PACIFIC JOURNAL OF MATHEMATICS
Vol. 15, No. 2, 1965

LAPLACE'S METHOD FOR TWO PARAMETERS

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The behavior for large h and k of the integral

$$I(h, k) = \int_0^a f(t) \exp [- h\phi(t) + k\psi(t)]\, dt$$

is considered under hypotheses which are fulfilled, for example, if f, ϕ, ψ are real analytic, ϕ is strictly increasing, and $\phi(0) = \psi(0) = 0$. In most cases it is assumed that $k = o(h)$ as $h, k \to \infty$. If ν and μ are the respective orders of the first nonvanishing derivatives of ϕ and ψ at the origin, it is found that the behavior of $I(h, k)$ depends on whether:

(1) $0 < \liminf k^\nu h^{-\mu}$ **and** $\limsup k^\nu h^{-\mu} < \infty$,
(2) $k^\nu h^{-\mu} \to 0$, (3) $k^\nu h^{-\mu} \to \infty$ **and** $\psi^{(\mu)}(0) < 0$, **or**
(4) $k^\nu h^{-\mu} \to \infty$ **and** $\psi^{(\mu)}(0) > 0$.

In case (1) it is shown that $I(h, k)$ is asymptotic to a power series in $(k/h)^{1/(\nu-\mu)}$ with coefficients depending on $k^\nu h^{-\mu}$. In case (2) it is shown that $I(h, k)$ is asymptotic to a double power series in $h^{-1/\nu}$ and $kh^{-\mu/\nu}$. In case (3) it is shown that $I(h, k)$ is asymptotic to a double power series in $k^{-1/\mu}$ and $hk^{\nu-\mu}$. In case (4) it is shown that there exist two parameters σ, τ tending to zero as $h, k \to \infty$ such that $\exp(\sigma^{-2})\, I(h, k)$ is asymptotic to a double power series in σ and τ. If $\mu \leqq \nu$ it is proved that the coefficients of the above power series are unique.

It is the purpose of this paper to obtain asymptotic expansions of the integral $I(h, k)$, for $a > 0$, as $k, h \to \infty$. In most cases we assume that h and k are bound by the relation $k = o(h)$. We assume, roughly speaking, that $\phi(t) \sim a_0 t^\nu$ $(a_0 > 0)$, $\psi(t) \sim b_0 t^\mu$, and $f(t) \sim c_0 t^\lambda$ as $t \to 0$. If $k = 0$ and $\nu = 2$ this is the classical Laplace's Method. We will show that the problem divides naturally into four cases: $k^\nu h^{-\mu} \to 0$, $k^\nu h^{-\mu} \to \infty\ (b_0 < 0)$, $k^\nu h^{-\mu} \to \infty\ (b_0 > 0)$, and $k^\nu h^{-\mu}$ is bounded away from both zero and infinity. Tricomi [4] and Fulks [3] have obtained results along this line when $\nu = 2$, $\mu = 1$, and $\lambda = 0$. Tricomi considered a specific integral of this type (related by a change of variable to the incomplete gamma function) and obtained complete expansions in three of the four above cases. Fulks considered a general class of integrals and obtained the first term in all four cases. The methods of both authors depend quite strongly on the quadratic nature of the exponent near the origin. In this paper we will consider aribtrary ν, μ, λ and obtain complete asymptotic expansions in all four cases. The

Received February 18, 1964. This work for this project was sponsored by the National Science Foundation, Grant NSF-G25060.

results of Fulks have been extended by Thomsen [2] in another direction. The author would like to thank Professor W. Fulks for suggesting this problem.

1. Statement of results. Let $f(x)$ and $g(x) \neq 0$ be defined for $x = (x_1, x_2, \cdots, x_n) \varepsilon S$ where S is a subset of Euclidean n space having the origin as a limit point. For each $j = 0, 1, \cdots, N$ let $p_j(x)$ be a homogeneous polynomial in x of degree j. We will use

$$f(x) \sim g(x) \sum_{j=0}^{N} p_j(x)$$

to mean that

$$f(x)[g(x)]^{-1} = \sum_{j=0}^{N} p_j(x) + 0(|x|^{N+1})$$

where $|x| = (x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2}$. If $f(x)$ and $g(x)$ depend on a parameter y we require that the big 0 constant and the coefficients of the polynomials should be uniformly bounded in y.

