PACIFIC JOURNAL OF MATHEMATICS
Vol. 16, No. 1, 1966

SOME CHARACTERIZATIONS OF EXPONENTIAL-TYPE DISTRIBUTIONS

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Let $\mathscr{F} = \{f(x; \delta) = \exp[x\delta + q(\delta)], \delta \in (a, b)\}$ be a family of exponential-type probability density-functions (exp. p.d.f.'s) with respect to a σ-finite measure μ. Let $M(t; \delta), a - \delta < t < b - \delta$, denote the moment generating function (m.g.f.) corresponding to $f(x; \delta) \in \mathscr{F}$, and let $c(t; \delta) = \ln M(t; \delta) = \sum_{k=1}^{\infty} \lambda_k(\delta)t^k/k!$ be the cumulative generating function. The main results pertain to characterizations of certain exp. p.d.f.'s in terms of the cumulants $\lambda_k(\delta)$. First, it is shown that if $M(t; \delta_0)$ is the m.g.f., respectively, of a degenerate, Poisson, or normal law for some $\delta_0 \in (a, b)$, then $M(t; \delta)$ is the m.g.f. of the given law for all $\delta \in (a, b)$, and that infinite divisibility (inf. div) of $M(t; \delta_0)$ for some δ_0 implies inf. div. for all δ. Further, it is shown that if $\varphi(t)$ is a nondegenerate, inf. div. characteristic function (ch. f.) with finite fourth cumulant λ_4, then $\lambda_4 = 0$ if and only if $\varphi(t)$ is the ch.f. of a normal law, while if $\lambda_4 = a\lambda_3 = a^2\lambda_2 \neq 0$, then $\varphi(t)$ is the ch.f. of a Poisson law. Combining these results, it follows that if $M(t; \delta_0)$ is inf. div., and nondegenerate, with $\lambda_4(\delta_0) = 0$, then $M(t; \delta)$ is the m.g.f. of a normal law for all $\delta \in (a, b)$. A similar result characterizes the Poisson law. Finally, it is proved that the normal law is the unique exp. p.d.f. which is symmetric.

An exponential-type family of distributions is defined by probability densities of the form

$$f(y; \delta) = \exp[y\delta + q(\delta)], \qquad a < \delta < b \tag{1}$$

with respect to a σ-finite measure μ over a Euclidean sample space $(\mathfrak{X}, \mathfrak{A})$. It is known ([1], p. 51) that the set of parameter points δ such that $\int \exp[\delta y]d\mu(y) < \infty$, is an interval (finite or not). The binomial, Poisson, normal, gamma, and negative binomial disiributions provide familiar examples of exponential-type distributions.

A few structural properties for this family are considered. Section 2 contains some useful lemmas which are applied in § 3 to obtain some characterizations of the Poisson and normal distributions.

2. Some lemmas.

Patil [3] has shown that a collection of d.f.'s $\{F(x; \delta): \delta \in (a, b)\}$ is of exponential-type if and only if the

Received March 12, 1964 and in revised form July 27, 1964.

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cumulants, $\lambda_k(\delta)$, exist for all k and satisfy

$$\lambda_k(\delta) = \frac{d\lambda_{k-1}(\delta)}{d\delta} \qquad \text{for } k = 2, 3, 4, \cdots \tag{2}$$

Further, he has shown [3, equation (12)] that $M(t; \delta)$ is the moment generating function of an exponential d.f. if and only if $M(t; \delta) = \exp\{q(\delta) - q(\delta + t)\}$. Lehmann ([1], p. 52) has shown that $e^{-q(\delta)}$ is an analytic function of δ for $a < Re\,\delta < b$. It follows that $q(\delta)$ is analytic for $a < Re\,\delta < b$. Then $\lambda_k(\delta)$ is analytic for $a < Re\,\delta < b$ and $k \geqq 1$. Hence, if $\delta_0 \in (a, b)$, there is a neighborhood Δ of δ_0 such that

$$\lambda_j(\delta) = \sum_{k=0}^{\infty} \frac{\lambda_{j+k}(\delta_0)(\delta - \delta_0)^k}{k!} \qquad \text{for } \delta \in \Delta .$$

