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AN EXTENSION OF HAIMO'S FORM OF HANKEL CONVOLUTIONS

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The real inversion formula for Hankel convolutions which is due to D. T. Haimo, is extended to certain generalized functions. That is accomplished by transferring the real inversion formula of D. T. Haimo onto the testing function space for the generalized function under consideration and then showing that the limiting process in the resulting formula converges with respect to the topology of the testing function space.

The Hirschman-Widder convolution transformation [3] has recently been extended to certain classes of generalized functions [7], [4] and their inversion formulae [3; pp. 127–132 and Theorems 7.1 b, p. 231] have been shown to be still valid when the limiting operation in those formulae is understood as weak convergence in the space D' of Schwartz distributions [5]. The purpose of the present work is to extend the inversion formula of D. T. Haimo for Hankel convolutions [1, p. 332] in a similar way to a certain space of generalized functions.

The notation and terminology of this work follows that of [7], [4]. I denotes the open interval $(0, \infty)$ and all testing functions herein are defined on I. Throughout this work x and y are variables over I unless otherwise mentioned. Finally, $D(I)$ is the space of smooth functions defined on I having compact supports. The topology of $D(I)$ is that which makes its dual the space $D'(I)$ of Schwartz distributions [5; Vol. I, p. 65] [8] on I. Thus, a sequence of functions $\{\phi_\nu\}_{\nu=1}^\infty$ is said to converge in $D(I)$ if and only if the supports of ϕ_ν are all contained within a fixed compact subset of I and for each k, $\{\phi_\nu^{(k)}\}_{\nu=1}^\infty$ converges uniformly on I.

2. The testing function space $\mathscr{G}(I)$. Let Δ_x stand for the operator $(D_x^2 + (2\gamma/x)D_x)$ where D_x is the differentiation operator and γ is a positive number. We say that a smooth function $\phi(x)$ defined over I belongs to $\mathscr{G}(I)$ if

$$\gamma_k(\phi) = \sup_{0<x<\infty} | e^{cx}\Delta_x^{(k)}\phi(x) | < \infty \tag{1}$$

for all k assuming values $0, 1, 2, \cdots$. Here, c is a fixed real number; but in our later discussion we will choose c to be a fixed real number less than a_1. The topology in $\mathscr{G}(I)$ is generated by the collection of seminorms $\{\gamma_k\}_{k=0}^\infty$. Since γ_0 is a norm the collection of seminorms

$\{\gamma_k\}_{k=0}^{\infty}$ is separating. We say that a sequence $\{\phi_\nu\}_{\nu=1}^{\infty}$ converges to ϕ in $\mathscr{G}(I)$ if for each $\phi_\nu \in \mathscr{G}(I)$ and for every fixed k, $\gamma_k(\phi_\nu - \phi)$ goes to zero as ν goes to ∞. $\mathscr{G}(I)$ is a locally convex Hausdorff topological linear space. $D(I)$ is a subspace of $\mathscr{G}(I)$ and the topology of $D(I)$ is stronger than the topology induced on $D(I)$ by $\mathscr{G}(I)$. Consequently the restriction of any member of $\mathscr{G}'(I)$ to $D(I)$ is in $D'(I)$.

DEFINITION: We say that a sequence $\{\phi_\nu(x)\}_{\nu=1}^{\infty}$ where each $\phi_\nu(x) \in G(I)$ is a Cauchy sequence in $G(I)$ if $\gamma_k(\phi_\nu - \phi_\mu) \to 0$ as ν and μ both go to ∞ independently where k assumes values $0, 1, 2, 3, \cdots$.

LEMMA 1. *$\mathscr{G}(I)$ is sequentially complete.*

Proof. By hypothesis the sequences

$$\{\phi_\nu(x)\}_{\nu=1}^{\infty}, \qquad \left\{\int_a^x \phi_\nu(t)dt\right\}_{\nu=1}^{\infty}$$

$$\{\Delta_x\phi_\nu(x)\}_{\nu=1}^{\infty}, \qquad \left\{\int_a^x [\Delta_t\phi_\nu(t)]dt\right\}_{\nu=1}^{\infty}$$