While in one dimension the polynomials $p_j(x)$ are of necessity unique, in higher dimensions they need not be. In our application of the above definition we will be able to prove a uniqueness result which covers all cases where ν and μ are integers with $\mu \leqq \nu$.

We will consider the integral $I(h, k)$ under the following hypotheses.

H_1. $\phi(t)$ is positive and nondecreasing in $0 < t < a$, and

$$\phi(t) \sim t^{\nu} \sum_{j=0}^{N} a_j t^j \qquad t \to 0 \tag{1.1}$$

where $\nu > 0$ and $a_0 > 0$.

H_2. $\psi(t)$ is measurable and bounded from above in $0 \leqq t \leqq a$, and

$$\psi(t) \sim t^{\mu} \sum_{j=0}^{N} b_j t^j \qquad t \to 0 \tag{1.2}$$

where $\mu > 0$.

H_3. $f(t)$ is Lebesgue integrable in $0 \leqq t \leqq a$, and

$$f(t) \sim t^{\lambda} \sum_{j=0}^{N} c_j t^j \qquad t \to 0 \tag{1.3}$$

where $\lambda \geqq 0$.

We first consider the case where $k^{\nu}h^{-\mu}$ is bounded away from both zero and infinity when h and k are large. We obtain a one dimensional expansion of $I(h, k)$ with coefficients depending on a parameter.

THEOREM 1. *Assume that* $k = o(h)$, $0 < \liminf k^{\nu}h^{-\mu}$ *and that* $\limsup k^{\nu}h^{-\mu} < \infty$. *Let* $x = (k/h)^{1/(\nu-\mu)}$ *and* $y = (k^{\nu}h^{-\mu})^{1/(\nu-\mu)}$. *There* *exist unique functions* $A_n = A_n(y)$ *such that*

$$I(h, k) \sim x^{\lambda+1} \sum_{n=0}^{N} A_n(y) x^n \qquad h, k \to \infty .$$

In particular

$$A_0(y) = c_0 \int_0^\infty t^\lambda \exp\{-y[a_0 t^\nu - b_0 t^\mu]\} dt .$$

In the remaining cases we obtain two dimensional expansions of $I(h, k)$. We next take up the case where $k^\nu h^{-\mu} \to 0$.

THEOREM 2. *Assume that* $k^\nu = o(h^\mu)$ *and that either* $k = o(h)$ *or* $\psi(t) \leqq 0$. *Let* $\xi = h^{-1/\nu}$ *and* $\eta = kh^{-\mu/\nu}$. *There exist constants* B_{mn} *such that*

$$I_{hk} \sim \xi^{\lambda+1} \sum_{m+n \leqq N} B_{mn} \xi^m \eta^n \qquad h, k \to \infty .$$

In particular

$$B_{00} = \nu^{-1} c_0 a_0^{-(\lambda+1)/\nu} \Gamma((\lambda + 1)/\nu) .$$

If $\mu \geqq 1$ *and either* $\mu \leqq \nu$ *or* $\psi(t) \leqq 0$, *the constants* B_{mn} *are unique.*

If $k^\nu h^{-\mu} \to \infty$ we must distinguish the cases $b_0 < 0$ and $b_0 > 0$. We next take up the case where $b_0 < 0$.

THEOREM 3. *Assume that* $h^\mu = o(k^\nu)$, $b_0 < 0$, *and that* $k = o(h)$. *Let* $p = k^{-1/\mu}$ *and* $q = hk^{-\nu/\mu}$. *There exist constants* C_{mn} *such that*

$$I(h, k) \sim p^{\lambda+1} \sum_{m+n \leqq N} C_{mn} p^m q^n \qquad h, k \to \infty .$$

In particular

$$C_{00} = \mu^{-1} c_0 (-b_0)^{(\lambda+1)/\mu} \Gamma((\lambda + 1)/\mu) .$$

If $\nu \geqq \mu + 1$ *the constants* C_{mn} *are unique.*

If $b_0 > 0$ we must make stronger regularity assumptions about the functions ϕ, ψ, and f. We expand $I(h, k)$ in terms of parameters σ and τ which depend less simply on the parameters h and k.

