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POWER-ASSOCIATIVE ALGEBRAS IN WHICH EVERY SUBALGEBRA IS AN IDEAL

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By an H-algebra we mean a nonassociative algebra (not necessarily finite-dimensional) over a field in which every subalgebra is an ideal of the algebra.

In this paper we prove

MAIN THEOREM. **Let A be a power-associative algebra over a field F of characteristic not 2. A is an H-algebra if and only if A is one of the following;**

(1) **a one-dimensional idempotent algebra;**

(2) **a zero algebra;**

(3) **an algebra with basis $u_0, u_i,\ i \in I$ (an index set of arbitrary cardinality) satisfying $u_i u_j = \alpha_{ij} u_0, \alpha_{ij} \in F,\ i, j \in I$, all other products zero. Moreover if J is a finite subset of I, then $\sum_{i,j \in J} \alpha_{ij} x_i x_j$ is nondegenerate in that $\sum_{i,j \in J} \alpha_{ij} \alpha_i \alpha_j = 0,\ \alpha_i, \alpha_j \in F,\ i \in J$ implies $\alpha_i = 0,\ i \in J$;**

(4) **direct sums of algebras of types (1), (2), (3) with at most one from each.**

This is an extension of a result of Liu Shao-Xue who established it for alternative and Jordan H-algebras of characteristic not 2 [1; Theorem 1].

An immediate corollary is that a power-associative H-algebra over a field of characteristic not 2 is associative [1; Cor. 1].

Some results on H-rings are also determined in this paper. By an H-ring we mean a nonassociative ring in which every subring is an ideal.

1. Preliminaries.

The *associator* (x, y, z) is defined by $(x, y, z) = (xy)z - x(yz)$. We will use the *Teichmüller identity* which holds in an arbitrary ring,

$$(1.1)\quad (wx, y, z) - (w, xy, z) + (w, x, yz) = w(x, y, z) + (w, x, y)z\ .$$

In a power-associative ring we have the identities $(x, x, x) = 0$ and $(x^2, x, x) = 0$ which when linearized yield, respectively,

$$(1.2)\qquad \sum_{\sigma \in S_3} (w_{\sigma(1)}, w_{\sigma(2)}, w_{\sigma(3)}) = 0$$

and

$$(1.3)\qquad \sum_{\sigma \in S_4} (w_{\sigma(1)} w_{\sigma(2)}, w_{\sigma(3)}, w_{\sigma(4)}) = 0$$

providing $2x = 0$ implies $x = 0$ in the ring.

Let a be an element of a ring (algebra). By $\{a\}$ is meant the subring (subalgebra) generated by a. Shao-Xue has established [1; Lemma 1]

LEMMA 1.1. *If a is an element of an H-algebra, then $\{a\}$ is finite-dimensional.*

2. Main section. To prove the main theorem we will first show that an H-algebra with unit is associative, then that a nil power-associative H-algebra is alternative, finally that a power-associative H-algebra is the direct sum of an H-algebra with unit and a nil H-algebra from which the theorem follows by Shao-Xue's result.

Separate statements for the ring case and the algebra case are needed where the results are also true of H-rings since there are ring ideals of an algebra which are not algebra ideals.

THEOREM 2.1. *If A is a ring with unit* 1, *and if A is an H-ring or an H-algebra, then A is associative.*

Proof. The nucleus N of A is defined by

$$N = \{u \in A \mid (u, x, y) = (x, u, y) = (x, y, u) = 0 \ \forall x, y \in A\} .$$

It follows easily from (1.1) and the linearity of the associator that N is a subring or subalgebra of A, as the case may be. Hence N is an ideal of A. But then $N(A, A, A) = 0$ by (1.1). The theorem follows immediately from the fact that $1 \in N$.

THEOREM 2.2. *Let A be a nil power-associative ring which is either*

(1) *an H-ring in which $px = 0$ implies $x = 0$, $x \in A$, if $p = 2$ or if $p = k^i$, k, i integers, $k \neq 0$, $i \geqq 2$, or*

(2) *an H-algebra where the characteristic of F is not* 2.

Then A is alternative.

Proof. We first show as in [1; Lemma 3] that for all $a \in A$,

$$a^3 = 0 . \tag{2.3}$$

Suppose $a^n = 0$, $a^{n-1} \neq 0$ for $n \geqq 4$, $a \in A$. Let $m = [(n+1)/2]$ where $[x]$ denotes the greatest integer in x. Then $m + 1 \leqq n - 1$. Now, $a^{m+1} = a^m a \in \{a^m\}$, hence $a^{m+1} = ja^m$, j an integer in case (1) or $j \in F$ in case (2), since $(a^m)^2 = 0$. If $ja^m \neq 0$, then a is not nilpotent (using the restriction on characteristic in case (1)), a contradiction.

Hence $ja^m = 0$ which implies $a^{m+1} = 0$, which is also a contradiction. Thus we have (2.3).

Let $b \in A$ such that $b^2 = 0$. We next establish

$$bA = 0 = Ab \,. \tag{2.4}$$

Choose $a \neq 0$ in A. Since $ab \in \{b\}$ and $b^2 = 0$, $ab = kb$. Similarly, since $a^3 = 0$ by (2.3) and $ab \in \{a\}$, $a^2b \in \{a^2\}$, we have $ab = la + ma^2$, $a^2b = na^2$. In case (1) k, l, m, n are integers, and in case (2) they are elements of F. Since $a^2b \in \{b\}$ and $b^2 = 0$, we have

$$0 = (a^2b)b = (na^2)b = n^2a^2 \,.$$

Hence $a^2b = 0$. But then since $ab \in \{b\}$ and $b^2 = 0$

$$0 = (ab)b = (la + ma^2)b = lab = l^2a + lma^2 \,.$$

Thus $l^2a^2 = 0$ since $a^3 = 0$, which implies $l = 0$ since $a \neq 0$. Therefore

$$0 = a(ma^2) = a(ab) = a(kb) = k^2b \,.$$

Hence $Ab = 0$.

