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OPERATORS COMMUTING WITH TRANSLATIONS

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This paper is concerned with the representation, in terms of convolutions with pseudomeasures, of continuous linear operators which commute with translations and which transform continuous functions with compact supports on a Hausdorff locally compact Abelian group G into restricted types of Radon measures on G. The two main theorems each assert that any such operator T is of the form $Tf = s * f$ for a suitably chosen pseudomeasure s on G; the assertions differ in detail in respect of the hypotheses imposed on the range of T. The second theorem is an extension of Proposition 2 of [1] from the case in which G is a finite product of lines and/or circles to the general situation.

Preliminaries. The notations are as described in § 1 of [1], with G in place of X, and with the following additions. If $K \subset G$, $C_K(G)$ denotes the set of $f \in C_c(G)$ satisfying supp $f \subset K$. The symbol $M_b(G)$ will denote the set of all bounded Radon measures on G. Continuity of the operators T considered will, in the absence of any indication to the contrary, refer to the inductive limit topology on $C_c(G)$ and the vague topology $\sigma(M(G), C_c(G))$ on $M(G)$ and its subsets. No distinction is drawn between a locally integrable function f on G and the associated measure $fdx \in M(G)$, dx denoting the element of Haar measure on G. In this paper, X will denote the character group of G, the Haar measure $d\xi$ on X being chosen so that the Fourier transformation is an isometry of $L^2(G)$ onto $L^2(X)$.

Prior to stating the representation theorems, we make some remarks about pseudomeasures on G.

Let $A(G)$ denote the space of functions u on G which are inverse Fourier transforms of functions $v \in L^1(X)$:

$$u(x) = \int_X v(\xi)\xi(x)d\xi ;$$

$A(G)$ is a Banach space under the norm

$$\| u \|_A = \int_X | v(\xi) | d\xi \equiv \| v \|_1 .$$

By a pseudomeasure on G is meant a continuous linear functional on $A(G)$, and we denote by $P(G)$ the set of pseudomeasures on G. By $\| \cdot \|_P$ is meant the usual norm on $P(G)$ qua dual of $A(G)$. The Fourier transformation can be defined for pseudomeasures s in such a way that

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$s \to \hat{s}$ is an isometric isomorphism of $P(G)$ onto $L^\infty(X)$. There is an obvious sense in which $M_b(G)$ can be regarded as a subset of $P(G)$.

If G is a finite product of lines and/or circles, one may think of $P(G)$ as comprising exactly those temperate distributions on G whose Fourier-Schwartz transform is an essentially bounded function. It is this identification which provides the link between Proposition 2 of [1] and Theorem 2 below.

If $s \in P(G)$, the mapping $f \to s * f$ is a continuous endomorphism of $L^2(G)$. In connection with Theorem 1 we shall be concerned with the case in which the restriction of this mapping to $C_c(G)$ has a range lying in $M_b(G)$, i.e., equivalently, in $L^1(G)$. The pseudomeasures s having this property form a subset $P^1(G)$ of $P(G)$. Naturally, $P^1(G)$ contains the set $P_c(G)$ of all pseudomeasures with compact supports (in particular, $P^1(G) = P(G)$ when G is compact) and contains also $M_b(G)$. The closed graph theorem shows that, if $s \in P^1(G)$, then to each compact set $K \subset G$ corresponds a number $m_k > 0$ such that

$$\| s * f \|_1 \leqq m_K \| f \| \qquad (f \in C_K(G)) \,, \tag{1.1}$$

where $\| \cdot \|$ denotes the supremum norm. Further comments on $P^1(G)$ are given in § 5 *infra.*

We can now state the two main theorems.

THEOREM 1. *The continuous linear operators T from $C_c(G)$ into $M_b(G)$ which commute with translations are precisely those of the form*

$$Tf = s * f \,, \tag{1.2}$$

where $s \in P^1(G)$.

THEOREM 2. *The continuous linear operators T from $C_c(G)$ into $M_c(G)$ which commute with translations are precisely those of the form* (1.2), *where now $s \in P_c(G)$.*

Theorem 2, combined with the basic properties of pseudomeasures, shows that any continuous linear operator T from $C_c(G)$ into $M_c(G)$ which commutes with translations admits an extension which maps $L^2_c(G)$ into $L^2_c(G)$ and $L^2(G)$ into $L^2(G)$, $L^2_c(G)$ denoting $L^2(G) \cap M_c(G)$, i.e., the set of functions in $L^2(G)$ which vanish a.e. outside a compact subset of G (a property equivalent to saying that the associated measure has a compact support). In § 5 (B) we shall see that, by virtue of Theorem 1, each continuous linear operator from $C_c(G)$ into $M_b(G)$ admits somewhat similar but less evident extensions.