THEOREM 4. *Assume that* $k = o(h)$, $h^\mu = o(k^\nu)$, *and that* λ, μ, *and* ν *are integers. Also assume that* $b_0 > 0$ *and that in some neighborhood of the origin* $t^{-\lambda} f(t) \varepsilon c^{N+1}$ *and that* $t^{-\nu}\phi(t)$, $t^{-\mu}\psi(t) \varepsilon c^{N+3}$ $(N \geqq 0)$. *There exist parameters* σ, τ *which tend to zero as* $h, k \to \infty$, *and unique constants* D_{mn} *such that*

$$I(h, k) \sim \tau^{\lambda+1}\sigma \exp(1/\sigma^2) \sum_{m+n \leqq N} D_{mn}\rho^m \tau^n$$

as $h, k \to \infty$. *In particular*

$$D_{00} = c_0[2\pi/(\nu\mu)]^{1/2} .$$

In a neighborhood of the origin and for sufficiently large h and k τ *is the unique positive solution of*

$$h\phi'(\tau) = k\psi'(\tau) \tag{1.4}$$

and σ *is defined by the relation*

$$\sigma^{-2} = -h\phi(\tau) + k\psi(\tau) . \tag{1.5}$$

In terms of h and k, τ *and* σ *are given by*

$$\tau = [(k\mu b_0)/(h\nu a_0)]^{1/(\nu-\mu)}[1 + 0(\tau)] \tag{1.6}$$

and

$$\sigma = [(h\nu a_0)^{\mu}/(k\mu b_0)^{\nu}]^{1/2(\nu-\mu)}[1 + 0(\tau)] . \tag{1.7}$$

In (1.6) and (1.7) the big 0 term possesses an expansion to the Nth power in τ.

2. Preliminary Lemmas.

The key to our proof will be to express $I(h, k)$ in the form suggested by the following Lemma.

LEMMA 1. *Let*

$$J(x) = \int_0^{a(x)} \alpha(x, t) \exp[-\beta(t) + \gamma(x, t)]dt$$

be defined for $x = (x_1 x_2, \cdots, x_n)$ *in a deleted neighborhood of the origin in* E_n. *Assume that:*

(2.1) $\alpha(x, t)$, $\beta(t)$ *and* $\gamma(x, t)$ *are measurable functions of t for each fixed x.*

(2.2) $\exp[-\beta(t)] \leqq K \exp[-bt^{\lambda}]$ *for some positive* b, λ, K.

(2.3) *There exists a* μ, $0 < \mu < 1$, *and an L such that*

$$\exp[\gamma(x, t)] \leqq L \exp[\mu b t^{\lambda}] , \qquad 0 \leqq t \leqq a(x) .$$

(2.4) *For each fixed t*

$$\alpha(x, t) \sim \sum_{j=0}^{N} \alpha_j(x, t) , \qquad x \to 0 ,$$

$$\gamma(x, t) \sim \sum_{j=1}^{N} \gamma_j(x, t) , \qquad x \to 0$$

where $\alpha_j(x, t)$ and $\gamma_j(x, t)$ are homogeneous polynomials in x of degree j. The coefficients of α_j and γ_j as well as the big 0 constants are uniformly bounded by a polynomial $M(t)$ (which may depend on N).

(2.5) $a(x) \geqq |x|^{-c}$ *for some $c > 0$ and all sufficiently small x.*

Then there exist homogeneous polynomials $p_j(x)$ (of degree j) such that

$$J(x) \sim \sum_{j=0}^{N} p_j(x)\,, \qquad p_0 = \int_0^\infty \alpha(0, t) \exp[-\beta(t)]dt\,.$$

If $J(x)$ depends on a parameter y and if $\mu, \lambda, b, c, K, L, M(t)$ are independent of y then the conclusion of Lemma 1 remains valid (in the sense that the coefficients of $p_j(x)$ and the big 0 constant are uniformly bounded).

