

A TRANSPLANTATION THEOREM FOR ULTRASPHERICAL COEFFICIENTS

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Let $f(\theta)$ be integrable on $(0, \pi)$ and define

$$a_n = \int_0^\pi f(\theta) \cos n\theta \, d\theta, \quad b_n = n^{1/2} \int_0^\pi f(\theta) P_n(\cos \theta) (\sin \theta)^{1/2} d\theta$$

where $P_n(x)$ is the Legendre polynomial of degree n . Then

$$(1) \quad c \leq \sum_{n=0}^{\infty} |a_n|^{p(n+1)^\alpha} / \sum_{n=0}^{\infty} |b_n|^{p(n+1)^\alpha} \leq C$$

for $1 < p < \infty$, $-1 < \alpha < p - 1$, where C and c depend on p and α but not on f . From this we obtain a form of the Marcinkiewicz multiplier theorem for Legendre coefficients. Also an analogue of the Hardy-Littlewood theorem on Fourier coefficients of monotone coefficients is obtained. In fact, any norm theorem for Fourier functions can be transplanted by (1) to a corresponding theorem for Legendre coefficients.

Actually, the main theorem of this paper deals with ultraspherical coefficients and (1) is just a typical special case, which is stated as above for simplicity.

Let $P_n^\lambda(x)$ be defined by $(1 - 2rx + r^2)^{-\lambda} = \sum_{n=0}^{\infty} P_n^\lambda(x) r^n$ for $\lambda > 0$. The functions $P_n^\lambda(\cos \theta)$ are orthogonal on $(0, \pi)$ with respect to the measure $(\sin \theta)^{2\lambda} d\theta$ and

$$(1) \quad \int_0^\pi [P_n^\lambda(\cos \theta)]^2 (\sin \theta)^{2\lambda} d\theta = \frac{\Gamma(n + 2\lambda)\Gamma(1/2)\Gamma(\lambda + 1/2)}{n!(n + \lambda)\Gamma(\lambda)\Gamma(2\lambda)} = [t_n^\lambda]^{-2}.$$

Observe that $t_n^\lambda = An^{1-\lambda} + O(n^{-\lambda})$ where A will denote a constant whose numerical value is of no interest to us. For simplicity we set $\varphi_n^\lambda(\theta) = t_n^\lambda P_n^\lambda(\cos \theta) (\sin \theta)^\lambda$. The functions $\{\varphi_n^\lambda(\theta)\}_{n=0}^{\infty}$ form a complete orthonormal sequence of functions on $(0, \pi)$ which for $\lambda=1$ reduce to $\{A \sin(n+1)\theta\}_0^\infty$. Also $\lim_{\lambda \rightarrow 0} \varphi_n^\lambda(\theta) = A \cos n\theta$ so the functions $\varphi_n^\lambda(\theta)$ are generalizations of the trigonometric functions which are used in classical Fourier series. For uniformity we define $\varphi_n^0(\theta) = (2/\pi)^{1/2} \cos n\theta$. Later we shall state an asymptotic formula for $\varphi_n^\lambda(\theta)$ which shows another close connection with trigonometric functions. In essence it says that $\varphi_n^\lambda(\theta)$ looks like $\cos[(n + \lambda)\theta - \pi(\lambda/2)]$. All of the facts about φ_n^λ that are quoted without reference are in [15]. Since $\varphi_n^\lambda(\theta)$ are a bounded orthonormal sequence we may consider their Fourier coefficients. Let $f \in L^1(0, \pi)$ and define

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$$a_n^\lambda = \int_0^\pi f(\theta) \varphi_n^\lambda(\theta) d\theta .$$

Let $\|a_n\|_p = [\sum_{n=0}^\infty |a_n|^p]^{1/p}$. Then using M. Riesz's inequality [12] for $b_n = \sum_{k \neq n} a_k / (n - k)$, i.e. $\|b_n\|_p \leq A_p \|a_n\|_p$, $1 < p < \infty$, and Hilbert's inequality, i.e. if $c_n = \sum a_k / (n + k)$ then $\|c_n\|_p \leq A_p \|a_n\|_p$, $1 < p < \infty$, it is easy to show that $\|a_n^1\|_p \leq A_p \|a_n^0\|$ and conversely $\|a_n^0\|_p \leq A_p \|a_n^1\|_p$, $1 < p < \infty$. It is this inequality that we generalize to all $\lambda > 0$. For some of the applications we actually want a slight generalization of the above. Instead of considering the l^p norm we work in a weighted l^p norm,

$$(2) \quad \|a_n\|_{p,\alpha} = \left[\sum_{n=0}^\infty |a_n|^p (n+1)^\alpha \right]^{1/p} .$$

These applications will be given in the last section.

Our main theorem is as follows.

