$$J(z_{j-1}, z_{j-1}, n_\rho) = \cdots = J(z_{j-1+k}, z_{j-1}, n_\rho) = 0$$

for $j - 1 < j - 1 + k < m$; we then have

$$\begin{aligned}
J(z_{j-1}, z_{j+k}, n_\rho) &= J(z_j(R(n_\rho) + \rho I), z_{j+k}, n_\rho) \\
&= J(z_j n_\rho, z_{j+k}, n_\rho) + \rho J(z_j, z_{j+k}, n_\rho) \\
&= J(z_j n_\rho, z_{j+k}, n_\rho), \text{ using the first induction hypothesis.} \\
&= -J(z_j n_\rho, n_\rho, z_{j+k}) \\
&= J(z_{j+k} n_\rho, n_\rho, z_j), \text{ using (3.3)} \\
&= -\rho J(z_{j+k}, n_\rho, z_j) + J(z_{j-1+k}, n_\rho, z_j) \\
&= 0\,.
\end{aligned}$$

PROPOSITION 5.7. For all $n \in N$, $J(z_i, z_j, n) = 0$ and $U^2 \subset A(-2\rho)$.

Proof. It suffices to show that for any weight $\sigma \neq 0$ we have $J(z_i, z_j, n_\sigma) = 0$ where n_σ is determined in Theorem 4.8.4; for we see that the elements n_σ span N as a vector space. Now if $\sigma = \lambda\rho$ where $\lambda \in F$, it is easy to see that $n_\sigma = \lambda n_\rho$ and so from Proposition 5.6 $J(z_i, z_j, n_\sigma) = 0$. Next assume σ is not a scalar multiple of ρ and choose elements x_σ and $x_{-\sigma}$ as in Theorem 4.8.6 so that $x_{-\sigma}x_\sigma = n_\sigma$. Now noting that $z_i z_j \in A(-\rho)^2 \subset A(\rho) \oplus A(-2\rho)$ we have, using 3.15,

$$\begin{aligned}
J(z_i z_j, x_{-\sigma}, x_\sigma) \in J(A(\rho), A(-\sigma), A(\sigma)) \\
+ J(A(-2\rho), A(-\sigma), A(\sigma)) = 0.
\end{aligned}$$

Thus we have

$$\begin{aligned}
J(n_\sigma, z_i, z_j) &= J(x_{-\sigma}x_\sigma, z_i, z_j) + J(z_i z_j, x_{-\sigma}, x_\sigma) \\
&= J(x_{-\sigma}z_i, z_j, x_\sigma) + J(z_j x_\sigma, x_{-\sigma}, z_i) \\
&= 0, \text{ using (3.15)}\,.
\end{aligned}$$

Next let $u, v \in U$, then by linearity $J(u, v, n_\rho) = 0$ and since U is $R(n_\rho)$-invariant subspace, the usual argument shows $(uv)(R(n_\rho) + 2\rho I)^k = 0$ for k large enough. But we know $uv = x + y$ where $x \in A(\rho)$, $y \in A(-2\rho)$ and therefore for k large enough,

$$0 = (uv - y)(R(n_\rho) + 2\rho I)^k = x(R(n_\rho) + 2\rho I)^k.$$

If $x \neq 0$, this implies there exists $0 \neq z \in A(\rho)$ so that $z(R(n_\rho) + 2\rho I) = 0$ and therefore $-3\rho z = z(R(n_\rho) - \rho I)$. Iterating this formula and using $\rho \equiv \rho(n_\rho) \neq 0$ we obtain $z = 0$, a contradiction. Thus $x = 0$ which proves the second part.

PROPOSITION 5.8. Let $x \in A(\rho)$ be such that $xR(n_\rho) = \rho x$, then $J(z_i, z_j, x) = 0$ for $i, j = 1, \cdots, m$.

Proof. It suffices to show $J(z_i, z_j, x) = 0$ for all $i \geqq j$ and

$j = m, \cdots, 1$. The case $j = m$ is trivial and for $j = m - 1$ we have

$$\begin{aligned} J(z_m, z_{m-1}, x) &= J(z_m, z_m(R(n_\rho) + \rho I), x) \\ &= J(z_m, z_m n_\rho, x) \\ &= J(n_\rho z_m, z_m, x) \\ &= -J(xz_m, z_m, n_\rho), \text{ using (3.3)} \\ &= 0, \text{ using Lemma 5.5.} \end{aligned}$$

Assume we have shown for $j = m, m-1, \cdots, m-k$

$$J(z_i, z_j, x) = 0 \text{ for } i = j, j+1, \cdots, m .$$

We shall show for $j = m - (k+1)$ that

$$J(z_i, z_j, x) = 0 \text{ for } i = j, j+1, \cdots, m .$$

As above, the cases $i = j$ and $i = j + 1$ are clear. So assume

$$J(z_i, z_j, x) = 0 \text{ for } i = j, j+1, \cdots, j+t-1$$

where $j < j + t - 1 < m$. Then for $i + 1 = j + t$ we have

$$\begin{aligned} J(z_{j+t}, z_j, x) &= J(z_{j+t}, z_{j+1}(R(n_\rho) + \rho I), x) \\ &= J(z_{j+t}, z_{j+1}n_\rho, x) + \rho J(z_{j+t}, z_{j+1}, x) \\ &= J(z_{j+t}, z_{j+1}n_\rho, x) , \\ &\qquad \text{using the first induction hypothesis} \\ &= -[J(z_{j+1}n_\rho, z_{j+t}, x) + J(z_{j+t}x, z_{j+1}, n_\rho)] , \\ &\qquad \text{using Lemma 5.5} \\ &= -[J(z_{j+1}z_{j+t}, x, n_\rho) + J(xn_\rho, z_{j+1}, z_{j+t})] \\ &= -J(xn_\rho, z_{j+1}, z_{j+t}) , \text{ using Proposition 5.7} \\ &= \rho J(z_{j+t}, z_{j+1}, x) \\ &= 0 , \text{ using induction hypothesis.} \end{aligned}$$

This proves the result.