and $\left\{\int_a^x\int_a^y [\Delta_t\phi_\nu(t)]dtdy\right\}_{\nu=1}^{\infty}$ converge uniformly with respect to x over any compact subset of I. We assume that a is a fixed positive quantity. All these combined together mean that the sequence $\{\phi_\nu^{(k)}(x)\}_{\nu=1}^{\infty}$ for all k converges uniformly over any compact subset of I where k assumes values $0, 1, 2$. Proceeding in the same way we can prove by induction that the sequence $\{\phi_\nu^{(k)}(x)\}_{\nu=1}^{\infty}$ for all k, where k assumes values $0, 1, 2, 3, 4, \cdots$ converges uniformly over any compact subset of I. Here, uniformity is assumed with respect to the variable x and not with respect to k. Therefore, by a classic result it follows that there exists a smooth function $\phi(x)$ such that the sequence $\{\phi_\nu^{(k)}(x)\}_{\nu=1}^{\infty}$ converges uniformly to $\phi^{(k)}(x)$ on every compact subset of I for $k = 0, 1, 2, \cdots$. Our object is now to show that $\phi(x) \in \mathscr{G}(I)$. By hypothesis for $\varepsilon > 0$ $\exists N$.

$$(2) \qquad \gamma_k(\phi_\nu - \phi_\mu) < \varepsilon \qquad \text{for all } \nu > N,\ \mu > N\,.$$

As $\mu \to \infty$ (2) reduces to

$$(3) \qquad \gamma_k(\phi_\nu - \phi) < \varepsilon \qquad \text{for all } \nu > N\,.$$

Now,

$$(4) \qquad \gamma_k(\phi) = \gamma_k(\phi - \phi_\nu + \phi_\nu) \leqq \gamma_k(\phi_\nu - \phi) + \gamma_k(\phi_\nu)\,.$$

By fixing a value of ν greater than N, (4) reduces to

$$(5) \qquad \gamma_k(\phi) < \varepsilon + \gamma_k(\phi_\nu) = \beta < \infty$$

$$\therefore \quad \phi \in \mathscr{G}(I) .$$

This completes the proof of our lemma.

3. **The testing function space $\mathscr{H}(I)$.** We say that a smooth function $\psi(x)$ defined on I, $\in \mathscr{H}(I)$ if $\{\psi(x)/\mu'(x)\} \varepsilon \mathscr{G}(I)$ where $\mu(x)$ is the same as will be defined in (7). The topology in $\mathscr{H}(I)$ is generated by a sequence of seminorms $\{\beta_k\}_{k=0}^{\infty}$ where,

$$\beta_k\{\psi(x)\} = \gamma_k\left\{\frac{\psi(x)}{\mu'(x)}\right\} . \tag{6}$$

The concept of convergence and completeness in $\mathscr{H}(I)$ is defined in a way similar to that in $\mathscr{G}(I)$. $\mathscr{H}(I)$ is also a sequentially complete, Hausdorff, locally convex, topological linear space. $D(I)$ is a subspace of $\mathscr{H}(I)$ and restriction of any member of $\mathscr{H}'(I)$ to $D(I)$ is in $D'(I)$. The mapping $\psi(x) \rightarrow \psi(x)/\mu'(x)$ in (6) is an isomorphism from $\mathscr{H}(I)$ onto $\mathscr{G}(I)$. It can be further proved that if $f \in \mathscr{H}'(I)$ then $\mu' f \in \mathscr{G}'(I)$ and vice-versa [9, p. 28].

4. **The generalized Hankel convolution of Haimo's type.** Let us first specify the type of kernel for which our Hankel convolution has been constructed. We assume γ to be a fixed positive number, and set as in [1],

$$\mu(x) = \frac{x^{2\gamma+1}}{2^{\gamma+1/2}\Gamma\left(\gamma + \frac{3}{2}\right)} \tag{7}$$

$$J(x) = 2^{\gamma-1/2}\Gamma\left(\gamma + \frac{1}{2}\right)x^{1/2-\gamma}J_{\gamma-1/2}(x) , \tag{8}$$

where $J_{\gamma-1/2}(x)$ is the ordinary Bessel function of order $(\gamma - 1/2)$.

$$G(x, y) = \int_0^\infty \frac{J(xt)J(yt)d\mu(t)}{\prod_{k=1}^{\infty}\left(1 + \frac{t^2}{a_k^2}\right)} , \tag{9}$$

the a_k's being real, satisfying $0 < a_1 \leqq a_2 \leqq a_3 \cdots$ with

$$\sum_{k=1}^{\infty} a_k^{-2} < \infty .$$

$$G_N(x, y) = \int_0^\infty \frac{J(xt)J(yt)d\mu(t)}{\prod_{k=N+1}^{\infty}\left(1 + \frac{t^2}{a_k^2}\right)} \tag{10}$$