LEMMA 1. *If $M(t; \delta_0)$ is degenerate for some $\delta_0 \in (a, b)$, then $M(t; \delta)$ is degenerate for all $\delta \in (a, b)$.*

Proof. $M(t; \delta_0)$ degenerate implies $\lambda_j(\delta_0) = 0$ for $j \geqq 2$. Write

$$\lambda_2(\delta) = \sum_{j=0}^{\infty} \frac{\lambda_{2+j}(\delta_0)(\delta - \delta_0)^j}{j!} \qquad \text{for } \delta \in \Delta .$$

Thus, $\lambda_2(\delta) \equiv 0$ for $\delta \in \Delta$. Since $\lambda_2(\delta)$ is analytic for $a < Re\,\delta < b$, we have $\lambda_2(\delta) \equiv 0$ for $\delta \in (a, b)$ and the conclusion follows.

COROLLARY. *If $\lambda_2(\delta_0)$ is different from zero for at least one $\delta_0 \in (a, b)$, then $\lambda_2(\delta)$ is different from zero for all $\delta \in (a, b)$.*

LEMMA 2. *If $M(t; \delta_0)$ is the m.g.f. of a Poisson type distribution for some $\delta_0 \in (a, b)$, then $M(t; \delta)$ is the m.g.f. of a Poisson type distribution for all $\delta \in (a, b)$.*

Proof. By assumption.

$$M(t; \delta_0) = \exp\left\{\frac{\lambda_2(\delta_0)}{c^2}(e^{ct} - 1) + \left(\lambda_1(\delta_0) - \frac{\lambda_2(\delta_0)}{c}\right)t\right\} ;$$

and

$$\lambda_j(\delta_0) = c^{j-2}\lambda_2(\delta_0) \qquad \text{for } j \geqq 2 .$$

If it can be shown that

$$\lambda_j(\delta) = c^{j-2}\lambda_2(\delta) \qquad \text{for } j \geqq 2 \tag{3}$$

and all $\delta \in (a, b)$, then the Lemma will follow. The proof of (3) is by induction on j. Let $h(\delta) = \lambda_3(\delta) - c\lambda_2(\delta)$. Now $h(\delta)$ is analytic for $a < Re\,\delta < b$. Furthermore, $h(\delta_0) = 0$, and

$$\begin{aligned} h^{(k)}(\delta_0) &= \lambda_{3+k}(\delta_0) - c\lambda_{2+k}(\delta_0) \\ &= c^{k+1}\lambda_2(\delta_0) - cc^k\lambda_2(\delta_0) \\ &= 0 . \end{aligned}$$

It follows that $h(\delta) \equiv 0$ for $\delta \in (a, b)$. So $\lambda_3(\delta) = c\lambda_2(\delta)$. Now, assume $\lambda_j(\delta) = c^{j-2}\lambda_2(\delta)$. Differentiation of both sides yields

$$\lambda_{j+1}(\delta) = c^{j-2}\lambda_3(\delta) = c^{j-2}c\lambda_2(\delta) = c^{(j+1)-2}\lambda_2(\delta) .$$

This completes the proof of (3). It follows that

$$M(t; \delta) = \exp\left\{\frac{\lambda_2(\delta)}{c^2}(e^{ct} - 1) + \left(\lambda_1(\delta) - \frac{\lambda_2(\delta)}{c}\right)t\right\} .$$

LEMMA 3. *If $M(t; \delta_0)$ is normal for some $\delta_0 \in (a, b)$, then $M(t; \delta)$ is normal for all $\delta \in (a, b)$.*

Proof. Since $M(t; \delta_0)$ is normal, $\lambda_2(\delta_0) \neq 0$ and $\lambda_j(\delta_0) = 0$ for $j \geqq 3$. Write for $\delta \in \Delta$,

$$\lambda_3(\delta) = \sum_{j=0}^{\infty} \frac{\lambda_{3+j}(\delta_0)(\delta - \delta_0)^j}{j!} = 0 .$$

Then $\lambda_3(\delta) \equiv 0$ for $\delta \in (a, b)$. Because of (2) it follows that $\lambda_j(\delta) = 0$ for $j \geqq 3$. Finally, $\lambda_2(\delta_0) \neq 0$ implies $\lambda_2(\delta) \neq 0$ for any $\delta \in (a, b)$.