THEOREM 1. *Let $f \in L^1(0, \pi)$ and define a_n^λ as above. Then if $\|a_n\|_{p,\alpha}$ is defined by (2) we have*

$$(3) \quad A \leq \|a_n^\lambda\|_{p,\alpha} / \|a_n^\mu\|_{p,\alpha} \leq A$$

for all $\lambda, \mu \geq 0$ and $1 < p < \infty$, $-1 < \alpha < p - 1$.

It will be sufficient to prove the inequalities (3) when $\mu < \lambda < \mu + 1$. We first give in detail the proof when $\mu = 0$ and $0 < \lambda < 1$. The formulas that we use in this case are all in the literature and are reasonably well known. Also this proof is easier to follow than the proof of the general case. Then we will sketch the proof for general λ, μ , $\mu < \lambda < \mu + 1$. For simplicity we set $a_n^0 = a_n$ and use $\cos n\theta$ instead of $\varphi_n^0(\theta)$.

Let $f_r(\theta) = \sum_{n=0}^\infty a_n r^n \cos n\theta$. Since $f_r(\theta) \rightarrow f(\theta)$ almost everywhere and boundedly in L^1 we have

$$\begin{aligned} a_n^\lambda &= \lim_{r \rightarrow 1} \int_0^\pi f_r(\theta) \varphi_n^\lambda(\theta) d\theta = \lim_{r \rightarrow 1} t_n^\lambda \int_0^\pi f_r(\theta) P_n^\lambda(\cos \theta) (\sin \theta)^\lambda d\theta \\ &= \lim_{r \rightarrow 1} \sum_{k=0}^\infty a_k r^k t_n^\lambda \int_0^\pi P_n^\lambda(\cos \theta) \cos k\theta (\sin \theta)^\lambda d\theta . \end{aligned}$$

We break the sum up into three parts, $0 \leq k \leq [n/2]$, $[n/2] < k < 2n$ and $2n \leq k$. What we need in each of these intervals is a good estimate for $t_n^\lambda \int_0^\pi P_n^\lambda(\cos \theta) \cos k\theta (\sin \theta)^\lambda d\theta = G(k, n)$.

Consider first the case $k \geq 2n$. We use the following well-known representation for $P_n^\lambda(\cos \theta)$ in terms of $\cos j\theta$.

$$(4) \quad P_n^\lambda(\cos \theta) = \sum_{j=0}^n \alpha_j \alpha_{n-j} \cos(n-2j)\theta$$

where $\alpha_j = (j)_\lambda/j! = Aj^{\lambda-1} + O(j^{\lambda-2})$. Then

$$G(k, n) = \sum_{j=0}^n \alpha_j \alpha_{n-j} t_n^\lambda \int_0^\pi (\sin \theta)^\lambda \times [\cos(k - n + 2j)\theta + \cos(k + n - 2j)\theta] d\theta .$$

Since $\int_0^\pi (\sin \theta)^\lambda \cos r\theta d\theta = O(r^{-1-\lambda})$ and $k \geq 2n$ we see that

$$|G(k, n)| \leq A \sum_{j=1}^{n-1} j^{\lambda-1} (n - j)^{\lambda-1} k^{-1-\lambda} n^{1-\lambda} = O((n/k)^\lambda k^{-1}) = O(k^{-1}) .$$

For the theorem that we want the last estimate $O(k^{-1})$ is sufficient. Observe however that we actually have a better estimate. Because of this it is possible to change Theorem 1 to get similar theorems where the Fourier coefficients are defined by $\int_0^\pi f(\theta) P_n^\lambda(\cos \theta) (\sin \theta)^\lambda n^\alpha d\theta$ for various values of α . A possible transplantation then goes to $\int_0^\pi f(\theta) P_n^{\lambda+\beta}(\cos \theta) (\sin \theta)^{\lambda+\beta} n^{\alpha-\beta} d\theta$. Or the $(\sin \theta)^\lambda$ can be omitted from both of these integrals. We mention these facts only because in the dual case different transplantation theorems have been considered by Muckenhoupt and Stein [11] and by the authors [3]. The reason that both types of theorems are true is best seen in the proof of the present theorem, which is essentially easier than either of the theorems in [11] or [3].

