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TOEPLITZ OPERATORS ON H_p

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A Toeplitz operator is an operator with a matrix representation $(\alpha_{m-n})_{m,n=0}^{\infty}$ where the α_n are the Fourier coefficients of a bounded function φ. The operator may be considered as acting on any of the Hardy spaces $H_p(1 < p < \infty)$ and it is the purpose of this note to show that the spectrum of any such operator is a connected set.

The Hardy space $H_r(1 \leqq r \leqq \infty)$ consists of those functions in $L_r(-\pi, \pi)$ whose Fourier coefficients corresponding to negative values of the index all vanish. If $f \in L_p(1 < p < \infty)$ with

$$f \sim \sum_{n=-\infty}^{\infty} c_n e^{in\theta}$$

then by a well-known theorem of M. Riesz the series

$$\sum_{n=0}^{\infty} c_n e^{in\theta}$$

is the Fourier series of a function Pf belonging to L_p (and so to H_p), and moreover

$$\| Pf \|_p \leqq A_p \| f \|_p$$

where A_p is a constant depending only on p. Thus P is a bounded projection from L_p to H_p.

(We use the following convention. When we speak of L_r or H_r then we assume only $1 \leqq r \leqq \infty$; but when we speak of L_p or H_p then we require $1 < p < \infty$.)

Now let $\varphi \in L_\infty$. We define the Toeplitz operator T_φ on H_p by

$$T_\varphi f = P(\varphi f) .$$

Clearly T_φ is a bounded operator with norm at most $A_p \| \varphi \|_\infty$. In a previous paper [3] it was shown that for $p = 2$ the spectrum of T_φ is connected for all φ. The proof made use of a theorem of Helson and Szegö [2] which characterized those measures $d\mu$ with the property that P (restricted to the trigonometric polynomials) is bounded in the norm of $L_2(d\mu)$. It is not at present known whether the analogue of this theorem holds for $p \neq 2$, but we shall present here a new proof which avoids using the Helson-Szegö theorem and which holds for arbitrary p.

Here is an outline of the proof. It suffices to show that if C is any simple closed curve in the complex plane which is disjoint from $\sigma(T_\varphi)$, the spectrum of T_φ, then $\sigma(T_\varphi)$ lies entirely inside or entirely outside C. For $\lambda \in C$ the equation $T_\varphi f = \lambda f + 1$ has a solution $f = f_\lambda \in H_p$ which can be shown to satisfy a differential equation whose solution is

$$f_\lambda = f_{\lambda_0} \exp\left(\int_{\lambda_0}^{\lambda} P \frac{1}{\varphi - \mu} d\mu\right) \tag{1}$$

where λ_0 is a fixed point of C. (This fact, in a somewhat different setting, was observed by Atkinson [1] and used by him to obtain very simply the solution of a large class of operator equations.) If one takes the path of integration to be the entire curve C then it can be shown very easily from (1) that $R(\varphi)$, the essential range of φ, lies either entirely inside or entirely outside C. In the latter case, say, (1) shows how to continue f_λ analytically to the inside of C. Now there is an explicit formula which gives the solution of the equation

$$T_\varphi h = \lambda h + k \tag{2}$$

in terms of f_λ for $\lambda \notin \sigma(T_\varphi)$. But then this formula shows us how to continue $h = h_\lambda$ analytically to the inside of C and this continuation will provide the unique solution of (2). Thus we shall have shown that $\sigma(T_\varphi)$ lies entirely outside C.

The f_λ we have been speaking about is an analytic function of λ whose values are measurable functions, and we must develop a little bit of theory of such things.

Let Ω be an open set in the complex plane and assume that for each $\lambda \in \Omega$ there is associated a measurable function f_λ on a finite measure space E. (All functions considered will tacitly be assumed to be finite a.e.) We shall say that f is analytic in Ω if for each $\lambda_0 \in \Omega$ there is a disc

$$D(\lambda_0, \delta) = \{\lambda\colon |\lambda - \lambda_0| < \delta\}$$

and a sequence $a_0, a_1, \cdots$ of measurable functions such that for all $\lambda \in D(\lambda_0, \delta)$ the series

$$\sum_{n=0}^{\infty} a_n(\lambda - \lambda_0)^n \tag{3}$$

converges a.e. to f_λ. we shall say that f is L_r-analytic if each a_n belongs to L_r and for each $\lambda \in D(\lambda_0, \delta)$ the series (3) converges to f_λ in the norm of L_r.