To use the preceding results we need the following remark: Let V be an $R(n_\rho)$-invariant subspace of the weight space $A(\sigma)$, then $\mathrm{trace}_V R(n_\rho) = \sigma(n_\rho)\mu$ where μ is the dimension of V.

PROPOSITION 5.9. Let $x \in A(\rho)$ be such that $xR(n_\rho) = \rho x$ and suppose $xU \neq 0$, where we use the notation $U \equiv \{z_1, \cdots, z_m\}$ preceding Lemma 5.5. Then the dimension of U is one.

COROLLARY 5.10. *If x is given as above and $xU(j, -\rho) \neq 0$, then the dimension of $U(j, -\rho)$ is one.*

Proof. Let $U_1 \equiv U = \{z_1, \cdots, z_m\}$ denote the set of elements of

$A(-\rho)$ we have been considering. Set $U_2 = UU \subset A(-2\rho)$, using Proposition 5.7; and set $U_k = U_{k-1}U \subset A(-k\rho)$. Then since there are finitely many weights, there exists an integer M so that $U_M \neq 0$ but $U_{M+1} = 0$. Let $P = xF \oplus n_\rho F \oplus \sum_{k=1}^{M} U_k$, then we shall show: (1) P is R_x-, $R(n_\rho)$- and $R(z_m)$-invariant; (2) $J(z, x, z_m) = 0$ for all $z \in P$. Then from Lemma 5.5 and the hypothesis we see $xz_m = \lambda n_\rho$ where $\lambda \neq 0$ and using the standard Lie algebra trace argument we shall show that the dimension of U is one.

Suppose (1) and (2) have been proved and consider the trace argument. From (2) we see

$$0 = J(z, x, z_m) = z([R(x), R(z_m)] - R(xz_m)) .$$

Thus as linear transformations on P we have

$$\begin{aligned} R(n_\rho) &= 1/\lambda R(xz_m), \text{ since } \lambda \neq 0 \\ &= 1/\lambda [R(x), R(z_m)] \end{aligned}$$

and so $\text{trace}_P R(n_\rho) = 0$. But from the matrix of $R(n_\rho)$ on P we see $\text{trace}_P R(n_\rho) = (\rho + 0 - \mu_1\rho - 2\mu_2\rho - \cdots - M\mu_M\rho)$ where μ_i is the dimension of U_i. Thus since $\rho \equiv \rho(n_\rho) \neq 0$, $\mu_1 = 1$, $\mu_2 = \cdots = \mu_M = 0$.

We now prove (2). If $z = x$ or $z = n_\rho$, the result follows. If $z \in U_1$, then use Proposition 5.8. Finally if $z \in U_i$ for $i > 1$, then $z \in A(-i\rho)$ and so $J(z, x, z_m) \in J(A(-i\rho), A(\rho), A(-\rho)) = 0$. Now for an arbitrary element $z \in P$ use the above and linearity.

Next we prove (1). Since $z_m \in U_1$, each $U_i = U_{i-1}U_1$ is such that $U_i z_m \subset U_i U_1 = U_{i+1}$. Furthermore since $xz_m = \lambda n_\rho$ and $n_\rho z_m \in U_1$ we see P is $R(z_m)$-invariant. Next we shall show P is $R(n_\rho)$-invariant. First $xR(n_\rho)$, $n_\rho R(n_\rho)$ and $U_1 R(n_\rho)$ are all contained in P. We have $U_1 R(n_\rho) \subset U_1$; next assume $U_i R(n_\rho) \subset U_i$. Then for $U_{i+1}R(n_\rho)$ we note that any $a \in U_{i+1}$ is of the form $a = \sum_j a_{ij} z_j$ where $a_{ij} \in U_i$, $z_j \in U_1$. Since $U_i \subset A(-i\rho)$ we have

$$\begin{aligned} 0 &= J(a_{ij}, z_j, n_\rho), \text{ using (3.15) or Prosposition 5.6} \\ &= (a_{ij}z_j)n_\rho + (z_j n_\rho)a_{ij} + (n_\rho a_{ij})z_j . \end{aligned}$$

Therefore

$$\begin{aligned} aR(n_\rho) &= \textstyle\sum_j (a_{ij}z_j)n_\rho \\ &= \textstyle\sum_j a_{ij}(z_j n_\rho) + z_j(n_\rho a_{ij}) . \end{aligned}$$

Now $a_{ij}(z_j n_\rho) \in U_i U_1 \subset U_{i+1}$ and by induction hypothesis $n_\rho a_{ij} \in U_i$ and therefore $z_j(n_\rho a_{ij}) \in U_{i+1}$. Thus $U_{i+1}R(n_\rho) \subset U_{i+1}$ and P is $R(n_\rho)$-invariant.

Finally we shall show P is $R(x)$-invariant. The elements $xR(x)$, $n_\rho R(x)$ and $z_i R(x) = \lambda_i n_\rho$ are all in P. Thus it suffices to show $U_i R(x) \subset$

P for $i > 1$. Now from Proposition 5.8 we have

$$\begin{aligned} 0 &= J(z_i, z_j, x) \\ &= (z_i z_j)x + (z_j x)z_i + (x z_i)z_j \\ &= (z_i z_j)x + \lambda_j n_\rho z_i - \lambda_i n_\rho z_j . \end{aligned}$$

Thus $(z_i z_j)R(x) = \lambda_i n_\rho z_j - \lambda_j n_\rho z_i \in U_1$ which yields $U_2 R(x) \subset U_1$. Now assume $U_i R(x) \subset U_{i-1}$, then since $U_{i+1} = U_i U_1$ and $U_i \subset A(-i\rho)$ we have for any $a = \sum a_{ij} z_j \in U_{i+1}$ that

$$\begin{aligned} 0 &= J(a_{ij}, z_j, x) \\ &= (a_{ij} z_j)x + (z_j x)a_{ij} + (x a_{ij})z_j \\ &= (a_{ij} z_j)x + \lambda_j n_\rho a_{ij} + b_{ij} z_j \end{aligned}$$

where by the induction hypothesis $b_{ij} = x a_{ij} \in U_{i-1}$. Thus we conclude $U_{i+1} R(x) \subset U_i$.