Next consider $G(k, n)$ for $k \leq [n/2]$. This time we need a formula of Szegö. For $0 < \lambda < 1$

$$(5) \quad (\sin \theta)^{2\lambda-1} P_n^\lambda(\cos \theta) = \frac{2^{2-2\lambda}}{\Gamma(\lambda)} \frac{\Gamma(n + 2\lambda)}{\Gamma(n + \lambda + 1)} \sum_{j=0}^\infty f_{j,n}^\lambda \sin(n + 2j + 1)\theta$$

where $f_{0,n}^\lambda = 1$ and

$$f_{j,n}^\lambda = \frac{(1 - \lambda)(2 - \lambda) \cdots (j - \lambda)}{j!} \frac{(n + 1) \cdots (n + j)}{(n + \lambda + 1) \cdots (n + \lambda + j)} .$$

See [15, p. 96]. A simple estimate shows that

$$f_{j,n}^\lambda = O(j^{-\lambda}(n + j)^{-\lambda n^\lambda}) .$$

Then

$$\begin{aligned} G(k, n) &= O \left[\sum_{j=0}^\infty n^{1-\lambda} f_{j,n}^\lambda n^{\lambda-1} \int_0^\pi (\sin \theta)^{1-\lambda} \right. \\ &\quad \left. \times [\sin(n + 2j - k + 1)\theta + \sin(n + 2j + k + 1)\theta] d\theta \right] \\ &= O \left[\sum_{j=0}^\infty f_{j,n}^\lambda (n + j)^{-2+\lambda} \right] \\ &= O(n^{-2+\lambda}) + O \left(\sum_{j=1}^\infty j^{-\lambda} (n + j)^{-2} n^\lambda \right) = O(n^{-1}) . \end{aligned}$$

As usual in results of this nature the region where k and n overlap is harder to handle. This is because a Hilbert transform of some sort always seems to arise. This time we not only have the usual Hilbert transform but we also get a strange variant of it. The transformation we encounter is

$$b_n = \frac{1}{n} \sum_{k=\lceil n/2 \rceil}^{2n} a_k \log \frac{n}{|n - k + \lambda + 1|} .$$

In § 2 we prove the following lemma, which we will use in the following argument.

LEMMA 1. If $\{a_n\} \in l^{p,\alpha}$, $1 < p < \infty$, $-1 < \alpha < p - 1$, and

$$b_n = \frac{1}{n} \sum_{k=\lceil n/2 \rceil}^{2n} a_k \log \frac{n}{|n - k + \lambda + 1|}$$

then $\|b_n\|_{p,\alpha} \leq A_p \|a_n\|_{p,\alpha}$.

For reference we state a form of the asymptotic formula for $P_n^\lambda(\cos \theta)$ which we will use, [15, p. 195].

For $0 < \lambda < 1$, $1/n \leq \theta \leq \pi/2$

$$(6) \quad P_n^\lambda(\cos \theta) = \frac{\Gamma(n + 2\lambda)}{\Gamma(n + \lambda + 1)} \left[\frac{A \cos \left\{ (n + \lambda)\theta - \frac{\lambda\pi}{2} \right\}}{(\sin \theta)^\lambda} + \frac{B \cos \left\{ (n + \lambda + 1)\theta - (\lambda + 1)\frac{\pi}{2} \right\}}{(n + \lambda + 1)(\sin \theta)^{\lambda+1}} + O(n^{-2}(\sin \theta)^{-\lambda-2}) \right] .$$

where A and B depend upon λ but not on n .

From this we have

$$(7) \quad t_n^\lambda P_n^\lambda(\cos \theta)(\sin \theta)^\lambda = A \cos \left\{ (n + \lambda)\theta - \frac{\lambda\pi}{2} \right\} + \frac{B \cos \left\{ (n + \lambda + 1)\theta - (\lambda + 1)\frac{\pi}{2} \right\}}{n \sin \theta} + O((n\theta)^{-2}) + O(n^{-1}) ,$$

where $1/n \leq \theta \leq \pi/2$ and the O terms are uniform in n and θ . Also we shall use the fact that $t_n^\lambda P_n^\lambda(\cos \theta)(\sin \theta)^\lambda$ are uniformly bounded functions, [15, 7. 33, 6]. Instead of considering

$$t_n^\lambda \int_0^\pi P_n^\lambda(\cos \theta)(\sin \theta)^\lambda \cos k\theta d\theta$$

we may consider $\int_0^{\pi/2}$ since the integrand is either even or odd with respect to $\theta = \pi/2$. Using (7) we get

$$\begin{aligned}
 t_n^\lambda \int_0^{\pi/2} P_n^\lambda(\cos \theta)(\sin \theta)^\lambda \cos k\theta d\theta &= t_n^\lambda \int_{1/n}^{\pi/2} P_n^\lambda(\cos \theta)(\sin \theta)^\lambda \cos k\theta d\theta + O\left(\frac{1}{n}\right) \\
 &= A \int_{1/n}^{\pi/2} \cos \left\{ (n + \lambda)\theta - \frac{\lambda}{2} \right\} \cos k\theta d\theta \\
 &\quad + B \int_{1/n}^{\pi/2} \frac{\cos \left\{ (n + \lambda + 1)\theta - (\lambda + 1)\frac{\pi}{2} \right\} \cos k\theta d\theta}{n \sin \theta} \\
 &\quad + O\left[\int_{1/n}^{\pi/2} \frac{d\theta}{n^2 \theta^2} \right] + O\left(\frac{1}{n}\right).
 \end{aligned}$$