LEMMA 1. *If f is L_r analytic then it is analytic.*

Proof. Since L_r-analyticity implies L_1-analyticity we may assume

$r = 1$. It suffices to show that if (3) converges L_1 for all $\lambda \in D(\lambda_0, \delta)$ then it converges a.e. for all $\lambda \in D(\lambda_0, \delta)$. Suppose $\delta_1 < \delta$. Then there is a constant A such that $\| a_n \|_1 \leqq A\delta_1^{-n}$ for all n. Let $\delta_2 < \delta_1$. Then if we set

$$E_n = \{\theta\colon | a_n(\theta) | \geqq \delta_2^{-n}\}$$

we have

$$A\delta_1^{-n} \geqq \int_{E_n} | a_n(\theta) | \, d\theta \geqq \delta_2^{-n} | E_n | ,$$

where $|E_n|$ denotes the measure of E_n. Thus

$$| E_n | \leqq A\left(\frac{\delta_1}{\delta_2}\right)^{-n}$$

and so $\sum | E_n | < \infty$. This shows that almost all θ belong to only finitely many E_n; that is, for almost all θ we have $| a_n(\theta) | < \delta_2^{-n}$ for sufficiently large n. Therefore for almost all θ the series (3) converges for each $\lambda \in D(\lambda_0, \delta_2)$. But δ_2 was an arbitrary number smaller than δ. If we take for δ_2 successively $(1 - k^{-1})\delta \, (k = 1, 2, \cdots)$ we deduce that for almost all θ the series (3) converges for all $\lambda \in D(\lambda_0, \delta)$.

The next lemma is a partial converse of Lemma 1.

LEMMA 2. *Suppose f is analytic in Ω. Then for any $\varepsilon > 0$ there is a set E_ε whose complement in E has measure at most ε such that f, when restricted to E_ε, is L_∞-analytic in Ω.*

Proof. First consider a disc $D(\lambda_0, \delta)$ throughout which (3) converges a.e. to f_λ. Then the series

$$\sum_{n=0}^{\infty} a_n \left(\frac{\delta}{2}\right)^n \tag{4}$$

converges a.e. and so by Egoroff's theorem there is a set F_ε whose complement has measure at most ε on which (4) converges uniformly. There is a constant M such that for all $\theta \in F_\varepsilon$ and all n we have

$$| a_n(\theta) | \leqq \left(\frac{\delta}{2}\right)^{-n} M . \tag{5}$$

Now let λ_1 be any point in the disc $D(\lambda_0, \delta/2)$. Then (5) shows that for

$$\lambda \in D\left(\lambda_1, \frac{\delta}{2} - | \lambda_1 - \lambda_0 |\right)$$

the series (3), which converges a.e. to f_λ, may be rearranged into a power series in $\lambda - \lambda_1$ which converges uniformly for $\theta \in F_\varepsilon$. This shows that f restricted to F_ε is L_∞-analytic in $D(\lambda_0, \delta/2)$.

Now we can find a countable set of discs $D(\lambda_j, \delta_j)$ $(j = 1, 2, \cdots)$ of the type just considered and such that

$$\Omega = \bigcup_{j=1}^{\infty} D(\lambda_j, \left(\frac{\delta_j}{2}\right) .$$

For each j there is a set $F_{\varepsilon,j}$ whose complement has measure at most $2^{-j}\varepsilon$ and such that f restricted to $F_{\varepsilon,j}$ is L_∞-analytic in

$$D\left(\lambda_j, \frac{\delta_j}{2}\right) .$$

But then

$$E_\varepsilon = \bigcap_{j=1}^{\infty} F_{\varepsilon,j}$$

has complement of measure at most ε and f restricted to E_ε is L_∞-analytic throughout Ω.