$G(x, y)$ will be the kernel of Haimo's type of generalized Hankel convolutions. We briefly review some of the paramount properties of the kernel $G(x, y)$.

$$\text{(11)} \qquad \text{A.} \quad \frac{\partial^{m+n}}{\partial x^m \partial y^n} G(x, y) , \qquad m, n = 0, 1, 2, \cdots$$

is bounded and continuous for $0 \leqq x,\ y < \infty$ [1, p. 344].

$$\text{(12)} \qquad \text{B.} \quad \int_0^\infty G(x, y) d\mu(y) = 1 , \qquad 0 \leqq x < \infty \quad [1, \text{ p. } 344] .$$

$$\text{(13)} \qquad \text{C.} \quad \begin{aligned} \lim_{N\to\infty} \int_a^b G_N(x, y) d\mu(y) &= 1 , \quad 0 \leqq a < x < b \leqq \infty \\ &= 0 \quad\ \ 0 \leqq a \leqq b < x < \infty \\ &= 0 \quad\ \ 0 < x < a \leqq b \leqq \infty \quad [1, \text{ p, } 345] . \end{aligned}$$

(14) D. With x fixed, in $0 \leqq x < \infty$

$$G(x, y) = Ce^{-a_1 y} y^{m_1 - \gamma} \oint (a_1 x)\left[1 + 0\left(\frac{1}{y}\right)\right] , \qquad y \to \infty$$

where $C > 0$, [1, p. 348]. m_1 and $\oint(x)$ are defined in a way similar to that defined by D. T. Haimo. When N is fixed, (11), (12) and (14) also hold for $G_N(x, y)$.

LEMMA 2. *With x fixed in $0 \leqq x < \infty$*

$$\text{(15)} \qquad \frac{\partial^r G(x, y)}{\partial x^r} = Ce^{-a_1 y} y^{m_1 - \gamma} a_1^r \oint^{(r)} (a_1 x)\left[1 + 0\left(\frac{1}{y}\right)\right] , \ y \to \infty$$
$$r = 0, 1, 2, 3, \cdots .$$

Proof. For $r = 0$ see (14); we will deal exclusively with the case where $r > 0$. Let a be a positive quantity such that $a_1 < a < a_2$. One can readily show by a techique similar to that of D. T. Haimo that

$$\text{(16)} \qquad \begin{aligned} G(x, y) = {} & y^{1/2-\gamma}\left[\left(\frac{d}{da}\right)^{m_1}\left\{\psi(a) \oint (ax) K_{\gamma-1/2}(ay)\right\}\right]_{a=a_1} \\ & + y^{1/2-\gamma} \int_{-\infty+ai}^{\infty+ai} \frac{z^{1/2+\gamma} J(xz) H_{\gamma-1/2}^{(1)}(yz) dz}{E(z)} . \end{aligned}$$

$1 + m_1$ is the multiplicity of the zero of $E(z)$ at $z = \pm a_1 i$ [1, pp. 347–349]. Therefore,

$$\text{(17)} \qquad \begin{aligned} \frac{\partial^r G(x, y)}{\partial x^r} = {} & y^{1/2-\gamma}\left[\left(\frac{d}{da}\right)^{m_1}\left\{\psi(a) a^r \oint^{(r)} (ax) K_{\gamma-1/2}(ay)\right\}\right]_{a=a_1} \\ & + y^{1/2-\gamma} \int_{-\infty+ai}^{\infty+ai} \frac{z^{1/2+\gamma+r} J^{(r)}(xz) H_{\gamma-1/2}^{(1)}(yz) dz}{E(z)} . \end{aligned}$$

Differentiation within the integral sign in (17) is justified by virtue of [6, pp. 197–203] and it is also proved thereby that the aforesaid integral is $0[y^{-1/2}e^{-ay}]$, $y \to \infty$ uniformly for x lying in any compact subset of I. This result coupled with [1, Corollary 2.2, p. 347] establishes (15). This asymptotic order can also be established for $G_N(x, y)$ with fixed N where a_1 will be replaced by a_N.