LEMMA 4. *If $M(t; \delta_0)$ is infinitely divisible for some $\delta_0 \in (a, b)$, then $M(t; \delta)$ is infinitely divisible for all $\delta \in (a, b)$.*

Proof. If $\lambda_2(\delta_0) = 0$, the result follows from Lemma 1. So assume $\lambda_2(\delta) \neq 0$ for any $\delta \in (a, b)$. Now, (Lukacs [2]), there exists a distribution $G(x; \delta_0)$ such that

$$\lambda_2(\delta_0 + t)/\lambda_2(\delta_0) = \int e^{xt} dG(x; \delta_0)$$

for $t \in (a - \delta_0, b - \delta_0)$. Let δ_1 be an arbitrary element of (a, b). If $t \in (a - \delta_1, b - \delta_1)$, then $t + \delta_1 \in (a, b)$ and $t + \delta_1 - \delta_0 \in (a - \delta_0, b - \delta_0)$. Hence, for $t \in (a - \delta_1, b - \delta_1)$

$$\begin{aligned} \frac{\lambda_2(\delta_1 + t)}{\lambda_2(\delta_1)} &= \frac{\lambda_2[\delta_0 + (t_1 + \delta_1 - \delta_0)]}{\lambda_2(\delta_1)} \\ &= \frac{\lambda_2(\delta_0)}{\lambda_2(\delta_1)} \int e^{(t+\delta_1-\delta_0)} dG(x; \delta_0) = \int e^{tx} dG_1(x; \delta_0) \end{aligned}$$

where $dG_1(x;\delta_0) = (\lambda_2(\delta_0)/\lambda_2(\delta_1))e^{(\delta_1-\delta_0)x}dG(x;\delta_0)$. It is easy to see that $G_1(x;\delta_0)$ is a distribution function. Thus,

$$\lambda_2(\delta_1 + t)/\lambda_2(\delta_1)$$

is a moment generating function for $t \in (a - \delta_1, b - \delta_1)$. Hence, $M(t;\delta_1)$ is infinitely divisible. Since δ_1 is an arbitrary element of (a, b), $M(t;\delta)$ is infinitely divisible for all $\delta \in (a, b)$.

In the following two lemmas, we assume that $f(t)$ is a non-degenerate, infinitely divisible characteristic function (ch. f.) and $\varphi(t) = \log f(t)$ has four derivatives at $t = 0$. Let

$$\lambda_j = \frac{i^j d^j\varphi(0)}{dt^j}, \qquad j = 1, 2, 3, 4 .$$

From the results of Shapiro [4], it is easily deduced that $-(1/\lambda_2)(d^2\varphi(t)/dt^2)$ is the characteristic function of a d.f. with mean λ_3/λ_2 and variance $(\lambda_2\lambda_4 - \lambda_3^2)/\lambda_2^2$.

LEMMA 5. *If $\lambda_4 = 0$, then $f(t)$ is the characteristic function of a normal distribution.*

Proof. $-(1/\lambda_2)(d^2\varphi(t)/dt^2)$ is a characteristic function of a distribution with mean λ_3/λ_2 and variance $(\lambda_2\lambda_4 - \lambda_3^2)/\lambda_2^2$. Thus $\lambda_4 = 0$ implies $\lambda_3 = 0$ since the variance is nonnegative. Therefore, $-(1/\lambda_2)(d^2\varphi(t)/dt^2)$ is the ch. f. of a degenerate distribution with mean 0. Hence,

$$\frac{-1}{\lambda_2}\frac{d^2\varphi(t)}{dt^2} \equiv 1 ;$$

and, it follows that $\varphi(t) = i\lambda_1 t - (\lambda_2 t^2/2)$ for all t.