LEMMA 3. *Let C be a simple closed curve contained in a simply connected open set Ω. Suppose f is analytic in Ω and*

$$\sup_{\mu \in C} \| f_\mu \|_r = M < \infty .$$

Then f is L_r-analytic inside C and for all λ inside C we have

$$\| f_\lambda \|_r \leqq M.$$

Proof. Let λ_0 be inside C and let δ be so small that $D(\lambda_0, \delta)$ is entirely inside C and

$$f_\lambda = \sum_{n=0}^{\infty} a_n (\lambda - \lambda_0)^n$$

a.e. for each $\lambda \in D(\lambda_0, \delta)$. The beginning of the proof of Lemma 2 showed that if we restrict ourselves to an appropriate set E_ε, with complement of measure at most ε, the series in (6) converges uniformly as long as $\lambda \in D(\lambda_0, \delta/2)$. Take any $g \in L_\infty$. Then we can conclude

$$\int_{E_\varepsilon} f_\lambda g d\theta = \sum_{n=0}^{\infty} \left(\int_{E_\varepsilon} a_n g d\theta \right) \left(\lambda - \lambda_0 \right)^n \qquad \lambda \in D\left(\lambda_0, \frac{\delta}{2}\right) .$$

It follows from the Cauchy inequalities that

$$\left| \int_{E_\varepsilon} a_n g d\theta \right| \leqq \left(\frac{\delta}{2} \right)^{-n} \max_{|\lambda - \lambda_0| = \delta/2} \left| \int_{E_\varepsilon} f_\lambda g d\theta \right| .$$

But since f restricted to E_ε is L_∞-analytic in Ω,

$$\int_{E_\varepsilon} f_\lambda g d\theta$$

is a complex-valued analytic function in Ω, and so for any λ inside C we have

$$\left|\int_{E_\varepsilon} f_\lambda g d\theta\right| \leqq \max_{\mu \in C}\left|\int_{E_\varepsilon} f_\mu g d\theta\right| \leqq M \| g \|_s , \tag{7}$$

where $s = r/(r-1)$. Consequently

$$\left|\int_{E_\varepsilon} a_n g d\theta\right| \leqq \left(\frac{\delta}{2}\right)^{-n} M \| g \|_s$$

for all $g \in L_\infty$, and so

$$\left\{\int_{E_\varepsilon} | a_n |^r d\theta\right\}^{1/r} \leqq \left(\frac{\delta}{2}\right)^{-n} M .$$

Since $\varepsilon > 0$ was arbitrary it follows that

$$\| a_n \|_r \leqq \left(\frac{\delta}{2}\right)^{-n} M ,$$

and so the series in (6) converges in L_r for each $\lambda \in D(\lambda_0, \delta/2)$. Thus f is L_r-analytic inside Ω. Finally (7), with E_ε replaced by E, gives $\| f_\lambda \|_r \leqq M$.

We shall have to deal later with the derivative of analytic function. If f is analytic in Ω we define f' as follows: if f_λ is given a.e. as the sum of the series (3) for $\lambda \in D(\lambda_0, \delta)$ then we set

$$f'_\lambda = \sum_{n=0}^{\infty} n a_n (\lambda - \lambda_0)^{n-1} \qquad \lambda \in D(\lambda_0, \delta) .$$

We leave as exercises for the reader the verification that for each $\lambda \in D(\lambda_0, \delta)$ the above series converges a.e. and that if

$$\lambda \in D(\lambda_0, \delta_0) \cap D(\lambda_1, \delta_1)$$

then the two possible interpretations of f'_λ agree a.e., so that f'_λ is well defined and, of course, analytic. We also leave it to the reader to show that if f is L_r-analytic then the same is true of f'.

Let us return to our Toeplitz operators T_φ acting on L_p. We denote by $\rho(T_\varphi)$ the resolvent set of T_φ, that is, the complement of $\sigma(T_\varphi)$. Recall that the essential range of φ is denoted by $R(\varphi)$.

LEMMA 4. *$\sigma(T_\varphi)$ contains $R(\varphi)$.*

Proof. Suppose $\lambda \in \rho(T_\varphi)$. Then for some constant A we have

$$\| P(\varphi - \lambda)f \|_p \geqq A \| f \|_p$$

for all $f \in H_p$, so with another constant A' we have

$$\| (\varphi - \lambda)f \|_p \geqq A' \| f \|_p .$$

If g is an arbitrary trigometric polynomial we shall have $f = e^{im\theta}g \in H_p$ for some m. Then

$$\| (\varphi - \lambda)e^{im\theta}g \|_p \geqq A' \| e^{im\theta}g \|_p$$

and of course this is exactly

$$\| (\varphi - \lambda)g \|_p \geqq A' \| g \|_p .$$

It follows that $|\varphi - \lambda| \geqq A'$ almost everywhere.