In Theorem (3.2) of [3] G. I. Gaudry has shown that there is a valid analogue of Theorem 1 for the case in which $M_b(G)$ is replaced by the space $M(G)$ of all Radon measures on G, the pseudomeasure s being then replaced by a somewhat more general entity termed a "quasimeasure". Theorem 2 above is used in [3] as an aid in studying the local structure of quasimeasures.

2. In the proof of Theorem 1 we shall need a lemma.

LEMMA. *To each subset W of G containing interior points corresponds a number $c = c_W > 0$ such that*

$$\| F \| < c \cdot \mathrm{Sup}\, \{\| F\hat{f} \| : f \in C_W(G), \| f \| \leqq 1\}$$

for all functions F on X.

Proof. Define

$$N(F) = \mathrm{Sup}\, \{\| F\hat{f} \| : f \in C_W(G), \| f \| \leqq 1\}\ ,$$

which is possibly ∞. If F is unbounded on X, the lemma on p. 281 of [1] shows that $N(F) = \infty$, so that in this case any value of $c > 0$ will suffice (provided the usual conventions are adopted). Assume then that $F \in B(X)$, the space of bounded functions on X. The functional N is evidently a norm on $B(X)$. Moreover, $B(X)$ is complete for N. For suppose that (F_n) is an N-Cauchy sequence in $B(X)$. Evidently, to each $\xi \in X$ corresponds a number $b_\xi > 0$ such that

$$(2.1) \qquad | F(\xi) | \leqq b_\xi \cdot N(F)\ .$$

It follows that $(F_n(\xi))$ is Cauchy for each $\xi \in X$, so that $F = \lim F_n$ exists pointwise on X. For any $\varepsilon > 0$ there exists $n_0 = n_0(\varepsilon)$ such that

$$N(F_m - F_n) \leqq \varepsilon \qquad (m, n > n_0)\ .$$

That is, for any $f \in C_W(G)$ satisfying $\| f \| \leqq 1$,

$$\mathrm{Sup}_{\xi \in X} | F_m(\xi) - F_n(\xi) | | \hat{f}(\xi) | \leqq \varepsilon \qquad (m, n > n_0)\ .$$

On letting $m \to \infty$ it appears that

$$\mathrm{Sup}_{\xi \in X} | F(\xi) - F_n(\xi) | | \hat{f}(\xi) | \leqq \varepsilon \qquad (n > n_0)$$

and hence that

$$N(F - F_n) \leqq \varepsilon \qquad (n > n_0)\ .$$

This shows first that $N(F) < \infty$, and hence that $F \in B(X)$, and then that $F_m \to F$ in the sense of the norm N. Thus $B(X)$ is N-complete.

Reference to (2.1) shows that the supremum norm is lower semicontinuous relative to N. Therefore, this supremum norm is actually continuous relative to N, which is precisely what the lemma asserts.

3. Proof of Theorem 1. The inequality (1.1) makes it plain that, if $s \in P^1(G)$, then (1.2) defines T as a continuous linear operator from $C_c(G)$ into $L^1(G) \subset M_b(G)$ which commutes with translations. Actually T, thus defined, maps $C_c(G)$ into $L^2(G)$ and is continuous for the L^2-topologies.

Turning to the converse, let us first show that the seminorm $f \to \int_G d(|Tf|)$ is continuous on $C_c(G)$. Indeed, integration theory shows that

$$\int_G d(|Tf|) = \operatorname{Sup} \left| \int_G f d(Tf) \right| ,$$

the supremum being taken with respect to those $f \in C_c(G)$ satisfying $\|f\| \leqq 1$. It thus appears that the seminorm $f \to \int_G d(|Tf|)$ is lower semicontinuous on the barrelled space $C_c(G)$, and is therefore continuous.