PROPOSITION 5.11. Let $A(-\rho) = U(1, -\rho) \oplus \cdots \oplus U(k, -\rho)$ be the decomposition of $A(-\rho)$ when $R(n_\rho)$ is put into Jordan canonical form. Then the dimension of all the $U(i, -\rho)$ is one.

Proof. Suppose there exists $U(j, -\rho) \equiv \{y_1, \cdots, y_m\}$ of dimension $m > 1$. Then putting $R(n_\rho)$ into its Jordon canonical form on $A(\rho)$, that is, writing $A(\rho) = U(1, \rho) \oplus \cdots \oplus U(q, \rho)$ we see $y_1 U(k, \rho) = 0$ for every $k = 1, \cdots, q$. For otherwise by an argument similar to that in Proposition 5.9 we have the situation $y_1 R(n_\rho) = -\rho y_1$ and $y_1 U(k,\rho) \neq 0$ for some k and therefore the dimension of $U(k, \rho)$ is one. But this means $U(k, \rho) = xF$ where $xR(n_\rho) = \rho x$ and $0 \neq x y_1 \in x U(j, -\rho)$. So again by Proposition 5.9, the dimension of $U(j, -\rho)$ is one, a contradiction. Thus we have $y_1 U(k, \rho) = 0$ for all k and therefore $y_1 A(\rho) = 0$. But using the nondegenerate invariant form (u, v) this yields for every $z \in A(\rho)$, $-\rho(z, y_1) = (z, y_1 n_\rho) = (z y_1, n_\rho) = 0$. Thus $(A(\rho), y_1) = 0$ and since $y_1 \in A(-\rho)$ we have $y_1 = 0$, a contradiction.

Proposition 5.11 means $R(n_\rho)$ can be diagonalized on $A(-\rho)$ and by a symmetrical argument $R(n_\rho)$ can be diagonalized on $A(\rho)$. Similarly $R(n_{k\rho})$ can be diagonalized on $A(\pm k\rho)$. But since $n_{k\rho} = k n_\rho$ we see $R(n_\rho)$ can be diagonalized on $A(\pm k\rho)$ and therefore on the weight space subalgebra $B(\rho)$. We use these facts in the remainder of this section to show that $B(\rho) = A(-\rho) \oplus n_\rho F \oplus A(\rho)$.

Let v be a fixed element in $A(\rho)$ where ρ is any nonzero weight and let

$$E(v) = \{x \in A(\rho) \colon xv \in A(2\rho)\} .$$

Since $v^2 = 0 \in A(2\rho)$, $v \in E(v)$. We shall now show

LEMMA 5.12. $$E(v) = vF\,.$$

Proof. Clearly $E \equiv E(v)$ is a subspace. Now let $y \in A(-\rho)$ be such that $n_\rho = yv$ and let

$$Q = yF \oplus n_\rho F \oplus E \oplus ER(v) \oplus \cdots \oplus ER(v)^t$$

where t is such that $ER(v)^t \neq 0$ but $ER(v)^{t+1} = 0$. We shall show (1) Q is $R(n_\rho)$-, $R(y)$- and $R(v)$-invariant; (2) $J(q, v, y) = 0$ for all $q \in Q$. Then by the usual trace argument we shall show the dimension of E is one which proves the result.

(1). Q is $R(n_\rho)$-invariant. First $yR(n_\rho)$, $n_\rho R(n_\rho)$ and $ER(n_\rho)$ are all in Q. Now if $x \in E$ then $z = xR^j(v) \in ER^j(v) \subset A((j+1)\rho)$. Since $R(n_\rho)$ acts diagonally in $A((j+1)\rho)$ we have $zR(n_\rho) = (j+1)\rho z \in ER^j(v)$. Next Q is $R(v)$-invariant by the choice of y and since $v \in E$. Finally Q is $R(y)$-invariant. We know $yR(y)$, $n_\rho R(y)$ and $ER(y)$ are all in Q, noting $ER(y) \subset n_\rho F$. So let $x \in E$, then we shall show $(xR(v))R(y) \in E$ and therefore $(ER(v))R(y) \subset E$. We have

$$\begin{aligned} -\rho J(y, x, v) &= J(yn_\rho, x, v) \\ &= J(yn_\rho, x, v) + J(xv, y, n_\rho), \text{ using } xv \in A(2\rho) \text{ and } (3.15) \\ &= J(yx, v, n_\rho) + J(vn_\rho, y, x) \\ &= \rho J(v, y, x), \text{ since } yx \in n_\rho F \\ &= \rho J(y, x, v)\,. \end{aligned}$$

Therefore $J(y, x, v) = 0$ and from this, $(xv)y = (xy)v + x(vy) \in vF + xF \subset E$. Now assume $(ER(v)^j)R(y) \subset ER(v)^{j-1}$, then for $w = zR(v) \in ER(v)^{j+1}$ where $z \in ER(v)^j \subset A((j+1)\rho)$ we have $0 = J(z, v, y) = (zv)y + (vy)z + (yz)v$. From this we obtain

$$wR(y) = (zv)y = -(vy)z - (yz)v = n_\rho z + z'v \in ER(v)^j$$

where $z' = yz \in ER(v)^{j-1}$ using the induction hypothesis. Thus by the choice of w we have $(ER(v)^{j+1})R(y) \subset ER(v)^j$.