LEMMA 5. *If* $\lambda \in \rho(T_\varphi)$ *then* $T_{(\varphi-\lambda)^{-1}}$, *as an operator on* $H_q(q = p/p - 1)$, *is invertible.*

Proof. The adjoint of $T_\varphi - \lambda I$ is the operator $T_{\overline{\varphi - \lambda}}$ acting on H_q. (Here we use the identification of H_q with H_p^* obtained by identifying the function $g \in H_q$ with the linear functional $f \to \int f\bar{g}d\theta$ on H_p.) Therefore $T_{\overline{\varphi-\lambda}}$ is invertible on H_q. Let

$$u = \exp(-2P\log|\varphi - \lambda|)$$

Then $c|\varphi - \lambda|^{-2} = u\bar{u}$ for some constant c, and since by Lemma 4 $|\varphi - \lambda|^{-1} \in L_\infty$ both u and u^{-1} belong to H_∞. For $g \in H_q$ we have

$$\begin{aligned} c(\varphi - \lambda)^{-1}g &= \overline{\varphi - \lambda}u\bar{u}g \\ &= \bar{u}P\overline{\varphi - \lambda}ug + \bar{u}\bar{v} \qquad v \in H_q^\circ \end{aligned}$$

(H_r° denotes the H_r functions with mean zero) and so

$$cP(\varphi - \lambda)^{-1}g = P(\bar{u}P\overline{\varphi - \lambda}ug) .$$

This shows that

$$cT_{(\varphi-\lambda)^{-1}} = T_{\bar{u}}T_{\overline{\varphi-\lambda}}T_u . \tag{8}$$

We have seen that $T_{\overline{\varphi-\lambda}}$ is invertible on H_q. Since $u^{-1} \in H_\infty$ the same is true of T_u. Since similarly T_u is invertible on H_p, its adjoint $T_{\bar{u}}$ is invertible on H_q. Thus the three operators on the right of (8) are all invertible and the lemma is established.

For any $\lambda \in \rho(T_\varphi)$ we shall denote by f_λ, g_λ the unique solutions of

$$T_{(\varphi-\lambda)}f_\lambda = 1,\ T_{(\varphi-\lambda)^{-1}}g_\lambda = 1 \tag{9}$$

in H_p, H_q respectively. The existence and uniqueness of g_λ are guaranteed by Lemma 5.

In the following lemma we shall be integrating $P(\varphi-\mu)^{-1}$ over a path lying in $\rho(T_\varphi)$. It follows from Lemma 4 that $(\varphi-\mu)^{-1}$ is L_p-continuous on this path and consequently the same is true of $P(\varphi-\mu)^{-1}$. Therefore there is no difficulty making sense of the integral. We shall interpret it as a weak integral.

LEMMA 6. *Let Γ be a rectifiable curve lying in $\rho(T_\varphi)$ and having initial and terminal points λ_0, λ respectively. Then*

$$f_\lambda = f_{\lambda_0}\exp\left\{\int_\Gamma P(\varphi-\mu)^{-1}d\mu\right\}, \tag{10}$$

$$g_\lambda = g_{\lambda_0}\exp\left\{-\int_\Gamma P(\varphi-\mu)^{-1}d\mu\right\}. \tag{11}$$

Proof. It follows from (9) that

$$(\varphi-\lambda)f_\lambda = 1+\bar{u}_\lambda \qquad u_\lambda \in H_p^\circ \tag{12}$$

$$(\varphi-\lambda)^{-1}g_\lambda = 1+\bar{v}_\lambda \qquad v_\lambda \in H_q^\circ\,. \tag{13}$$

Therefore $f_\lambda g_\lambda = 1+\bar{w}$ where $w \in H_1^\circ$. But since $f_\lambda g_\lambda \in H_1$ we conclude

$$f_\lambda g_\lambda = 1\,. \tag{14}$$

Now f_λ is L_p-analytic since, as is well-known, $(T_\varphi-\lambda I)^{-1}$ is analytic in $\rho(T_\varphi)$. Therefore $\bar{u}_\lambda$ is also L_p-analytic and differentiation of both sides of (12) gives

$$(\varphi-\lambda)f'_\lambda - f_\lambda = \bar{u}'_\lambda\,.$$

If we multiply both sides of this identity by $(\varphi-\lambda)^{-1}g_\lambda$ and use (13) and (14) we obtain

$$(\varphi-\lambda)^{-1} = g_\lambda f'_\lambda - (1+\bar{v}_\lambda)\bar{u}'_\lambda\,. \tag{15}$$