Accordingly, if $K \subset G$ is compact, there exists a number $m_K > 0$ such that

$$\int_G d(|Tf|) \leqq m_K \|f\| \qquad (f \in C_K(G)) . \tag{3.1}$$

Take now a net (e_i) of nonnegative functions in $C_c(G)$ such that $\int_G e_i dx = 1$ and supp $e_i \subset N_i$, where the N_i form a neighbourhood base at the origin in G. We may assume that all the N_i are contained in some compact set N. If $f \in C_K(G)$, then $\lim e_i * f = f$ uniformly on G and supp $(e_i * f) \subset N + K$. Since T is continuous and commutes with translations, $T(e*f) = Te*f$ for $e, f \in C_c(G)$. So, if $\mu_i = Te_i$, it follows from (3.1) that

$$Tf = \lim T(e_i * f) = \lim \mu_i * f \text{ in } M_b(G) , \tag{3.2}$$

and that

$$\int_G d(|\mu_i * f|) \leqq m_{N+K} \|f\| .$$

Taking the Fourier transform of this relation, it follows that for $f \in C_K(G)$ we have

$$\|\hat{\mu}_i \cdot \hat{f}\| \leqq m_{N+K} \|f\| .$$

Fixing K as any compact set with interior points, and applying the lemma in § 2, we conclude that

$$\mathrm{Sup}_i \, \| \hat{\mu}_i \| < \infty \, .$$

This in turn ensures that the net (μ_i) has a weak limiting point $s \in P(G)$. The net $(\mu_i * f)$ then has $s * f$ as a weak limiting point in $L^2(G)$ and a comparison with (3.2) shows that Tf must coincide with $s*f$, i.e., that (1.2) must hold. Since T maps $C_c(G)$ into $M_b(G)$, s must belong to $P^1(G)$. The proof is complete.

4. **Proof of Theorem 2.** Once again it is evident that, if $s \in P_c(G)$, then (1.2) defines T as a continuous linear map of $C_c(G)$ into $M_c(G)$ which commutes with translations.

For the converse, note that Theorem 1 implies the existence of a pseudomeasure s such that (1.2) holds. The proof of Theorem 1 shows moreover that s is a weak limiting point in $P(G)$ of the measures $\mu_i = Te_i$. Now supp $e_i \subset N$, a compact subset of G. Lemmas 2 and 3 of [2] show that accordingly there is a compact subset K' of G such that supp $\mu_i \subset K'$ for all i. But then it follows that supp $s \subset K'$ too, showing that $s \in P_c(G)$.

REMARK. In Theorem 4.2 of [3] it is remarked that Theorem 2 entails that every quasimeasure with a compact support is a pseudomeasure. Theorem 1 leads to an analogous result, as we now show.

Reference to the proof of Theorem 4.5 of [3] confirms that if q is a quasimeasure on G, then $f \to q*f$ maps $L^2_c(G)$ continuously into $L^2_{loc}(G)$. Let us write

$$\| h \|_1 = \int_G^* | h(x) | \, dx \quad (\leqq \infty)$$

for an arbitrary complex-valued function h on G, so that $h \in L^1(G)$ if and only if h is measurable and $\| h \|_1 < \infty$. Then we have the

COROLLARY. *If q is a quasimeasure on G such that*

$$\| q * f \|_1 < \infty \quad (f \in C_c(G)) \, , \tag{4.1}$$

then q is a pseudomeasure belonging to $P^1(G)$.

Proof. Since $q*f \in L^2_{loc}(G)$, (4.1) shows that $q*f \in L^1(G)$. The preceding remarks show that the mapping $f \to q*f$ has a graph which is closed in $C_c(G) \times L^1(G)$ and is therefore continuous. The assertion therefore follows from Theorem 1.

5. **Concerning $P^1(G)$.** We collect a few results about $P^1(G)$ and its elements.

(A) When G is compact, $P^1(G) = P(G)$ (see § 1). The situation is much more complex when G is noncompact, and we know of no effective and direct characterisation of $P^1(G)$ as a subset of $P(G)$. It is easy to see that if $s \in P^1(G)$, then $\hat{s}$ coincides l.a.e. on each compact subset H of X with the transform of an (H-dependent) function in $L^1(G)$; in particular, $\hat{s}$ is equal l.a.e. on X to a continuous function on X. This shows that $P^1(G)$ is dense in $P(G)$ if and only if G is compact. More elaborate arguments (based on properties of Helson subsets of X; see [4], Chapter 5) will show also that $P^1(G)$ is closed in $P(G)$ if and only if G is compact.

We turn next to a positive assertion which adds interest and weight to Theorem 1.