Next we prove (2). First if $q = y$ or $q = n_\rho$, then $J(q, v, y) = 0$. Now for $q \in E$ we see from the above proof that $J(q, v, y) = 0$ (by the change of notation of x to q). Now for $q \in ER(v)^j \subset A((j+1)\rho)$ we have by (3.15), $J(q, v, y) = 0$. Thus by linearity we have for all $q \in Q$, $J(q, v, y) = 0$.

Next we apply the standard trace argument. On Q we have from (2) that

$$q([R(v), R(y)] - R(vy)) = 0$$

so that on Q we have $R(n_\rho) = [R(y), R(v)]$ and therefore $\operatorname{trace}_Q R(n_\rho) = 0$.

But from the matrix of $R(n_\rho)$ on Q, remembering $R(n_\rho)$ acts diagonally on $B(\rho)$, we see $\operatorname{trace}_Q R(n_\rho) = -\rho + 0 + \mu_0\rho + 2\mu_1\rho + \cdots + (t+1)\mu_t\rho$ where μ_j is the dimension of $ER(v)^j$. Thus since $\rho \neq 0$, $\mu_0 = 1$ and $\mu_1 = \cdots = \mu_t = 0$.

THEOREM 5.13. *Let ρ be a nonzero weight, then the only integral multiples of ρ which are weights are $0, \pm\rho$. Thus the weight space subalgebra $B(\rho) = A(-\rho) \oplus n_\rho F \oplus A(\rho)$ and therefore $A(\rho)^2 \subset A(-\rho)$.*

Proof. Let $x \in A(\rho)$ and $y \in A(-\rho)$ be such that $xy = n_\rho$ and let

$$R = xF \oplus n_\rho F \oplus yF \oplus \sum_{k=2}^{\infty} A(-k\rho) .$$

Then R is $R(n_\rho)$- and $R(y)$-invariant. Next we shall show R is $R(x)$-invariant. Clearly the elements $xR(x)$, $n_\rho R(x)$, $yR(x)$ and $A(-k\rho)R(x)$ for $k \geqq 3$ are all in R. So we must show $A(-2\rho)R(x) \subset R$. For any $z \in A(-2\rho)$ we have

$$0 = J(z, x, y) = (zx)y + (xy)z + (yz)x = (zx)y + n_\rho z + (yz)x$$

and from this $(zx)y \in A(-2\rho) + A(-3\rho)A(\rho) \subset A(-2\rho)$. Thus the element $zx \in A(-2\rho)A(\rho) \subset A(-\rho)$ is contained in the set $E(y)$ discussed in Lemma 5.12 and $E(y)$ equals yF. Thus $A(-2\rho)R(x) \subset yF \subset R$ so that R is $R(x)$-invariant.

Now for any $q \in R$, $J(q, x, y) = 0$ and therefore $R(n_\rho) = R(xy) = [R(x), R(y)]$ on R so that $\operatorname{trace}_R R(n_\rho) = 0$. But from the matrix of $R(n_\rho)$ on R we see

$$\operatorname{trace}_R R(n_\rho) = \rho + 0 - \rho + 2\mu_2\rho + 3\mu_3\rho + \cdots$$

where μ_k is the dimension of $A(-k\rho)$, $k \geqq 2$. Thus since $\rho \neq 0$ we must have $\mu_k = 0$ for $k \geqq 2$ which shows $A(-k\rho) = 0$ and by Theorem 4.8.3, $A(k\rho) = 0$ for $k \geqq 2$. The fact that $A(\rho)^2 \subset A(-\rho)$ now follows from Theorem 4.10.

6. More on weight space subalgebras.

In this section we continue the discussion of weight space subalgebras and prove the theorem mentioned in the introduction. Let $B(\rho) = A(\rho) \oplus nF \oplus A(-\rho)$ be a weight space subalgebra and $\{x_1, \cdots, x_m\}$ a basis for $A(\rho)$. Since $A(-\rho)$ is dual to $A(\rho)$ relative to $(x, y) = \operatorname{trace} R_x R_y$ choose $\{y_1, \cdots, y_m\}$ to be a dual basis for $A(-\rho)$ so that we have $(x_i, y_j) = \delta_{ij}$. From this we also have

$$y_j x_i = \delta_{ij} n_\rho . \tag{6.1}$$

For by Proposition 5.1 we have $y_j x_i = \lambda n_\rho$ and therefore $\lambda\rho = \lambda(n_\rho, n_\rho) = (\lambda n_\rho, n_\rho) = (y_j x_i, n_\rho) = (y_j, x_i n_\rho) = \rho(y_j, x_i) = \rho\delta_{ij}$.

Next let $x \in A$ be such that for all $a \in B(\rho)$, $ax \in B(\rho)$ and set $\bar{R}_x : a \longrightarrow ax$ (see Proposition 5.1) and let $B(x, y) = (1/2m\rho)\operatorname{trace} \bar{R}_x \bar{R}_y$, then we have

THEOREM 6.2. *$B(\rho)$ is a simple subalgebra of A and $B(x, y)$ is a nondegenerate invariant form on $B(\rho)$.*

Proof. Let $C \neq 0$ be an ideal of $B(\rho)$ containing an element $c = \Sigma\, c_i x_i + c_0 n_\rho + \Sigma\, c_i' y_i$. First assume $c_0 \neq 0$, then from the multiplicative relations in $B(\rho)$ we have $\rho^2 c - (cn_\rho)n_\rho = \rho^2 c_0 n_\rho \in C$ and therefore $n_\rho \in C$ which implies $C = B$, using $A(\pm\rho)n_\rho \subset A(\pm\rho)$. Next we shall show that C always contains an element with a nonzero coefficient for n_ρ. Suppose $c = \Sigma c_i\, x_i + \Sigma\, c_i' y_i \in C$ and assume some $c_k \neq 0$, then $cy_k = \Sigma\, c_i x_i y_k + \Sigma c_i' y_i y_k = c_k x_k y_k + c_\rho = -c_k n_\rho + c_\rho$ is in C where $c_\rho \in A(\rho)$ and $c_k \neq 0$. Similarly if some $c'_k \neq 0$. Thus by the first of the proof $B(\rho)$ is simple.