LEMMA 3. *Let Δ_x, $G(x, y)$ and $G_N(x, y)$ be defined precisely in a way similar to that of D. T. Haimo. Then;*

$$\Delta_k^{(n)}G(x, y) = A_0G(x, y) + A_1G_1(x, y) + A_2G_2(x, y) + \cdots A_nG_n(x, y) . \tag{18}$$

where $A_0, A_1, A_2, \cdots A_n$ are all polynomials of degree $2n$ each in $a_0, a_1, a_2, \cdots a_n$.

Proof.

$$\Delta_xG(x, y) = \int_0^\infty \frac{(-t^2)J(xt)J(yt)d\mu(t)}{E(t)} \qquad [1, \text{ p. } 334]$$
$$= -\int_0^\infty \frac{\left[\dfrac{t^2}{a_1^2} + 1 - 1\right]a_1^2J(xt)J(yt)d\mu(t)}{E(t)}$$

or

$$\Delta_xG(x, y) = a_1^2[G(x, y) - G_1(x, y)] . \tag{19}$$

In the same way we can show that

$$\Delta_xG_m(x, y) = a_{m+1}^2[G_m(x, y) - G_{m+1}(x, y)] . \tag{20}$$

Using the technique of induction and operating by Δ_x repeatedly on $G(x, y)$, (18) follows readily in view of (19) and (20).

THEOREM 1. *Let $f \in \mathscr{H}'(I)$ and define $F(x)$ by*

$$F(x) = \langle \mu'(y)f(y), G(x, y)\rangle = \langle f(y), \mu'(y)G(x, y)\rangle . \tag{21}$$

Then,

$$F^{(m)}(x) = \left\langle \mu'(y)f(y), \frac{\partial^m}{\partial x^m}G(x, y)\right\rangle , \qquad m = 1, 2, 3, \cdots . \tag{22}$$

Proof. In this theorem and from now on c is assumed to be $< a_1$. In view of (1), (15) and (18), $\mathrm{G}(x, y) \in \mathscr{G}(I)$ for a fixed x. Therefore, by (6) $\mu'(y)G(x, y) \in \mathscr{H}(I)$, and as such (21) is meaningful. For a similar reason (22) also has meaning. Now,

$$(23)\quad \frac{F(x+\Delta x)-F(x)}{\Delta x} - \left\langle \mu'(y)f(y),\ \frac{\partial G(x,y)}{\partial x}\right\rangle = \langle \mu'(y)f(y),\ \theta_{\Delta x}(y)\rangle$$

where

$$\theta_{\Delta x}(y) = \frac{G(x+\Delta x, y) - G(x,y)}{\Delta x} - \frac{\partial G}{\partial x}(x,y)\ .$$

We will prove the result only for $m = 1$, but the proof for any positive integral values of m can be given very simply by induction. Our objective will therefore, be achieved if we can show that

$$\gamma_k[\theta_{\Delta x}(y)] \to 0 \quad \text{as} \quad \Delta x \to 0 \quad \text{for a fixed } y$$

where $k = 0, 1, 2, 3, \cdots$.

In view of (18), it is enough to show that

$$(24)\qquad \sup_{0<y<\infty}\left| e^{cy}\left[\frac{G_n(x+\Delta x, y) - G_n(x,y)}{\Delta x} - \frac{\partial}{\partial x}G_n(x,y)\right]\right| \to 0 \qquad \text{as } \Delta x \to 0$$

for a fixed x and n where n assumes values $0, 1, 2, \cdots$. Again using (15), (24) can be readily proved by the technique precisely similar to followed by Zemanian in [7, Th. 4.1]. This completes the proof of Theorem 7.

LEMMA 4. *Let* $\phi(x) \in D(I)$ *and assume that*

$$(25)\qquad \Delta_x^{(k)}\phi(x) = \psi_k(x)\ .$$

Then

$$(26)\qquad e^{cy}\int_0^\infty G_N(x,y)[\psi_k(y) - \psi_k(x)]d\mu(x) \stackrel{\Delta}{=\!=} I \to 0$$

uniformly on $0 < y < \infty$ *as* $N \to \infty$.