Proof. We expand $\exp\gamma(x, t)$ to N terms in order to obtain

$$J(x) = \sum_{j=0}^{N} \frac{1}{j\,!} \int_0^{a(x)} \exp[-\beta(t)]\ \alpha(x, t)[\gamma(x, t)]^j dt + R$$

where

$$R = \frac{1}{(N+1)\,!} \int_0^{a(x)} \exp[-\beta(t)]\ \alpha(x, t)[\gamma(x, t)]^{N+1} \exp A\, dt$$

and A is between 0 and $\gamma(x, t)$. It follows from (2.2) and (2.3) that

$$\exp[-\beta(t) + A] \leqq \exp[-(1-\mu)bt^\lambda]$$

and from (2.4) that

$$|\alpha(x, t)[\gamma(x, t)]^{N+1}| \leqq M_1(t)\,|x|^{N+1}$$

where $M_1(t)$ is a polynomial in t. It follows from (2.4) and the fact that the asymptotic expansion of a product is the product of the asymptotic expansions that

$$\frac{1}{j\,!} \sum_{j=0}^{N} \alpha(x, t)[\gamma(x, t)]^j = \sum_{j=0}^{N} p_j(x, t) + R_1$$

where each $p_j(x, t)$ is a polynomial in x (homogeneous of degree j) whose coefficients are bounded by a polynomial in t and

$$|R_1| \leqq M_2(t)\,|x|^{N+1}\,,$$

where $M_2(t)$ is a polynomial in t. After substituting the preceding results into the expressions for J and R we see that

$$J(x) = \sum_{j=0}^{N} \int_0^{a(x)} \exp[-\beta(t)] p_j(x, t) dt + 0(|x|^{N+1}) .$$

It follows from (2.2) and (2.5) that replacing $a(x)$ by $+\infty$ introduces an exponentially small error. Hence

$$J(x) \sim \sum_{j=0}^{N} p_j(x)$$

where

$$p_j(x) = \int_0^{\infty} \exp[-\beta(t)] p_j(x, t) dt .$$

In particular it is easily shown that

$$p_0 = \int_0^{\infty} \alpha(0, t) \exp[-\beta(t)] dt .$$

This completes the proof of Lemma 1.

The following lemma will help to facilitate the proof of Theorem 4.

LEMMA 2. *If μ and ν are positive integers such that $\mu < \nu$, then*

$$-\mu(t^{\nu} - 1) + \nu(t^{\mu} - 1) \leqq (\mu - \nu)(t - 1)^2$$

for all $t \geqq 0$.

Proof. We assume that $\nu \neq 2$ in which case both sides of the above inequality are identical. Let

$$g(t) = -\mu(t^{\nu} - 1) + \nu(t^{\mu} - 1) - (\mu - \nu)(t - 1)^2 .$$

It is easily verified that $g'''(t)$ has at most one simple zero for positive t and that hence $g''(t)$ at most two simple zeros or one double zero. On the other hand

$$g''(t) = -\mu\nu(\nu - 1)t^{\nu-2} + \mu\nu(\mu - 1)t^{\mu-2} + 2(\nu - \mu)$$

is positive for small positive t and negative for large t from which it follows that $g''(t)$ has an odd number of zeros (including multiplicities). Hence $g''(t)$ has exactly one zero for positive t and $g'(t)$ has at most two zeros for positive t. Since $g(0) = g(1) = 0$, $g'(t)$ has one zero in $(0, 1)$ and it is easily verified that $g'(1) = 0$. It follows that $g(t)$ does not change sign in $(0, 1)$ or in $(1, \infty)$. Since

$$g''(1) = (2 - \mu\nu)(\nu - \mu) < 0$$

for $\nu \geqq 3$ it follows that $g(t) \leqq 0$ for all $t \geqq 0$ which completes the proof.