To show $B(x, y)$ is a nondegenerate invariant from on $B(\rho)$, it suffices by Lemma 5.2 to show $B(x, y)$ is an invariant form; for in this case $\{x \in B(\rho) : B(x, B(\rho)) = 0\}$ is an ideal of $B(\rho)$ which must be zero. From identities (3.4) we have

$$\operatorname{trace} \bar{R}(xy)\bar{R}(z) - \operatorname{trace} \bar{R}(x)\bar{R}(yz) = \operatorname{trace} \bar{R}(xy\cdot z + x\cdot yz)\ ,$$

since $B(\rho)$ is a subalgebra and therefore satisfies the same indentities as A. Thus it suffices to show $\operatorname{trace} \bar{R}(z) = 0$ for all $z \in B(\rho)$. We have $\operatorname{trace} \bar{R}(n_\rho) = 0$ since dimension $A(\rho) =$ dimension $A(-\rho)$ and by the action of $\bar{R}(n_\rho)$ on $B(\rho)$. Similarly by the multiplicative relations of any basis element x_ρ of $A(\rho)$ in $B(\rho)$ we see that $R(x_\rho)$ has a matrix of trace zero; the same holds for $R(x_{-\rho})$. Thus by linearity of the trace function we have the results.

Next we investigate the identities for $B(\rho)$ more closely.

$$(6.3)\qquad (xy)z = \begin{cases} B(xy, z)n_\rho & \text{if } x, y, z \in A(\rho) \\ -B(xy, z)n_\rho & \text{if } x, y, z \in A(-\rho) \end{cases}$$

For, by the multiplicative relations in $B(\rho)$, there exists $\lambda \in F$ with $(xy)z = \lambda n_\rho$. Since $B(n_\rho, n_\rho) = \rho \neq 0$ we have $\lambda\rho = B(\lambda n_\rho, n_\rho) = B(xy, z, n_\rho) = B(xy, zn_\rho)$ and this implies the result. Using (6.3) we have

$$(6.4)\qquad J(x, y, z) = \begin{cases} 3\, B(xy, z)n_\rho & \text{if } x, y, z \in A(\rho) \\ -3B(xy, z)n_\rho & \text{if } x, y, z \in A(-\rho) \end{cases}$$

Using (6.3) and (6.4) we have

(6.5) $J(x, y, z) = 3(xy)z$ for $x, y, z \in A(\sigma), \sigma = \pm \rho$

(6.6) $(xy)z + x(yz) = 0$ for $x, y, z \in A(\sigma), \sigma = \pm \rho$

(6.7) $(xy)x = 0$ for $x, y \in A(\sigma), \sigma = \pm \rho$

(6.8) $3(xy)z = 2J(x, y, z)$ for $x, y \in A(\sigma), z \in A(-\sigma)$ and $\sigma = \pm \rho$.

For the proof of (6.8) we set $\sigma \equiv \sigma(n_\rho)$ and note that $-\sigma J(x, y, z) = J(x, y, zn_\rho) = J(zn_\rho, x, y)$ and $J(xy \cdot z, n_\rho) = (xy, z)n_\rho + (zn_\rho)(xy) + (n_\rho \cdot xy)z = \sigma xy \cdot z - \sigma z \cdot xy + \sigma xy \cdot z = 3\sigma xy \cdot z$. Therefore

$$\begin{aligned} 3\sigma xy \cdot z - \sigma J(x, y, z) &= J(zn_\rho, x, y) + J(xy, z, n_\rho) \\ &= J(zx, y, n_\rho) + J(yn_\rho, z, x) \\ &= \sigma J(y, z, x) \end{aligned}$$

which completes the proof.

Next let $w, x, y, z \in A(\sigma), \sigma = \pm \rho$, then

$$\begin{aligned} wJ(x, y, z) - xJ(y, z, w) &+ yJ(z, w, x) - zJ(w, x, y) \\ &= 3/2[2J(wx, y, z) + 2J(yz, w, x)], \text{ using (3.5)} \\ &= 3/2[2J(y, z, wx) + 2J(w, x, yz)] \\ &= 3/2[3yz \cdot wx + 3wx \cdot yz] \text{ , using (6.8)} \\ &= 0 \text{ .} \end{aligned}$$

This proves the important indentity

(6.9) $$wJ(x, y, z) = xJ(y, z, w) - yJ(z, w, x) + zJ(w, x, y) \text{ for all } w, x, y, z \in A(\sigma), \sigma = \pm \rho \text{ .}$$

THEOREM 6.10. *If $\sigma = \pm \rho$, then the dimension of $A(\sigma)$ is one or three.*

Proof. If the dimension of $A(\sigma)$ is > 1, then there exist linearly independent elements x_i and x_j in $A(\sigma)$. Let y_k denote an element in the corresponding dual basis for $A(-\sigma)$, as in the first part of this section. From (6.8), $3(x_ix_j)y_j = 2J(x_i, x_j, y_j) = 2[(x_ix_j)y_j + (x_jy_j)x_i + (y_jx_i)x_j] = 2[(x_ix_j)y_j - n_\rho x_i]$ and therefore

(6.11) $$\begin{aligned} (x_ix_j)y_j &= -2n_\rho x_i \\ &= 2\sigma x_i \text{ .} \end{aligned}$$