The last two terms are $O(1/n)$ and the first is $A'/(n - k + \lambda) + O(1/n)$. We need to consider the second term. Using the addition theorem for $\cos \theta$ we get $B/n \int_{1/n}^{\pi/2} [\cos(n - k + 1)\theta / \sin \theta] d\theta$ + three more terms which are similar but easier to handle. Since $1/\sin \theta - 1/\theta$ is a bounded function for $0 < \theta \leq \pi/2$ we may instead consider

$$J = \frac{B}{n} \int_{1/n}^{\pi/2} \frac{\cos(n - k + \lambda + 1)\theta}{\theta} d\theta.$$

Assume first that $k < n + 1 + \lambda$. Then changing variables by $(n - k + \lambda + 1)\theta = y$, we find

$$J = \frac{B}{n} \int_{(n-k+\lambda+1)/n}^1 \frac{\cos y dy}{y} + \frac{B}{n} \int_1^{n-k+\lambda+1} \frac{\cos y dy}{y}.$$

The second term is $O(1/n)$ by an integration by parts. The first term is

$$\begin{aligned}
 \frac{B}{n} \int_{(n-k+\lambda+1)/n}^1 \frac{dy}{y} + O\left(\frac{1}{n} \int_{(n-k+\lambda+1)/n}^1 y dy\right) \\
 = \frac{B}{n} \log \frac{n}{n - k + \lambda + 1} + O\left(\frac{1}{n}\right).
 \end{aligned}$$

If $k > n + \lambda + 1$ we get instead that

$$J = \frac{B}{n} \log \frac{n}{k - n - \lambda - 1} + O\left(\frac{1}{n}\right).$$

Using all of the estimates, we have

$$\begin{aligned}
 \alpha_n^\lambda &= O\left[\frac{1}{n} \sum_{k=0}^{[n/2]} |a_k|\right] + A \sum_{k=[n/2]}^{2n} \frac{a_k}{n - k + \lambda} \\
 &\quad + \frac{B}{n} \sum_{k=[n/2]}^{2n} a_k \log \frac{n}{|n - k + \lambda + 1|} \\
 &\quad + O\left[\frac{1}{n} \sum_{k=[n/2]}^{2n} |a_k|\right] + \lim_{r \rightarrow 1} \sum_{k=2n}^\infty a_k r^k A(k, n)
 \end{aligned}$$

where $A(k, n) = O(k^{-1})$.

To show the $l^{p,\alpha}$ boundedness of these sums we need two forms of Hardy's inequality and M. Riesz's inequality for the discrete Hilbert transform as well as Lemma 1. The relevant forms of Hardy's inequality are in [6], p. 255, # 346 (a), (b), part (α). The continuous analogue of the $l^{p,\alpha}$ boundedness of the discrete Hilbert transform is in [5].

Using these inequalities we see that the first and fourth terms are bounded by Hardy's inequality. By dominated convergence we may let $r \rightarrow 1$ in the fifth term and it is bounded in $l^{p,\alpha}$ by Hardy's inequality. The second term is just the discrete Hilbert transform plus two terms like the first and last terms. Thus it is bounded in $l^{p,\alpha}$. The third term is handled by Lemma 1.

In actual fact the second and third terms given above are not exactly right since the terms in which k and n have opposite parity are zero. The notation to include this is too cumbersome to be worth including and this point causes no trouble.

To show that $\|a_n\|_{p,\alpha} \leq A \|a_n^\lambda\|_{n,\alpha}$ observe that (formally)

$$a_k = \int_0^\pi f(\theta) \cos k\theta d\theta = \sum_{n=0}^\infty a_n^\lambda t_n^\lambda \int_0^\pi P_n^\lambda(\cos \theta) \cos k\theta (\sin \theta)^\lambda d\theta .$$

We have the same $G(k, n)$ that we analysed above and so no more work need be done on it. However there is the problem of Abel summability of ultraspherical expansions. Estimates for the Poisson kernel which allows us to prove the dominated L^1 convergence of the Abel means are in [11, § 4]. The argument that is needed to prove this is well known.

We now consider the general case of Theorem 1 with $\mu < \lambda < \mu + 1$. The proof proceeds along the same general lines but the formulas for P_n^λ that we need are considerably more complicated. To take the place of (4) we need the following result of Gegenbauer [4].