3. *Proof.* Let $I(h, k) = I_1 + I_2$ corresponding to the intervals $[0, \delta]$ and $[\delta, a]$ respectively. Since $\phi(t)$ is positive and nondecreasing and $\psi(t)$ is bounded from above, say by M, we have

$$\exp[-h\phi(t) + k\psi(t)] \leqq \exp[-h\phi(\delta) + kM], \quad t\varepsilon[\delta, a].$$

If $k = o(h)$ we have

$$|I_2| \leqq \exp[-h\phi(\delta)/2]\int_\delta^a |f(t)|\,dt$$

for all sufficiently large h and k. If $\psi(t) \leqq 0$ the same result holds without the assumption $k = o(h)$. In all four of our theorems we assume either $k = o(h)$ or $k^\nu = o(h^\mu)$. It follows that I_2 is small with respect to any parameter which behaves like a product of powers of h and k. It is therefore sufficient to consider

$$I_1 = \int_0^\delta f(t)\exp[-h\phi(t) + k\psi(t)]dt$$

for arbitrary but fixed $\delta > 0$. We will assume from this point on that δ is so small that the expansions (1.1), (1.2), and (1.3) are valid in $[0, \delta]$.

We turn to the proof of Theorem 1.

Proof of Theorem 1. In addition to our general assumptions we have $k = o(h)$, $0 < \liminf k^\nu h^{-\mu}$, and $\limsup k^\nu h^{-\mu} < \infty$. In particular $x = (k/h)^{1/(\nu-\mu)} \to 0$ and there exist positive constants m, M such that $m < y = (k^\nu/h^\mu)^{1/(\nu-\mu)} < M$ for all large h, k. Let $u(t) = t^{-\lambda}f(t)$, $v(t) = t^{-\nu-1}[a_0t^\nu - \phi(t)]$ and $w(t) = t^{-\mu-1}[\psi(t) - b_0t^\mu]$. Then after replacing t by xs we have

$$x^{-\lambda-1}I_1 = \int_0^{\delta x^{-1}} s^\lambda u(xs)\exp\{-y[a_0t^\nu - b_0t^\mu] + E\}ds$$

where

$$E = xy[s^{\nu+1}v(xs) + s^{\mu+1}w(xs)].$$

The growth rates of h and k imply that $\mu < \nu$ and hence there exists a K such that

$$\exp\{-y[a_0s^\nu - b_0s^\mu]\} \leqq K\exp\{-ma_0s^\nu/2\}$$

for large h and k which shows that (2.2) is satisfied. If L is a bound for v and w we have

$$E \leqq ML\delta[s^\nu + s^\mu], \qquad 0 < s < \delta x^{-1}.$$

Hence (2.3) is satified if δ is sufficiently small. It follows from (1.1) that

$$v(t) = \sum_{j=1}^{N} a_j t^{j-1} + 0(t^N)$$

and that hence $xv(xs)$ has the type of expansion prescribed by (2.4). A similar remark applied to u and w shows that (2.4) is satisfied (with bounds which are independent of y). It is evident that (2.1), and (2.5) are satisfied. Thus by taking δ smaller, if necessary, we see that I_1 has the desired expansion. In particular it follows that $A_0 = A_0(y)$ has the prescribed form. The proof of uniqueness is standard.

Proof of Theorem 2. In addition to our general hypotheses we have $k^\nu = o(h)$ and either $k = o(h)$ or $\psi(t) \leqq 0$. In particular $\xi = h^{-1/\nu}$ and $\eta = kh^{-\mu/\nu} \to 0$ as $h, k \to \infty$. Let $u(t) = t^{-\lambda} f(t)$, $v(t) = t^{-\nu-1} [a_0 t^\nu - \phi(t)]$, and $w(t) = t^{-\mu}\psi(t)$. After replacing t by ξs we have

$$\xi^{-\lambda-1} I_1 = \int_0^{\delta\xi^{-1}} s^\lambda u(\xi s) \exp[-a_0 s^\nu + E] ds$$

where

$$E = \xi s^{\nu+1} v(\xi s) + \eta s^\mu w(\xi s) .$$

It is evident that (2.1), (2.2), and (2.5) are satisfied. In $0 \leqq s \leqq \delta\xi^{-1}$ the estimates (with M a bound for v and w)

$$\begin{aligned} &\xi s^{\nu+1} v(\xi s) \leqq M\delta s^\nu , && \\ &\eta s^\mu w(\xi s) \leqq M\delta^{\mu-\nu} kh^{-1} s^\nu , && \text{if } \mu \geqq \nu , \\ &\eta s^\mu w(\xi s) \leqq M\eta s^\mu , && \text{if } \mu < \nu , \\ \text{and} \quad &s^\mu w(\xi s) \leqq 0 , && \text{if } \psi(t) \leqq 0 , \end{aligned}$$

imply that existence of a constant K such that

$$\exp E \leqq K \exp[(a_0/2)s^\mu]$$

for sufficiently small δ and all large h and K. Hence (2.3) is satisfied. It can be shown that (2.4) is satisfied in the same manner as in the proof of the Theorem 1. This completes the proof that I_1 has the stated expansion.