This last equation shows that $x_ix_j \neq 0$. Now $0 \neq y = x_ix_j \in A(-\sigma)$ and therefore there exists $x_k \in A(\sigma)$ so that $yx_k \neq 0$; for, otherwise $yA(\sigma) = 0$ and from this we obtain $(y, A(\sigma)) = 0$ and so $y = 0$. From (6.7) we see that $x_k \neq x_i$ or $x_k \neq x_j$ and so we have three distinct elements $x_i, x_j, x_k \in A(\sigma)$ such that, using (6.5) and (6.4), $0 \neq 3(x_ix_j)x_k = J(x_i, x_j, x_k) = (3/\rho)B(x_ix_j, x_kn_\rho)n_\rho$. Now for any $w \in A(\sigma)$ we have

$$\begin{aligned}(3\sigma/\rho)B(x_ix_j, x_kn_\rho)w &= (3/\rho)B(x_ix_j, x_kn_\rho)wn_\rho \\ &= w[(3/\rho)B(x_ix_j, x_kn_\rho)n_\rho] \\ &= wJ(x_i, x_j, x_k) \\ &= x_iJ(x_j, x_k, w) - x_jJ(x_k, w, x_i) + x_kJ(w, x_i, x_j)\end{aligned}$$

using (6.9). Thus by (6.4) and the action of $n_\rho F$ and $A(\sigma)$ we may conclude that the dimension of $A(\sigma)$ is 3 provided x_i, x_j and x_k are linearly independent. This result is clear by the choice of x_i, x_j and x_k and the fact $(x_ix_j)x_k \neq 0$.

Now if the dimension of $A(\rho)$ is 1, we see that the weight space subalgebra $B(\rho)$ equals $x_\rho F \oplus n_\rho F \oplus x_{-\rho}F$ and is the usual split three dimensional Lie algebra. In the other case when the dimension of $A(\rho)$ is 3, rewrite $\{x_i, x_j, x_k\}$ as $\{x_1, x_2, x_3\}$ and letting $\{y_1, y_2, y_3\}$ be the corresponding dual basis for $A(-\rho)$ we have $B(\rho) = A(-\rho) \oplus n_\rho F \oplus A(\rho)$ where

$$\begin{aligned} &A(\rho) = \{x_1, x_2, x_3\}\ , && A(-\rho) = \{y_1, y_2, y_3\} \text{ and} \\ &x_in_\rho = \rho x_i\ , && y_in_\rho = -\rho y_i\ , \\ &y_ix_j = \delta_{ij}u_\rho && \text{for } i, j = 1, 2, 3\ ; \\ &A^2(\rho) \subset A(-\rho) && \text{and } A^2(-\rho) \subset A(\rho)\ .\end{aligned}$$

Next let x_i, x_j be as above and write $x_ix_j = \sum_{n=1}^3 a(i, j, n)y_n$, then for any other x_k we have

$$(x_ix_j, x_k) = \Sigma_n a(i, j, n)(y_n, x_k) = a(i, j, k)\ .$$

This formula implies $a(i, j, k)$ is a skew-symmetric function for $i, j, k = 1, 2, 3$. Thus since $a(i, j, i) = a(i, j, j) = 0$ we have

$$x_ix_j = a(i, j, k)y_k \quad \text{for} \quad k \neq i \text{ or } j \text{ and } i, j, k = 1, 2, 3\ .$$

Thus setting $a = a(1, 2, 3) \in F$, we have

$$x_1x_2 = ay_3\ , \qquad x_2x_3 = ay_1\ , \qquad x_3x_1 = ay_2\ .$$

Similarly if we set $y_iy_j = \sum_{n=1}^3 b(i, j, n)x_n$ and $b(1, 2, 3) = b \in F$ we see $y_1y_2 = bx_3$, $y_2y_3 = bx_1$ and $y_3y_1 = bx_2$. This gives the following table for multiplication in $B(\rho)$

	n_ρ	x_1	x_2	x_3	y_1	y_2	y_3
n_ρ	0	$-\rho x_1$	$-\rho x_2$	$-\rho x_3$	ρy_1	ρy_2	ρy_3
x_1		0	ay_3	$-ay_2$	n_ρ	0	0
x_2			0	ay_1	0	n_ρ	0
x_3				0	0	0	n_ρ
y_1					0	bx_3	$-bx_2$
y_2						0	bx_1
y_3							0

Now a and b are nonzero, otherwise $B(\rho)$ would not be simple and from (6.11) we see $2\rho x_1 = (x_1x_2)y_2 = ay_3y_2 = -abx_1$ and therefore $2\rho = -ab$.

Next we consider Malcev algebras [2]. This is an algebra which satisfies the identities obtained by introducing commutation $x \circ y = xy - yx$ as a new multiplicative operation in an alternative algebra. In particular if this is done in the split Cayley-Dickson algebra C, then we obtain an eight dimensional anti-commutative Malcev algebra C^-. In this algebra the identity 1 of C is such that $1 \circ x = 0$ for all $x \in C$ and so we set $C^0 = C^-/1F$. It can be shown that C^0 is a simple Malcev algebra which satisfies the identities

$$xy = -yx \quad \text{and} \quad J(x, y, xz) = J(x, y, z)x ,$$

omitting "$\circ$" as the notation for multiplication. Furthermore C^0 has a basis $\{u, e_1, e_2, e_3, e_1', e_2', e_3'\}$ which satisfies the relations

$$\begin{aligned} &ue_i = 2e_i , && ue_i' = -2e_i' , && e_ie_j' = \delta_{ij}u ; \\ &e_1e_2 = 2e_3' , && e_1e_3 = -2e_2' , && e_2e_3 = 2e_1' ; \\ &e_1'e_2' = -2e_3 , && e_1'e_3' = 2e_2 , && e_2'e_3' = -2e_1 . \end{aligned}$$

We now have

THEOREM 6.12. *The root algebra $B(\rho)$ is a seven dimensional Malcev algebra as described above.*

Proof. Since F is algebraically closed, we can find λ, μ, v in F such that $\lambda\rho = -2$, $-\lambda = \mu v$, $a\mu^2 = 2v$ and $bv^2 = -2\mu$. Then make the following change of basis in $B(\rho)$:

$$H = \lambda n_\rho , \quad X_i = \mu x_i , \quad Y_i = vy_i \qquad i = 1, 2, 3$$

and use the multiplication table for the n_ρ, x_i, y_i to see that the H, X_i, Y_i satisfy the above relations for a Malcev algebra with the identification $u = H$, $e_i = X_i$, $e_i' = Y_i$.