It is easy to see that if h_λ is L_r-analytic and h_λ belongs to a certain closed subspace of L_r for all λ then h'_λ belongs to the same subspace. Therefore f'_λ belongs to H_p and so $g_\lambda f'_\lambda \in H_1$. Similarly $\bar{u}'_\lambda \in \overline{H_p^0}$ and so $(1+\bar{v}_\lambda)\bar{u}'_\lambda \in \overline{H_1^\circ}$. Consequently (15) gives

$$P(\varphi-\lambda)^{-1} = g_\lambda f'_\lambda$$

and so by (14)

$$f'_\lambda = f_\lambda P(\varphi-\lambda)^{-1}\,. \tag{16}$$

Now consider a disc $D(\lambda_0, \delta)$ inside of which we have series representations

$$f_\lambda = \sum_{n=0}^{\infty} a_n(\lambda - \lambda_0)^n$$

$$P(\varphi - \lambda)^{-1} = \sum_{n=0}^{\infty} b_n(\lambda - \lambda_0)^n .$$

For each $\lambda \in D(\lambda_0, \delta)$ the two series converge a.e. and this implies that for all θ not belonging to some null set N the series converge for all $\lambda \in D(\lambda_0, \delta)$. Let us write $U(\theta, \lambda)$, $V(\theta, \lambda)$ for the sums of the two series; U and V are defined for $\theta \notin N$, $\lambda \in D(\lambda_0, \delta)$. The equation (16) is equivalent to the statement that for each $n \geqq 0$ the identity

$$(n+1)a_{n+1} = \sum_{m=0}^{n} a_m b_{n-m}$$

holds almost everywhere. It follows that for all θ not belonging to some null set N_1 the above identities hold for all n. Thus if $\theta \notin N \cup N_1$ we have

$$\frac{\partial}{\partial\lambda} U(\theta, \lambda) = U(\theta, \lambda)\, V(\theta, \lambda)$$

for all $\lambda \in D(\lambda_0, \delta)$. This implies that for any rectifiable curve Γ which lies in $D(\lambda_0, \delta)$ and has initial point λ_0 and terminal point λ

$$U(\theta, \lambda) = U(\theta, \lambda_0) \exp\left\{\int_\Gamma V(\theta, \mu) d\mu\right\} .$$

Since this holds for all $\theta \notin N \cup N_1$ and since for each λ, μ

$$f_\lambda = U(\theta, \lambda),\ P(\varphi - \mu)^{-1} = V(\theta, \mu) \quad \text{a.e.}$$

we conclude that (10) holds, at least for curves Γ of this special type. But any rectifiable curve lying in $\rho(T_\varphi)$ may be obtained by joining finitely many curves of the special type, so (10) holds in general. Formula (11) is an immediate consequence of (10) and (14).

THEOREM. *$\sigma(T_\varphi)$ is connected.*

Proof. It suffices to show that if C is a simple closed curve in $\rho(T_\varphi)$ the $\sigma(T_\varphi)$ is either entirely inside or entirely outside C. Let us apply Lemma 6 with $\Gamma = C$ and observe that by (14) f_λ is almost nowhere zero. Then we obtain

$$\exp\left\{\int_\sigma P(\varphi - \mu)^{-1} d\mu\right\} = 1 .$$

Thus if

$$\Phi(\theta) = \begin{cases} 1 & \varphi(\theta) \text{ inside } C \\ 0 & \varphi(\theta) \text{ outside } C \end{cases}$$

we have $e^{-2\pi i P\Phi} = 1$. Therefore $P\Phi$ is a real (in fact integer) valued H_2 function and so is constant. But since Φ is real valued this implies that Φ is itself constant, and so $R(\phi)$ lies entirely inside or entirely outside C. Assume the latter. The other case is quite similar, except that the point at infinity is involved; but this is handled in the usual way.