There remains the question of uniqueness of the coefficients. In terms of ξ and η the relations $k = o(h)$, $k^\nu = o(h^\mu)$, $h \to \infty$, and $k \to \infty$ are $\xi^{\nu-\mu}\eta = o(1)$, $\eta = o(1)$, $\xi^{-\nu} \to \infty$ and $\eta\xi^{-\mu} \to \infty$ respectively. Since uniqueness is asserted only if $\mu \geqq 1$ and $\nu \geqq \mu$ or $\psi(t) \leqq 0$ (in which case we do not need $k = o(h)$), we see that we need consider only the restriction $\mu \geqq 1$ and $\eta\xi^{-\mu} \to \infty$. By subtracting two supposed expansions of $\xi^{-\lambda-1} I(h, k)$ we obtain for some $N \geqq 0$

$$\sum_{j=0}^{N} Z_j \xi^j \eta^{n-j} + 0\left((\xi^2 + \eta^2)^{\frac{N+1}{2}}\right) \equiv 0 .$$

If $\mu > 1$ we may let $\eta = \theta\xi$ without violating $\eta\xi^{-\mu} \to \infty$ and prove that the Z's are all zero. If $\mu = 1$ we let $\eta = \xi^{1-\varepsilon}$. The above identity can then be written

$$\xi^{n(1-\varepsilon)} \sum_{j=0}^{N} Z_j \xi^{j\varepsilon} + 0(\xi^{(n+1)(1-\varepsilon)}) \equiv 0 .$$

If $0 < \varepsilon < 1/(N+1)$ the first term is of lower order that the error term and hence by letting $\xi \to 0$ we can again prove that the Z's are all zero. This completes the proof of Theorem 2.

Proof of Theorem 3. The proof is very similar to the proof of Theorem 2. It suffices to note that in the case $\psi(t) \geqq 0$ we used only the assumption $k^\nu = o(h^\mu)$ and the expansions of ϕ and ψ to prove that I_1 had the stated expansion. It is therefore clear that if $h^\mu = o(k^\nu)$ the same proof provides an expansion of I_1 in terms of the parameters $p = k^{-1/\mu}$ and $q = hk^{-\nu/\mu}$. The existence part of the proof of Theorem 2 is then completed by noting that $b_0 < 0$ implies that $\psi(t) \leqq 0$ in $[0, \delta]$ for small δ.

The uniqueness proof is also similar to that of Theorem 2. We leave it to the reader to carry out the details.

Proof of Theorem 4. In addition to our general hypotheses we assume that λ, ν and μ are integers and that some neighborhood of the origin $t^{-\lambda}f(t)\varepsilon C^{N+1}$ and that $t^{-\nu}\phi(t)$, $t^{-\mu}\psi(t)\varepsilon C^{N+3}$. We also assume that $h^\mu = o(k^\nu)$, $k = o(h)$, and that $b_0 > 0$. In particular it follows that $\mu < \nu$ and that the expansions of f, ϕ, and ψ can be differentiated a suitable number of times.