The anti-isomorphic copy A' of A satisfies the hypotheses of A, hence $A'b' = 0$ where b' is the anti-isomorphic copy of b. But then $bA = 0$, and we have (2.4).

In view of (2.4), the theorem will be established if we can show that the associators (a, a, b), (a, b, a), and (b, a, a) vanish whenever $a^2 \neq 0 \neq b^2$. Hence assume the latter.

By (2.3) and (2.4), for all $c \in A$

$$c^2A = 0 = Ac^2 \,. \tag{2.5}$$

Since $\{a\}$ and $\{b\}$ are ideals, $ab, ba \in \{a\}$ and $ab, ba \in \{b\}$, hence

$$\begin{aligned} ab &= k_1a + l_1a^2,\ ba = m_1a + n_1a^2 \,, \\ ab &= k_2b + l_2b^2,\ ba = m_2b + n_2b^2 \end{aligned} \tag{2.6}$$

by (2.3) where $k_1, k_2, l_1, l_2, m_1, m_2, n_1, n_2$ are integers in case (1) or elements of F in case (2). Computing, using (1.2) with $w_1 = a$, $w_2 = w_3 = b$, the restrictions on characteristic, and (2.5),

$$\begin{aligned} 0 &= (a, b, b) + (b, b, a) + (b, a, b) \\ &= (ab)b - b(ba) + (ba)b - b(ab) \\ &= (k_1a)b - b(m_2b) + (m_2b)b - b(k_2b) \\ &= k_1^2a + k_1l_1a^2 - k_2b^2 \,, \end{aligned}$$

which implies $k_1^2a^2 = 0$ by (2.5). Hence $k_1 = 0$ since $a^2 \neq 0$. Considering the anti-isomorphic copy A' of A similarly as before yields $m_1 = 0$. Finally, direct computation using (2.5) and (2.6) yields

$(a, a, b) = -k_1a^2$, $(a, b, a) = (k_1 - m_1)a^2$, and $(b, a, a) = m_1a^2$, which completes the proof.

Proof of main theorem. We will show that A is alternative, from which the theorem follows by [1; Theorem 1].

If A is nil, then A is alternative by Theorem 2.2. Hence assume A is not nil. Let a be an element of A which is not nilpotent. Then $\{a\}$ is finite-dimensional by Lemma 1.1. Thus $\{a\}$ contains an idempotent e. Define

$$A_1 = \{x \in A \mid ex = 0\} .$$

We will show that A_1 is nil and that $A = \{e\} \oplus A_1$ from which the theorem follows by Theorems 2.1 and 2.2 since $\{e\}$ has unit element e.

Because $\{e\}$ is an ideal with unit element e,

$$(x, e, e) = 0 = (e, e, x)$$

for all $x \in A$, hence if we let $w_1 = x$, $w_2 = w_3 = e$ in (1.2) we obtain the identities

$$0 = (e, x, e) = (x, e, e) = (e, e, x) . \tag{2.7}$$

Let $x_1 \in A_1$. Expanding $(e, x_1, e) = 0$ yields

$$x_1e = 0 . \tag{2.8}$$

Let $y_1 \in A_1$. By (2.8),

$$(x_1, e, y_1) = 0 . \tag{2.9}$$

In (1.3), let $w_1 = x_1$, $w_2 = y_1$, $w_3 = w_4 = e$ and use (2.7), (2.8), and (2.9) to obtain

$$0 = (e, x_1, y_1) + (e, y_1, x_1) . \tag{2.10}$$

Now, consider $\{x_1\}$. Using (2.8) and (2.10), we compute for $n > 1$

$$\begin{aligned} ex_1^n &= e(x_1x_1^{n-1}) = -(e, x_1, x_1^{n-1}) = (e, x_1^{n-1}, x_1) \\ &= (ex_1^{n-1})x_1 - ex_1^n . \end{aligned}$$

Hence

$$2ex_1^n = (ex_1^{n-1})x_1, n > 1 . \tag{2.11}$$

But then by an obvious induction argument we have from (2.11) that $ex_1^n = 0$ which implies that $\{x_1\} \subset A_1$. Hence $x_1y_1 \in A_1$ since $\{x_1\}$ is an ideal. Therefore A_1 is a subalgebra of A.

As in the proof of [1; Lemma 2], choose $x \in A$. Then $x = ex + (x - ex)$. Now, $e(x - ex) = 0$ by (2.7), hence $x - ex \in A_1$. Since

$\{e\}$ is an ideal, $ex \in \{e\}$. Moreover, $\{e\} \cap A_1 = 0$, thus $A = \{e\} \oplus A_1$.

If A_1 is not nil, then, as above, A_1 has an idempotent e_1, and $A_1 = \{e_1\} \oplus A_2$ where

$$\mathrm{A}_2 = \{x_2 \in A_1 \mid e_1 x_2 = 0\} .$$

Hence $A = \{e\} \oplus \{e_1\} \oplus A_2$. Let $f = e + e_1$. Since $e = ef \in \{f\}$ and $e_1 = fe_1 \in \{f\}$, e and e_1 are linearly dependent because $\{f\}$ is one dimensional, a contradiction. Hence A_1 is nil, which completes the proof of the theorem.

H-algebras which are not associative can be constructed. Let A be the two-dimensional algebra over a field F with basis a, b satisfying $a^2 = ab = b^2 = a$, $ba = 0$. It is easy to check that every subalgebra of A is an ideal. Also, since $(b, b, b) = a$, A is neither power-associative nor associative.

Bibliography

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