If $0 < \alpha < \beta$ then

$$(8) \quad P_n^\beta(\cos \theta) = \sum_{j=0}^{[n/2]} \alpha_j P_{n-2j}^\alpha(\cos \theta)$$

where

$$\alpha_j = \frac{\Gamma(\alpha)(n - 2j + \alpha)\Gamma(j + \beta - \alpha)\Gamma(n - j + \beta)}{\Gamma(\beta)\Gamma(\beta - \alpha)j!\Gamma(n - j + \alpha + 1)} .$$

Instead of (5) we need a result that follows from (8) and is given in [2]. If $(\beta - 1)/2 < \alpha < \beta$ then

$$(9) \quad (\sin \theta)^{2\alpha} P_n^\alpha(\cos \theta) = \sum_{j=0}^\infty \beta_j P_{n+2j}^\beta(\cos \theta) (\sin \theta)^{2\beta}$$

where

$$\beta_j = \frac{\Gamma(\beta)2^{2\beta-2\alpha}(n+2j)!(n+2j+\beta)\Gamma(n+2\alpha)\Gamma(n+j+\beta)\Gamma(j+\beta-\alpha)}{\Gamma(\beta-\alpha)\Gamma(\alpha)j!n!\Gamma(n+j+\alpha+1)\Gamma(n+2j+2\beta)}.$$

Observe that β_j is positive if $\alpha < \beta$. This result also holds for $\alpha > \beta$ but then the coefficients are no longer positive and changes must be made for $\alpha = \beta + 1, \beta + 2, \dots$, since the right hand side is then a finite sum. A simple computation shows that

$$(10) \quad \alpha_j \sim (n - 2j + \alpha)j^{\beta-\alpha-1}n^{\beta-\alpha-1}$$

and

$$(11) \quad \beta_j \sim (n + 2j)^{2-2\beta}n^{2\alpha-1}j^{\beta-\alpha-1}(n + j)^{\beta-\alpha-1} \sim (n + j)^{1-\beta-\alpha}n^{2\alpha-1}j^{\beta-\alpha-1},$$

for $\alpha < \beta$. For $\alpha > \beta$ and $\alpha \neq \beta + 1, \beta + 2, \dots$, we have

$$|\beta_j| \sim (n + j)^{1-\beta-\alpha}n^{2\alpha-1}j^{\beta-\alpha-1}.$$

By $a_j \sim b_j$ we mean $0 < c \leq a_j/b_j \leq C < \infty$.

If in (9) we let $n = 0$ and use (1) we have

$$(12) \quad \left| \int_0^\pi (\sin \theta)^{2\alpha} P_{2j}^\beta(\cos \theta) d\theta \right| = O(j^{2\beta-2\alpha-2}).$$

Next we need something to take the place of $\cos x \cos y = [\cos(x + y) + \cos(x - y)]/2$ and $\sin x \cos y = [\sin(x + y) + \sin(x - y)]/2$. For the first we use a formula of Dougall which is given in [9] and reduces to it for $\lambda \rightarrow 0$. If $\lambda > 0$ then

$$(13) \quad \frac{P_n^\lambda(x)}{P_n^\lambda(1)} \frac{P_m^\lambda(x)}{P_m^\lambda(1)} = \sum_{k=|n-m|}^{n+m} c_\lambda(k, m, n) \frac{P_k^\lambda(x)}{P_k^\lambda(1)}$$

where $c_\lambda(k, m, n) \geq 0$ and $\sum_k c_\lambda(k, m, n) = 1$. We define $c_\lambda(k, m, n) = 0$ if $k < |n - m|$ or $k > n + m$ and then we may sum on all non-negative k . The numbers c_λ are known [9], but we shall not need them in our argument.

For the second formula above we use the following substitute which again reduces to it for $\lambda \rightarrow 0$. If $\lambda > 0$ then

$$(14) \quad \frac{P_n^{\lambda+1}(x)}{P_n^{\lambda+1}(1)} \frac{P_m^\lambda(x)}{P_m^\lambda(1)} = \sum_{k=|n-m|-2}^{n+m} d_\lambda(k, m, n) \frac{P_k^{\lambda+1}(x)}{P_k^{\lambda+1}(1)}$$

where $d_\lambda \geq 0$ if $n \geq m - 1$. This is found in [1]. From (14) it follows that $\sum_{k=0}^\infty d_\lambda(k, m, n) = 1$ where $d_\lambda(k, m, n) = 0$ if $k < |n - m| - 2$ or $k > n + m$. Finally recall that

$$(15) \quad P_n^\lambda(1) \sim n^{2\lambda-1}.$$

These results are sufficient to allow us to estimate $\int_0^\pi \varphi_n^\lambda(\theta)\varphi_k^\mu(\theta)d\theta$ for $\mu < \lambda < \mu + 1$ and $k \leq n/2$ or $n \leq k/2$. To estimate this integral for $k/2 < n < 2k$ we use the following asymptotic formulas due to Szegö.