Note that the single assumption that $\lambda_4 = 0$ does not suffice to ensure normality since the binomial distribution, while not infinitely divisible, with $pq = 1/6$ has $\lambda_4 = 0$.

LEMMA 6. *If $\lambda_4 = a\lambda_3 = a^2\lambda_2 \neq 0$, and $f(t)$ is infinitely divisible, then $f(t)$ is the characteristic function of a Poisson type distribution.*

Proof. $-(1/\lambda_2)(d^2\varphi(t)/dt^2)$ is the ch.f. of a distribution with mean $\lambda_3/\lambda_2 = a$ and variance $(\lambda_2\lambda_4 - \lambda_3^2)/\lambda_2^2 = (a^2\lambda_2^2 - a^2\lambda_2^2)/\lambda_2^2 = 0$. So, $-(1/\lambda_2)(d^2\varphi(t)/dt^2)$ is a ch.f. of a degenerate distribution with mean a. That is,

$$-\frac{1}{\lambda_2}\frac{d^2\varphi(t)}{dt^2} = e^{iat} .$$

It follows that

$$\varphi(t) = \frac{\lambda_2}{a^2}(e^{iat} - 1) + i\left(\lambda_1 - \frac{\lambda_1}{a}\right)t .$$

REMARK 1. It is not sufficient to assume infinite divisibility and $\lambda_3 = \lambda_4 \neq 0$.

EXAMPLE. Let $\varphi(t) = \lambda(e^{it} - 1) + i\lambda t - (t^2/2)$. Then $\lambda_3 = \lambda_4 = \lambda \neq 0$. $\varphi(t)$ is the ch.f. of the composition of normal and Poisson distributions.

REMARK 2. It is not sufficient to assume infinite divisibility and $\lambda_2 = \lambda_3 \neq 0$.

EXAMPLE. Let $\varphi(t) = e^{2it} - 1 - 2t^2$. Then $\lambda_2 = \lambda_3 = 8$.

REMARK 3. It is not sufficient to assume $\lambda_2 = \lambda_3 = \lambda_4 \neq 0$.

EXAMPLE. Let $x_0 = (1 + \sqrt{13})/2$ and $x_1 = 1 - x_0$. Let $p_0 = (x_0 - 1)/(2x_0 - 1)$ and $p_1 = 1 - p_0$. It is easy to see that $0 < p_0, p_1 < 1$. Let $g_1(t) = e^{ix_0t}p_0 + e^{ix_1t}p_1$ and $g_2(t) \equiv 1$. Then, if

$$g(t) = \frac{1}{3}g_1(t) + \frac{2}{3}g_2(t) ,$$

it follows by direct computation that $\lambda_2 = \lambda_3 = \lambda_4 = 1$. Here, $g(t)$ is obviously not an infinitely divisible ch.f..

3. Characterization of the normal aud Poisson distributions.

THEOREM 1. *If $M(t; \delta_0)$ is infinitely divisible and nondegenerate, and if $\lambda_4(\delta_0) = 0$, then $M(t; \delta)$ is the m.g.f. of a normal distribution, for all $\delta \in (a, b)$.*

Proof. By Lemma 5, $M(t; \delta_0)$ is the m.g.f. of a normal distribution. Then by Lemma 3, the conclusion holds for all $\delta \in (a, b)$.

The family of normal distributions has the property that all its members are symmetric distributions. This means that all central moments of odd order vanish; in particular, the third central moment $\mu_3 = \lambda_3$, must vanish. The next theorem, which follows easily from equation (2) and Lemma 3, implies that the normal law is the unique exponential-type distribution which is symmetric.