We begin by proving the existence of a positive τ satisfying (1.4). Let $g(t) = \phi'(t)/\psi'(t)$, $t > 0$, $g(0) = 0$. It follows from the expansions of ϕ and ψ that there exists a $\delta > 0$ such that $g(t)\varepsilon C$, $0 \leqq t \leqq \delta$, and that $g'(t) > 0$, $0 < t < \delta$. Hence if k/h is sufficiently small there exists a unique τ, $0 < \tau < \delta$, such that $g(\tau) = k/h$ which is equivalent to (1.4). After substituting the expansions of ϕ and ψ into (1.4) and (1.5) we see that τ and σ possess the expansions (1.6) and (1.7). The following convenient expressions for h and k are easily proved from (1.6) and (1.7).

$$ha_0 = \sigma^{-2}\tau^{-\nu}[\mu/(\nu - \mu)][1 + 0(\tau)] . \tag{3.1}$$

$$hb_0 = \sigma^{-2}\tau^{-\mu}[\nu/(\nu - \mu)][1 + 0(\tau)] . \tag{3.2}$$

The fact that $\phi(t)$, $\psi(t)\varepsilon C^{N+3}$ implies that in (1.6), (1.7), (3.1) and

(3.2) the term $0(\tau)$ possesses an expansion to the Nth power of τ.

The integral defining I_1 may be written

$$I_1 = \exp(\sigma^{-2}) \int_0^\delta f(t) \operatorname{ext}\left[-\frac{\zeta}{2}(t-\tau)^2 + \Delta\right] dt$$

where

$$\zeta = h\phi''(\tau) - k\psi''(\tau) \tag{3.3}$$

and

$$\Delta = -h[\phi(t) - \phi(\tau)] + k[\psi(t) - \psi(\tau)] + \frac{\zeta}{2}(t-\tau)^2 . \tag{3.4}$$

We next prove the existence of an η, $0 < \eta < 1$, such that

$$\Delta \leqq \frac{\eta}{2}\zeta(t-\tau)^2 . \tag{3.5}$$

for $0 \leqq t \leqq \delta$ if δ is sufficiently small. We first note from (3.1), (3.2) (3.3) and the expansion of ϕ and ψ that

$$\zeta = \nu\mu\sigma^{-2}\tau^{-2}[1 + 0(\tau)] , \tag{3.6}$$

where $0(\tau)$ has an expansion to the Nth power of τ. We separate the proof of (3.5) into three cases: $\nu = 2$, $\nu \geqq 3$ and $t > \tau$, $\nu \geqq 3$ and $t \leqq \tau$.

It follows from Taylor's formula with the Lagrange form of the remainder that

$$\Delta = -[h\phi'''(t_1) - k\psi'''(t_1)]\frac{(t-\tau)^2}{6}$$

where t_1 is between t and τ. If $\nu = 2$ (and hence $\mu = 1$) we have

$$\Delta \leqq M(h+k)\,|\,t-\tau\,|^3$$

where M is an upper bound for $6\,|\,\phi'''(t)\,|$ and $6\,|\,\psi'''(t)\,|$. By substituting 3.1 and 3.2 into the above estimate we see that for $0 \leqq t\,\delta$, and some constant M'

$$\Delta \leqq M'\delta\zeta(t-\tau)^2$$

for h and k large. By choosing δ small we see that (3.5) follows.

If $\nu \geqq 3$ and $t > \tau$ we obtain from the expansion of $\phi'''(t_1)$, $\psi'''(t_1)$ that

$$\Delta = \frac{-(\nu\mu)}{(\nu-\mu)}\sigma^{-2}t_1^{-3}\left\{(\nu-1)(\nu-2)\left(\frac{t_1}{\tau}\right)^\nu - (\mu-1)(\mu-2)\left(\frac{t_1}{\tau}\right)^\mu\right.$$

$$+ [0(\delta) + 0(\tau)][(t_1/\tau)^{\nu} + (t_1/\tau)^{\mu}]\Big\} .$$

For δ and τ small and $t > \tau$ (hence $t_1 > \tau$) the above expression is negative. Since ζ is positive (3.5) follows trivially.