LEMMA 2. *Let $\mu > 0$, μ not an integer. Then*

$$P_n^\mu(\cos \theta) = \frac{2}{\pi} \sin \pi \mu \frac{\Gamma(n + 2\mu)}{\Gamma(\mu)} \times \left[\sum_{m=0}^{p-1} \frac{\Gamma(m + \mu)\Gamma(m - \mu + 1)}{\Gamma(n + m + \mu + 1)m!} \frac{\cos \left[(n + m + \mu)\theta - (m + \mu)\frac{\pi}{2} \right]}{(2 \sin \theta)^{m+\mu}} \right] + R_p$$

where

$$|R_p| = O[(\sin \theta)^{-p-\mu}n^{\mu-p-1}]$$

and the O holds uniformly for $0 < \theta < \pi$.

For $\mu = 1, 2, 3, \dots$ we have

LEMMA 3.

$$P_n^\mu(\cos \theta) = 2 \sum_{m=0}^{\mu-1} (-1)^m \binom{m + \mu - 1}{m} \binom{n + 2\mu - 1}{\mu - m + 1} \times \frac{\cos \left[(n + m + \mu)\theta - (m + \mu)\frac{\pi}{2} \right]}{[(2 \sin \theta)(m + \mu)]}.$$

The same estimates hold for an error term in Lemma 3 as in Lemma 2 if one stops before $m = \mu - 1$. These two lemmas are in [14, p. 49 and p. 59]. In fact we do not need the full force of either of these Lemmas but they are relatively inaccessible and not as well known as they should be.

Now to complete the proof of Theorem 1. Let $\mu < \lambda < \mu + 1$ and $f_r(\theta) = \sum_{n=0}^\infty a_n^\mu r^n \varphi_n^\mu(\theta)$. Then by dominated convergence and the boundedness of the Abel means of an ultraspherical expansion we have

$$\alpha_n^\lambda = \lim_{r \rightarrow 1^-} \int_0^\pi f_r(\theta)\varphi_n^\lambda(\theta)d\theta = \lim_{r \rightarrow 1^-} \sum_{k=0}^\infty a_k^\mu r^k \int_0^\pi \varphi_k^\mu(\theta)\varphi_n^\lambda(\theta)d\theta.$$

As above we need to estimate $\int_0^\pi \varphi_k^\lambda(\theta)\varphi_n^\mu(\theta)d\theta = G(k, n)$ for three cases, $k \leq n/2$, $n/2 < k < 2n$, and $2n \leq k$. Consider the third case first. Using (8) and (13) we have

$$\begin{aligned}
 G(k, n) &= t_n^\lambda t_k^\mu \int_0^\pi P_k^\mu(\cos \theta) P_n^\lambda(\cos \theta) (\sin \theta)^{\lambda+\mu} d\theta \\
 &= \sum_{j=0}^{[n/2]} t_n^\lambda t_k^\mu \alpha_j \int_0^\pi P_k^\mu(\cos \theta) P_{n-2j}^\mu(\cos \theta) (\sin \theta)^{\lambda+\mu} d\theta \\
 &= \sum_{j=0}^{[n/2]} t_n^\lambda t_k^\mu \alpha_j P_k^\mu(1) P_{n-2j}^\mu(1) \sum_{l=0}^{\infty} c_\mu(l, k, n-2j) [P_l^\mu(1)]^{-1} \\
 &\quad \cdot \int_0^\pi P_l^\mu(\cos \theta) (\sin \theta)^{\lambda-\mu} (\sin \theta)^{2\mu} d\theta .
 \end{aligned}$$

Then using (10), (12), (15), and recalling that $c_\mu(l, k, n-2j) = 0$ unless $k-n+2j \leq l \leq k+n-2j$ and so $l \sim k$.

For simplification of printing we use n, k, j in the following arguments instead of $n+1, k+1, j+1$, etc. This leads to some infinite terms which clearly aren't infinite and they are to be interpreted in the obvious way.

$$\begin{aligned}
 |G(k, n)| &\leq \sum_{j=0}^{[n/2]} (n)^{1-\lambda} (k)^{1-\mu} (n-2j+\mu) (j)^{\lambda-\mu-1} (n)^{\lambda-\mu-1} (n-2j)^{2\mu-1} (k)^{\mu-\lambda-2} \\
 &\leq A(n/k)^\lambda (k)^{-1} \leq A(k)^{-1} .
 \end{aligned}$$

Next we consider $G(k, n)$ for $k \leq n/2$. Using (9) and (14) we have

$$\begin{aligned}
 G(k, n) &= \sum_{j=0}^{\infty} t_n^\lambda t_k^\mu \beta_j \int_0^\pi P_k^\mu(\cos \theta) P_{n+2j}^{\mu+1}(\cos \theta) (\sin \theta)^{3\mu-\lambda+2} d\theta \\
 &= \sum_{j=0}^{\infty} t_n^\lambda t_k^\mu \beta_j P_k^\mu(1) P_{n+2j}^{\mu+1}(1) \sum_{l=0}^{\infty} d_\mu(l, k, n+2j) [P_l^{\mu+1}(1)]^{-1} \\
 &\quad \cdot \int_0^\pi P_l^{\mu+1}(\cos \theta) (\sin \theta)^{\mu-\lambda} (\sin \theta)^{2\mu+2} d\theta .
 \end{aligned}$$