We next show that the Lie algebra of linear transformations $R(N)$ can be simultaneously diagonalized in $A = \Sigma\rho A(-\rho) \oplus N \oplus \Sigma A(\rho)$, where the Cartan subalgebra N equals $\Sigma_\rho A(-\rho)A(\rho)$. From the preceding discussion we know the dimension of $A(\rho)$ is one or three. Now in either case $A(\rho)A(-\rho) = n_\rho F$ and therefore N is spanned by such elements n_ρ for $\rho \neq 0$. Thus to show $R(N)$ diagonalizable it suffices to show all the linear transformations $R(n_\sigma)$ have this property on each subalgebra $B(\rho)$.

If the dimension of $A(\rho)$ is one, then this is clear for each $R(n_\sigma)$, where σ is any nonzero weight. If the dimension of $A(\rho)$ is 3, then

$A(\rho) = \{x_1, x_2, x_3\}$ and $A(-\rho) = \{y_1, y_2, y_3\}$ where $\{y_1, y_2, y_3\}$ is a dual basis of $\{x_1, x_2, x_3\}$ and $\{x_1, x_2, x_3\}$ is a basis which simultaneously triangulates every $R(n_\sigma)$. We now show $\{y_1, y_2, y_3\}$ has the same property. For some $a_{ij} \in F$,

$$x_1R(n_\sigma) = \rho(n_\sigma)x_1 \tag{6.13}$$

$$x_2R(n_\sigma) = a_{21}x_1 + \rho(n_\sigma)x_2 \tag{6.14}$$

$$x_3R(n_\sigma) = a_{31}x_1 + a_{32}x_2 + \rho(n_\sigma)x_3 \,. \tag{6.15}$$

We shall show that the $a_{ij} = 0$. Using the properties of $B(x, y)$ and the fact that $J(x_2, n_\sigma, y_1) = 0$ we have

$$\begin{aligned} 0 &= B(J(x_2, n_\sigma, y_1), n_\rho) \\ &= B((x_2n_\sigma)y_1 + (n_\sigma y_1)x_2 + (y_1x_2)n_\sigma, n_\rho) \\ &= B((x_2n_\sigma)y_1, n_\rho) + B((n_\sigma y_1)x_2, n_\rho) \text{ , since } y_1x_2 = 0 \\ &= B((x_2n_\sigma)y_1, n_\rho) + \rho B(n_\sigma y_1, x_2) \text{ , since } x_in_\rho = \rho x_i \\ &= B((x_2n_\sigma)y_1, n_\rho) \end{aligned}$$

using

$$\begin{aligned} B(n_\sigma y_1, x_2) &= \text{trace } \bar{R}(n_\sigma y_1)\bar{R}(x_2) - \text{trace } \bar{R}(n_\sigma)\bar{R}(y_1x_2) \\ &= \text{trace } \bar{R}(n_\sigma y_1 \cdot x_2 + n_\sigma \cdot y_1x_2) = 0 \,, \end{aligned}$$

since $n_\sigma y_1 \cdot x_2 + n_\sigma \cdot y_1x_2 \in n_\rho F$ and trace $\bar{R}(n_\rho) = 0$. Therefore since $(x_2n_\sigma)y_1 \in n_\rho F$, we must have $(x_2n_\sigma)y_1 = 0$, since $B(n_\rho, n_\rho) \neq 0$. Now multiply (6.14) by y_1 to obtain $0 = (x_2n_\sigma)y_1 = a_{21}x_1y_1 + \rho(n_\sigma)x_2y_1 = -a_{21}n_\rho$ and therefore $a_{21} = 0$. Similarly we can show $(x_3n_\sigma)y_1 = (x_3n_\sigma)y_2 = 0$ and conclude $a_{31} = a_{32} = 0$ so that $R(n_\sigma)$ acts diagonally on $A(\rho)$.

Next suppose $y_iR(n_\sigma) = b_{i1}y_1 + b_{i2}y_2 + b_{i3}y_3$ for $i = 1, 2, 3$, then as before we can show $(y_1n_\sigma)x_2 = (y_1n_\sigma)x_3 = 0$ and conclude that $b_{12} = b_{13} = 0$. Similarly for $i = 2, 3$ to obtain that $R(n_\sigma)$ acts diagonally on $A(-\rho)$. Thus for any weights $\rho, \sigma \neq 0$, $R(n_\sigma)$ acts diagonally on $B(\rho)$. But since $N = \Sigma_\sigma n_\sigma F$ and $A = \Sigma_\rho A(-\rho) \oplus N \oplus \Sigma_\rho A(\rho) = \Sigma_\rho B(\rho)$ we have