Finally if $\nu \geqq 3$ and $t < \tau$ we use Taylor's formula with the integral form of the remainder to obtain

$$\begin{aligned} & - h[\phi(t) - \phi(\tau)] + k[\psi(t) - \psi(\tau)] \\ = & - \int_{\tau}^{t} (t - x)[h\phi''(x) - k\psi''(x)]dx \\ = & - \frac{\nu\mu}{(\nu - \mu)} \sigma^{-2}\tau^{-2} \int_{\tau}^{t} (t - x)[(\nu - 1)(x/\tau)^{\nu-2} - (\mu - 1)(x/\tau)^{\mu-2} + 0(\tau)]dx \end{aligned}$$

where since $x < \tau$ we have on occasion, replaced $0(x)$ by $0(\tau)$. After evaluating the integral, the above expression becomes

$$\begin{aligned} - [(\nu - \mu)^{-1}]\sigma^{-2}\{\mu[(t/\tau)^2 - 1] - \nu[(t/\tau)^{\mu} - 1]\} \\ + \sigma^{-2}\tau^{-2}0(\tau)(t - \tau)^2 . \end{aligned}$$

After applying Lemma 2 to the above expression we obtain

$$\varDelta \leqq \frac{\sigma^{-2}\tau^{-2}}{2} [\nu\mu - 2 + 0(\tau)](t - \tau)^2$$

from which (3.5) easily follows.

We next make the change of variable

$$t - \tau = \zeta^{-1/2}s$$

in order to obtain

$$\exp(\sigma^{-2})I_1 = \int_{-\zeta^{1/2}\tau}^{(\delta-\tau)\zeta^{1/2}} \zeta^{-1/2}f(\tau + \zeta^{-1/2}s) \exp\left[- \frac{s^2}{2} + \varDelta(t(s))\right]ds$$

which after breaking the integral at zero separates into two integrals to which Lemma 1 can be applied. It is evident that (2.1) and (2.2) are satisfied. (2.3) easily follows from (3.5) by expressing t in terms of s. (2.5) similarly follows from (3.5). It remains to show that (2.4) is satisfied. To this end we expand $\varDelta$ to $N + 2$ terms and obtain

$$\varDelta = - \sum_{j=3}^{N+2} [h\phi^{(j)}(\tau) - k\psi^{(j)}(\tau)]\, \zeta^{-j/2} \frac{s^j}{j\,!} + R .$$

We wish to show that for each fixed s

$$\varDelta = \sum_{1 \leqq m+n \leqq N} A_{mn}(s)\sigma^m\tau^n + 0\left((\sigma^2 + \tau^2)^{\frac{N+1}{2}}\right) .$$

It follows from (1.1) and (1.6) and (3.1) that

$$h\phi^j(\tau)\zeta^{-j/2} = \text{const. } \sigma^{j-2}[1 + 0(\tau)]$$

the $0(\tau)$ term possessing an expansion to the $(N + 2 - j)$th power of τ. In the same fashion it is easily shown that $k\psi^{(j)}(\tau)\zeta^{-j/2}$ has a similar expression. Hence there remains only to handle the remainder term. $N + 3 \geqq \nu$ we use the Lagrange form of the remainder to obtain

$$R = [-h\phi'''(t_1) + k\psi'''(t_1)]\zeta^{-\frac{(N+3)}{2}}\frac{s^{N+3}}{(N+3)!}$$

If M is a common bound for $\phi^{(N+3)}$ and $\psi^{(N+3)}$ we have for h and k so large that $k/h < 1$

$$|R| \leqq 2Mh\zeta^{-\frac{N+3}{2}}\frac{s^{N+3}}{(N+3)!}$$

from which it follows that

$$|R| \leqq K\sigma^{N+1}s^{N+3}$$

where K is constant. If $\mathrm{N} + 3 \leqq \nu$ the remainder requires a more delicate estimate. We write $\phi(t) = t^\nu\ [t^{-\nu}\phi(t)]$, expand t^ν about $t = \tau$, and expand $t^{-\nu}\phi(t)$ to $N + 2$ terms about $t = \tau$. If we then solve for R we will find that it is in a form for which it is easily shown that

$$|R| \leqq [\sigma^2 + \tau^2]^{\frac{N+1}{2}}p(s)$$

where $p(s)$ is a polynomial in s. This shows that Δ has the required expansion. In a similar fashion it is shown that

$$\zeta^{-1/2}f(\tau + \zeta^{-1/2}s) = c_0(\nu\mu)^{-1/2}\sigma\tau^{\nu+1} + \cdots$$

also satisfies the requirements of (2.4). Uniqueness presents no problem since (3.1) and (3.2) show that σ and τ can tend to zero through essentially all positive values. This completes the proof of Theorem 4.

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