This time $d_\lambda(l, k, n+2j) = 0$ unless $n/2+j \leq l \leq 2n+4j$ (actually it is zero for many values in this range also but that doesn't matter) and so $l \approx n+2j$ and thus using (11), (12), and (15) we obtain

$$\begin{aligned}
 |G(k, n)| &\leq \sum_{j=0}^{\infty} (n)^{1-\lambda} (k)^{1-\mu} (k)^{2\mu-1} \beta_j \int_0^\pi P_{n+2j}^{\mu+1}(\cos \theta) (\sin \theta)^{3\mu-\lambda+2} d\theta . \\
 &\leq \sum_{j=0}^{\infty} (n)^{1-\lambda} (k)^\mu (n)^{2\lambda-1} (n+j)^{-\lambda-\mu} (j)^{\mu-\lambda} \int_0^\pi P_{n+2j}^{\mu+1}(\cos \theta) (\sin \theta)^{3\mu-\lambda+2} d\theta \\
 &\leq (n)^\lambda (k)^\mu \sum_{j=0}^{\infty} (j)^{\mu-\lambda} (n+j)^{-\lambda-\mu} (n+j)^{-2+2(\mu+1)-2+\lambda-3\mu} \\
 &\leq (n)^\lambda (k)^\mu \left[\sum_{j=0}^n (j)^{\mu-\lambda} n^{-2-2\mu} + \sum_{j=n}^{\infty} (j)^{-2-\mu-\lambda} \right] \\
 &\leq (n)^\lambda (k)^\mu [(n)^{-1-\mu-\lambda} + (n)^{-1-\mu-\lambda}] \leq [k/n]^\mu (n)^{-1} .
 \end{aligned}$$

For the terms with $k/2 \leq n \leq 2k$ we use Lemmas 2 and 3. As in the case $\mu = 0$, $0 < \lambda < 1$ we first reduce the integral to

$$\int_{1/n}^{\pi/2} \varphi_n^\lambda(\theta) \varphi_k^\mu(\theta) d\theta + O(n^{-1})$$

and then terms of the same type as previously appear. The proof is then finished by the same appeal to Hardy's inequality, M. Riesz's inequality, and Lemma 1.

Theorem 1 then follows by a repeated application of the inequalities just proven.

2. A lemma. We now give a proof of Lemma 1. Recall that

$$b_n = \frac{1}{n} \sum_{k=\lceil n/2 \rceil}^{2n} a_k \log \frac{n}{|n + \lambda + 1 - k|}.$$

We define $A_k = \sum_{j=\lceil n/2 \rceil}^k a_j$. Then

$$\begin{aligned} b_n &= \frac{1}{n} \sum_{k=\lceil n/2 \rceil+1}^{2n} (A_k - A_{k-1}) \log \frac{n}{|n + \lambda + 1 - k|} + \frac{1}{n} a_{\lceil n/2 \rceil} \log \frac{n}{\frac{n}{2} + \lambda + 1} \\ &= \frac{1}{n} \sum_{k=\lceil n/2 \rceil+1}^{2n-1} A_k \left[\log \frac{n}{|n + \lambda + 1 - k|} - \log \frac{n}{|n + \lambda - k|} \right] \\ &\quad + O\left(\frac{1}{n} a_{\lceil n/2 \rceil}\right) + O\left(\frac{A_{2n}}{n^2}\right) \\ &= + \frac{1}{n} \sum_{k=\lceil n/2 \rceil}^{2n} A_k \log \left| \frac{n + \lambda - k}{n + \lambda + 1 - k} \right| + R_n \end{aligned}$$

where R_n is a bounded sequence in $l^{p,\alpha}$ if $\{a_n\} \in l^{p,\alpha}$. But

$$\begin{aligned} \log \left| \frac{n + \lambda - k}{n + \lambda + 1 - k} \right| &= -\log \left| 1 + \frac{1}{n + \lambda - k} \right| \\ &= + \frac{1}{k - n - \lambda} + O\frac{1}{(k - n - \lambda)^3}. \end{aligned}$$

So we have

$$b_n = -\frac{1}{n} \sum_{k=\lceil n/2 \rceil}^{2n} \frac{A_k}{n + \lambda - k} + \frac{1}{n} \sum_{k=\lceil n/2 \rceil}^{2n} \frac{A_k}{(k - n - \lambda)^2} + R_n.$$

The second term is a bounded sequence in $l^{p,\alpha}$ by [6, p. 198, # 274]. We write the first term as

$$-\frac{1}{n} \sum_{k=\lceil n/2 \rceil}^{2n} \frac{A_k}{n + \lambda - k} = \sum_{k=\lceil n/2 \rceil}^{2n} \frac{\frac{-A_k}{(k - \lambda)}}{n + \lambda - k} + \sum_{k=\lceil n/2 \rceil}^{2n} \frac{A_k}{n(k - \lambda)}.$$

But $A_k/(k - \lambda)$ is in $l^{p,\alpha}$ and so we have that $\{b_n\}$ is an $l^{p,\alpha}$ sequence by Hardy's inequality and M. Riesz's inequality for the discrete Hilbert transform.