THEOREM 6.16. *For any $n \in N$, $R(n)$ acts diagonally in A.*

We shall now prove the theorem stated in the introduction. Let $\sigma \neq 0$ and $\rho \neq 0, \pm\sigma$ and let $x \in A(\rho)$, $y \in A(\sigma)$ and suppose $0 \neq xy \in A(\rho)A(\sigma) \subset A(\rho + \sigma)$ so that $\rho + \sigma$ is a weight. Let $z \in A(\rho + \sigma)$, then for any $n \in N$ we have

$$\begin{aligned} J(xy, z, n) &= J(xy, z, n) + J(zn, x, y) \\ &= J(xz, n, y) + J(ny, x, z) \\ &= 0 \end{aligned}$$

using (3.15) for the first and third equalities. Since $z, xy \in A(\rho + \sigma)$ and $J(xy, z, n) = 0$ we can conclude $xy \cdot z \in A(2(\rho + \sigma)) = 0$, using Theorem 5.13. This proves

$$(6.17) \qquad A(\rho)A(\sigma) \cdot A(\rho + \sigma) = 0 \qquad \text{if } \sigma \neq 0 \text{ and } \rho \neq 0, \pm\sigma .$$

Next we show

LEMMA 6.18. *The dimension of* $A(\rho + \sigma)$ *is one if* $\sigma \neq 0$ *and* $\rho \neq 0, \pm\sigma$ *and* $A(\rho)A(\sigma) \neq 0$.

Proof. Suppose dimension $A(\rho + \sigma) = 3$, then form $B(\rho + \sigma) = A(-(\rho + \sigma)) \oplus n_{\rho+\sigma}F \oplus A(\rho + \sigma)$. From the hypothesis we can find an element $z = xy \in A(\rho)A(\sigma) \subset A(\rho + \sigma)$ which is not zero and such that $zA(\rho + \sigma) \neq 0$; this last statement follows from the multiplicative relations for the Malcev algebra $B(\rho + \sigma)$; (briefly: otherwise, $B(z, A(-(\rho + \sigma))) = B(z, A(\rho + \sigma)^2) = B(zA(\rho + \sigma), A(\rho + \sigma)) = 0)$. But the fact $zA(\rho + \sigma) \neq 0$ contradicts (6.17).

Now let $G(\sigma)$ denote a weight space of dimension one and $S(\sigma)$ denote a weight space of dimension three and set

$$G = \Sigma_\sigma(G(-\sigma) \oplus n_\sigma F \oplus G(\sigma))$$
$$S = \Sigma_\rho(S(-\rho) \oplus n_\rho F \oplus S(\rho)) .$$

Then $A = G + S$ and we shall show G is an ideal in A. For any weight ρ with weight space $G(\rho)$ we have $GG(\rho) \subset G$, since the product of two weight spaces of dimension one is at most of dimension one. Next if λ is a weight with weight space $S(\lambda)$ of dimension 3, then $G(\rho)S(\lambda) \subset G(\rho + \lambda)$ by Lemma 6.18. Now if $n_\sigma \in G$, then write $n_\sigma = yx$ where $y \in G(-\sigma)$, $x \in G(\sigma)$. If $z \in S(\lambda)$ (where we know $\lambda \neq \pm\sigma$), we have $0 = J(y, x, z) = yx \cdot z + xz \cdot y + zy \cdot x$ and therefore

$$n_\sigma z = yx \cdot z = - xz \cdot y - zy \cdot x \in G(\sigma)S(\lambda) \cdot G(-\sigma) + G(-\sigma)S(\lambda) \cdot G(\sigma) \subset G .$$

Thus $GS(\lambda) \subset G$. Since $GN \subset G$, we combine the above results to see that G is an ideal of A. But since A is simple $G = 0$ or $G = A$; in the latter case A is a Lie algebra.

So next suppose $G = 0$ and let $\rho \neq 0$ be a fixed weight and $\sigma \neq 0$, $\pm\rho$ any other weight and form $B(\rho) = A(-\rho) \oplus n_\rho F \oplus A(\rho)$.

Case 1. $A(\rho)A(\sigma) = 0$. Then $B(\rho)A(\sigma) = A(-\rho)A(\sigma) + n_\rho A(\sigma)$. Now if $A(-\rho)A(\sigma) = 0$ also, then $n_\rho A(\sigma) = 0$. For let $x \in A(\rho)$, $y \in A(-\rho)$ be such that $xy = n_\rho$ and let $z \in A(\sigma)$, then

$$0 = J(x, y, z) = xy \cdot z + yz \cdot x + zx \cdot y = xy \cdot z = n_\rho z$$

and we would have $B(\rho)A(\sigma) = 0$. Thus we want to show $A(-\rho)A(\sigma) = 0$, so suppose this is not true. Then $0 \neq A(-\rho)A(\sigma) \subset A(\sigma - \rho)$ and from Lemma 6.18 the dimension of $A(\sigma - \rho)$ is one and therefore $A(\sigma - \rho) \subset G = 0$. Thus we may actually conclude $A(-\rho)A(\sigma) = 0$ and this proves

$$B(\rho)A(\sigma) = 0 \quad \text{if } A(\rho)A(\sigma) = 0 .$$

Case 2. $A(\rho)A(\sigma) \neq 0$. Then $0 \neq A(\rho)A(\sigma) \subset A(\rho + \sigma)$ and again by Lemma 6.18 the dimension of $A(\rho + \sigma)$ is one. Thus, as above, this yields $A(\rho)A(\sigma) = 0$, a contradiction. Thus we may conclude from these cases that $B(\rho)A(\sigma) = 0$ if $\rho \neq 0, \sigma \neq 0, \pm\rho$. This yields $B(\rho)A \subset B(\rho)$ which means $B(\rho)$ is a nonzero ideal of A and so $A = B(\rho)$.

REFERENCES

1. N. Jacobson, *Lie Algebras*, Interscience, 1962.

2. A. Sagle, *On simple Malcev algebras over fields of characteristic zero*, Pacific J. Math. **12** (1962), 1057-1078.

3. A. Sagle, *On anti-commutative algebras and analytic loops*, to appear in Canad. J. Math.

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