Let Ω be a simply connected open set which contains C and such that any point of Ω not inside C belongs to $\rho(T_\varphi)$. Choose $\lambda_0 \in C$, keep it fixed, and use (10) and (11) to define f_λ and g_λ for all $\lambda \in \Omega$. Here Γ is always taken to lie in Ω. Notice that

$$\int_\Gamma P(\varphi - \mu)^{-1} d\mu$$

is independent of Γ (since Ω is simply connected and $P(\varphi - \mu)^{-1}$ is L_p-analytic for μ in Ω) and represents an L_p-analytic function of λ. Therefore f_λ and g_λ are analytic throughout Ω and by Lemma 3 even L_p-analytic and L_q-analytic respectively inside C. If $h \in H_q^\circ$ then

$$\int f_\lambda h d\theta = 0$$

whenever $\lambda \in \rho(T_\varphi)$, since $f_\lambda \in H_p$. But since f_λ is L_p-analytic throughout Ω this identity holds throughout Ω, and so $f_\lambda \in H_p$ for all $\lambda \in \Omega$. Similarly we have $g_\lambda \in H_q$ for all $\lambda \in \Omega$. Moreover the identities (9) and (14) which hold in $\rho(T_\varphi)$ persist in Ω.

We show now that $T_\varphi - \lambda I$ is invertible for each λ inside C. Suppose $h \in H_p$ and $(T_\varphi - \lambda I)h = 0$. Then

$$\overline{\varphi - \lambda h} \in H_p^\circ \ .$$

Since, by (9),

$$\overline{(\varphi - \lambda)^{-1} g_\lambda} \in H_q$$

we deduce $\overline{hg_\lambda} \in H_1^\circ$. But since $hg_\lambda \in H_1$ we must have $hg_\lambda = 0$ and so $h = 0$. We have shown that $T_\varphi - \lambda I$ is one-one.

Next let $k \in H_\infty$ be arbitrary and for $\lambda \in \rho(T_\varphi)$ let $h_\lambda \in H_p$ denote the solution of

$$(T_\varphi - \lambda I)h_\lambda = k \ . \tag{17}$$

Then

$$(\varphi - \lambda)h_\lambda = k + \bar{l}_\lambda \qquad l_\lambda \in H_p^\circ .$$

Multiplying both sides by $(\varphi - \lambda)^{-1}g_\lambda$ and using (13) we obtain

$$g_\lambda h_\lambda = (\varphi - \lambda)^{-1}g_\lambda k + (1 + \bar{v}_\lambda)\bar{l}_\lambda .$$

Since $g_\lambda h_\lambda \in H_1$ and $(1 + v_\lambda)l \in H_1^\circ$ we conclude that

$$g_\lambda h_\lambda = P(\varphi - \lambda)^{-1}g_\lambda k .$$

Therefore

$$h_\lambda = f_\lambda P(\varphi - \lambda)^{-1}g_\lambda k .$$

Let this identity, which holds for $\lambda \in \rho(T_\varphi)$, be used to define h_λ for $\lambda \in \Omega$. Note that since k is bounded $P(\varphi - \lambda)^{-1}g_\lambda k$ is L_q-analytic and so h_λ is analytic. But since

$$\sup_{\mu \in C} \| h_\mu \|_p \leqq \sup_{\mu \in C} \| (T_\varphi - \lambda I)^{-1} \| \, \| k \|_p$$

Lemma 3 tells us that h_λ is L_p-analytic inside C and satisfies the inequality

$$\| h_\lambda \|_p \leqq \sup_{\mu \in C} \| (T_\varphi - \lambda I)^{-1} \| \, \| k \|_p \tag{18}$$

there. By an argument already given $h_\lambda \in H_p$ and satisfies (17) there.

Finally let k be an arbitrary function belonging to H_p. Then we can find a sequence of functions k_n belonging to H_∞ and satisfying $\| k_n - k \|_p \to 0$. Let $h_{n,\lambda}$ denote the solution of

$$(T_\varphi - \lambda I)h_{n,\lambda} = k_n .$$

As $n, m \to \infty$ we have $\| k_n - k_m \|_p \to 0$, so by (18)

$$\| h_{n,\lambda} - h_{m,\lambda} \|_p \to 0 .$$

Then $\{h_{m,\lambda}\}$ converges in L_p to a function $h_\lambda \in H_p$ and

$$(T_\varphi - \lambda I)h_\lambda = k .$$

This completes the proof of the theorem.

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