A similar proof also shows that

$$c_n = \sum_{k=\lfloor n/2 \rfloor}^{2n} \frac{a_k}{k} \log \frac{n}{|n - k + \alpha|}$$

is a bounded operator for α not an integer. If α is an integer the transformations are bounded if the term when the logarithm is undefined is dropped.

A similar theorem is also true in the continuous case where an integration by parts takes the place of our summation by parts.

3. Applications. Our first application is an analogue of a theorem of Hardy and Littlewood concerning the Fourier coefficients of even functions, monotonically decreasing in $(0, \pi)$, [16, p. 130]. Their theorem is

THEOREM A. *If $f(\theta)$ is a decreasing integrable function on $(0, \pi)$ and if a_n are the Fourier cosine coefficients of f , then*

$$\left[\sum_{n=0}^{\infty} |a_n|^p (n+1)^\alpha \right]^{1/p}$$

is finite if and only if

$$\left[\int_0^\pi |f(\theta)|^p \theta^{p-2+\alpha} d\theta \right]^{1/p}$$

is finite, $1 < p < \infty$, $-1 < \alpha < p - 1$.

From this and Theorem 1 we obtain

THEOREM 2. *Let $f(\theta)$ be decreasing and integrable on $(0, \pi)$ and $a_n = t_n^\lambda \int_0^\pi f(\theta) P_n^\lambda(\cos \theta) (\sin \theta)^\lambda d\theta$, $0 < \lambda$. Then $\left[\sum_{n=0}^{\infty} |a_n|^p (n+1)^\alpha \right]^{1/p}$ is finite if and only if $\left[\int_0^\pi |f(\theta)|^p \theta^{p-2-\alpha} d\theta \right]^{1/p}$ is finite, $1 < p < \infty$, $-1 < \alpha < p - 1$.*

Another application is the analogue of the Marcinkiewicz Multiplier theorem. In the case of Fourier coefficients it is due to Sunouchi [13] for $\{a_n\} \in l^p$ and to Igari [10] for $\{a_n\} \in l^{p,\alpha}$.

THEOREM B. *Let $f(\theta) \in L^1(0, \pi)$, $a_n = \int_0^\pi f(\theta) \cos n\theta d\theta$, $|t(\theta)| \leq C$,*

$$\int_{\pi/2-n-1}^{\pi/2-n} |dt(\theta)| \leq C, \quad n = 0, 1, \dots$$

Then if $b_n = \int_0^\pi t(\theta) f(\theta) \cos n\theta d\theta$ and $\{a_n\} \in l^{p,\alpha}$, $1 < p < \infty$, $-1 < \alpha < p - 1$, we have $\{b_n\} \in l^{p,\alpha}$ and $\|b_n\|_{p,\alpha} \leq A \|a_n\|_{p,\alpha}$.

From this we get a form of the Marcinkiewicz theorem for ultra-

spherical coefficients.

THEOREM 3. Let $f(\theta) \in L^1(0, \pi)$, $a_n = t_n^\lambda \int_0^\pi f(\theta) P_n^\lambda(\cos \theta) (\sin \theta)^\lambda d\theta$, $\lambda > 0$, $|t(\theta)| \leq C$,

$$\int_{\pi^{2-n-1}}^{\pi^{2-n}} |dt(\theta)| \leq C, \quad n = 0, 1, \dots$$

Then if $b_n = t_n^\lambda \int_0^\pi t(\theta) f(\theta) P_n^\lambda(\cos \theta) (\sin \theta)^\lambda d\theta$ and if $\{a_n\} \in l^{p,\alpha}$, $1 < p < \infty$, $-1 < \alpha < p - 1$ then $\{b_n\} \in l^{p,\alpha}$ and $\|b_n\|_{p,\alpha} \leq A \|a_n\|_{p,\alpha}$.

For $p = 2$ Hirschman has already obtained a form of the Marcinkiewicz theorem. If we let

$$t_r(\theta) = \begin{cases} 1 & 0 \leq \theta \leq 1/r \\ 0 & 1/r < \theta \leq \pi \end{cases}$$

then we get the projection theorem of Hirschman [8] but only for ultraspherical coefficients. Hirschman proves his result for Jacobi coefficients and presumably Theorem 1 is also true for Jacobi polynomials. However this is still open.

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