THEOREM 2. *Let $\mathscr{F} = \left\{F(x; \delta) = \int_{-\infty}^{x} e^{y\delta + q(\delta)}\,d\mu(y);\ \delta \in (a, b)\right\}$ be a family of exponential-type distributions, and assume that $\lambda_3(\delta) = 0$*

for all $\delta \in (a, b)$ *and* $\lambda_2(\delta_0) > 0$ *for some* $\delta_0 \in (a, b)$. *Then* $\mathscr{F}$ *is a family of normal distributions.*

The following question now arises: If, for some $\delta_0 \in (a, b)$, $M(t; \delta_0)$ is infinitely divisible and $\lambda_3(\delta_0) = 0$, must $M(t; \delta)$ be normal? The answer is no.

EXAMPLE. Let $N(t) = e^{-t+t^2/2}$ for $-\infty < t < \infty$,

$$P(t) = \int_0^t\int_0^s N(y)dyds ,$$

and $N_1(t) = e^{P(t)}$. Then, (Lukacs [2]), $N_1(t)$ is an infinitely divisible moment generating function. Clearly,

$$M(t; \mu) = \frac{N_1(t+\mu)}{N_1(\mu)} = e^{-\log N_1(\mu)+\log N_1(\mu+t)}$$

is an exponential-type moment generating function. It is easy to see that $M(t; \mu)$ is infinitely divisible. Now

$$\begin{aligned}\lambda_3(\mu) &= \left.\frac{d^3 \log M(t; \mu)}{dt^3}\right|_{t=0} \\ &= \left.\frac{d^3P(t+\mu)}{dt^3}\right|_{t=0} = \left.\frac{dN(t+\mu)}{dt}\right|_{t=0} = \frac{dN(\mu)}{d\mu} \\ &= (-1+\mu)e^{-\mu+\mu^2/2}\end{aligned}$$

so that $\lambda_3(1) = 0$. However, $\lambda_3(\mu)$ is not identically zero so that $M(t; \mu)$ is not the m.g.f. of a normal distribution for any value of μ. [For $M(t; \mu_0)$ normal would imply $M(t; \mu)$ normal for all μ which, in turn, would imply $\lambda_3(\mu) \equiv 0$.]

THEOREM 3. *If* $M(t; \delta_0)$ *is infinitely divisible for some* $\delta_0 \in (a, b)$, *and if* $\lambda_4(\delta_0) = c\lambda_3(\delta_0) = c^2\lambda_2(\delta_0) \neq 0$, *then* $M(t; \delta)$ *is the m.g.f. of a Poisson type distribution for all* $\delta \in (a, b)$.

Proof. This follows directly from Lemmas 2 and 6.

THEOREM 4. *If* $\lambda_3(\delta) \equiv c\lambda_2(\delta)$ *for all* $\delta \in (a, b)$ *where* $\lambda_2(\delta)$ *and* $\lambda_3(\delta)$ *are cumulants of an exponential-type distribution, then* $M(t; \delta)$ *is the m.g.f. of a Poisson type distribution.*

Proof. First we show by induction that

$$\lambda_{j+2}(\delta) = c^j\lambda_2(\delta) .$$

By assumption, this is true for $j = 1$. Assume now that $\lambda_{j+2}(\delta) =$

$c^j\lambda_2(\delta)$. Differentiating both sides, we get

$$\lambda_{j+3}(\delta) = c^j\lambda_3(\delta) = c^{j+1}\lambda_2(\delta) .$$

Then,

$$\log M(t; \delta) = \frac{\lambda_2(\delta)}{c^2}(e^{ct} - 1) + \left(\lambda_1(\delta) - \frac{\lambda_2(\delta)}{c}\right)t .$$

REMARK. Let $\delta_0, \delta_1 \in (a, b)$. Many of the preceding results would be trivial if there existed constants c, d with $c \neq 0$ such that

$$M(t; \delta_0) = e^{dt}M(ct, \delta_1) .$$

However, that this is not always the case is shown by taking

$$M(t; \delta) = e^{e^{\delta}(e^t - 1)} , \qquad t, \delta \in (-\infty, \infty) .$$

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