(B) Suppose that $s \in P^1(G)$, that $2 \leqq p \leqq \infty$, and that p' is defined by $1/p + 1/p' = 1$. Let W be any relatively compact open subset of G, $(a_r)_{r=1}^{\infty}$ any sequence of points of G. Put e_r for the characteristic function of $a_r\bar{W}$. If f is a measurable function on G vanishing outside a compact subset of $E = \bigcup\{a_r\bar{W} : r = 1, 2, \cdots\}$ and such that

$$\|f\|_{*p} \equiv \sum_{r=1}^{\infty} \|fe_r\|_p < \infty\ , \tag{5.1}$$

then $s*f \in L^{p'}(G)$, and furthermore there exists a number $m'_W > 0$ such

$$\|s*f\|_{p'} \leqq m'_W \cdot \|f\|_{*p}\ . \tag{5.2}$$

Proof. Consider first the case in which f is essentially bounded and vanishes outside $\bar{W}$. There exists then a sequence $(f_n)_{n=1}^{\infty}$ of functions in $C_{\bar{W}}(G)$ such that $\|f_n\| \leqq \|f\|_\infty$ and $f_n \to f$ a.e. By (1.1), $\|s*f_n\|_1 \leqq m_{\bar{W}}\|f\|_\infty$ and so the $s*f_n$ have a weak limiting point $\mu \in M_b(G)$. On the other hand, since $f_n \to f$ in $L^2(G)$, $s*f_n \to s*f$ in $L^2(G)$. It follows that $\mu = s*f \in M_b(G) \cap L^2(G) \subset L^1(G)$ and

$$\|s*f\|_1 \leqq \varliminf_{n\to\infty} \|s*f_n\|_1 \leqq m_{\bar{W}}\|f\|_\infty\ . \tag{5.3}$$

We also know that

$$\|s*f\|_2 \leqq \|s\|_P \cdot \|f\|_2\ . \tag{5.4}$$

Now (5.3) and (5.4) and the Riesz convexity theorem combine to show that, for some number $m'_W > 0$ and all $p \geqq 2$, one has

$$\|s*f\|_{p'} \leqq m'_W \cdot \|f\|_p \tag{5.5}$$

whenever $f \in L^p(G)$ vanishes outside $\bar{W}$. By translation, (5.5) remains valid whenever $f \in L^p(G)$ vanishes outside a translated set $a\bar{W}$, where $a \in G$ is arbitary.

Now suppose that f vanishes outside a compact subset of E and and satisfies (5.1). Then $f = \sum_{r=1}^{\infty} f_r$, where $f_r = fe_r$ and where the series converges in $L^p(G)$ and a fortiori in $L^2(G)$. By (5.5),

$$\| s * f_r \|_{p'} \leqq m'_W \cdot \| f_r \|_p , \tag{5.6}$$

so that in particular $\sum_{r=1}^{\infty} (s * f_r)$ is convergent in $L^{p'}(G)$. This latter series is, however, convergent in $L^2(G)$ to $s * f$, whence it appears that $s * f \in L^{p'}(G)$ and, from (5.6), that (5.2) is true. This completes the proof.

REMARKS. (1) In the statement of (B) we assumed that f vanishes outside a compact subset of E merely to ensure that $s * f$ is defined *a priori*. Actually, the proof furnishes a method of extending the definition of $s * f$ to all cases in which f vanishes outside E and satisfies (5.1).

Notice that if $G = R^n$, we can always arrange that the $a_r \bar{W}$ form a covering of R^n by nonoverlapping congruent closed n-dimensional cubes; this is indeed one of the most natural choices of the $a_r \bar{W}$ in this case. Taking $n = 1$, we see that $s \in P^1(R)$ if and only if the operator $f \rightarrow s * f$ maps the Wiener class M_1 ([5], p. 73) into $L^1(R)$; and that any continuous linear operator from M_1 into $L^1(R)$ which commutes with translations is of the form $f \rightarrow s * f$ for a suitably chosen $s \in P^1(R)$.

(2) By virtue of Theorem 1, (B) expresses some nontrivial extension properties possessed by all continuous linear operators from $C_c(G)$ into $M_b(G)$ which commute with translations.

(C) In case $G = R^n = X$, it is simple to specify smoothness conditions on $\hat{s}$ ensuring that a given $s \in P(R^n)$ shall belong to $P^1(R^n)$. In fact, if we define m_n to be 1 if $n = 1$ and to be $2[n/4] + 2$ if $n > 1$ (square brackets denoting the integral part), it is sufficient that each partial derivative of $\hat{s}$ of order at most m_n be expressible as the sum of a function in $L(R^n)$ and a function in $L^2(R^n)$. (The partial derivatives are here understood in the distributional sense.)

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