Proof. We break up the integration in (26) into integrations on $0 < x < y - \delta$, $y - \delta < x < y + \delta$ and $y + \delta < x < \infty$ $(\delta > 0)$, and denote the corresponding quantities by I_1, I_2 and I_3 respectively. By (25) $\psi_k(x) \in D(I)$. Now

$$(27)\qquad I_2 = e^{cy}\int_{y-\delta}^{y+\delta} G_N(x,y)[\psi_k(y) - \psi_k(x)]d\mu(x)\ .$$

By virtue of (12) and (27)

$$\begin{aligned} |I_2| &\leqq \sup_{y-\delta<x<y+\delta} |e^{cy}(\psi_k(y) - \psi_k(x))| \\ &\leqq e^{cy}\delta \sup_{y-\delta<\tau<y+\delta} |\psi_k^{(1)}(\tau)|\ . \end{aligned}$$

Since $\psi_k(x) \in D(I)$, the last quantity is bounded by δB where B is a constant with respect to δ and y when δ is restricted to $0 < \delta < 1$. Therefore, given an $\varepsilon > 0$, $|I_2|$ is bounded by ε for $\delta = \min(1, \varepsilon/B)$. Fix δ this way. Next consider,

$$
\begin{aligned}
I_1 &= e^{cy}\int_0^{y-\delta} G_N(x, y)[\psi_k(y) - \psi_k(x)]d\mu(x) \\
&= e^{cy}\int_0^{y-\delta} G_N(x, y)\psi_k(y)d\mu(x) \\
&\quad - e^{cy}\int_0^{y-\delta} G_N(x, y)\psi_k(x)d\mu(x) .
\end{aligned}
\tag{28}
$$

Let the support of $\phi(x)$ be contained in $[A, B]$ where $0 < A < B$. Therefore, the support of $\psi_k(x)$ will be also contained in $[A, B]$. Suppose that,

$$\sup_{0<x<\infty} |e^{cx}\psi_k(x)| = M ,$$

and

$$\sup_{0<x<\infty} |\psi_k(x)| = m ,$$

then

$$
\begin{aligned}
|I_1| &\leqq M\int_0^{y-\delta} G_N(x, y)d\mu(x) + e^{cy}\int_0^{y-\delta} G_N(x, y)\,|\psi_k(x)|\,d\mu(x) \\
&\leqq M\int_\delta^{\infty} G_N(u)d\mu(u) + e^{cy}\int_0^{y-\delta} G_N(x, y)\,|\psi_k(x)|\,d\mu(x) .
\end{aligned}
$$

It is a fact that

$$\int_\delta^\infty G_N(u)d\mu(u) = 2\sum_{k=N+1}^{\infty} a_k^{-2} \longrightarrow 0 \qquad \text{as } N \longrightarrow \infty \quad [1, \text{pp. } 345\text{–}46] .$$

Again, in a way similar to what is done in (14) one can show that

$$G_N(x, y) = Ce^{-a_N y}y^{m_1-\gamma}\mathfrak{f}(a_N x)\left[1 + 0\left(\frac{1}{y}\right)\right] . \tag{29}$$

Since $c < a_1 \leqq a_N$, we can choose a fixed quantity $k > B$ so large that the asymptotic estimate (29) is valid for $y > k$. Therefore, $e^{cy}G_N(x, y) \longrightarrow 0$ as $N \longrightarrow \infty$ uniformly with respect to x lying in any compact subset of I and $y > k$. Now,

$$
\begin{aligned}
e^{cy}\int_0^{y-\delta} G_N(x, y)\,|\psi_k(x)|\,d\mu(x) &\leqq \mu'(B)m\int_A^B e^{cy}G_N(x, y)dx \\
&\longrightarrow 0 \text{ uniformly for all} \\
& y > k \quad \text{as} \quad N \longrightarrow \infty .
\end{aligned}
$$

When $0 < y \leqq k$ we have

$$e^{cy}\int_0^{y-\delta} G_N(x, y)\psi_k(x)d\mu(x) \leqq m[\sup_{0<y<k} e^{cy}]\int_0^{y-\delta} G_N(x, y)d\mu(x)$$
$$\to 0 \text{ as } N \to \infty \text{ uniformly}$$

for all $0 < y \leqq k$, [1, p. 346]. Therefore, $|I_1| \to 0$ as $N \to \infty$ uniformly for all $y > 0$. Again

(30) $$I_3 = e^{cy}\int_{y+\delta}^{\infty} G_N(x, y)[\psi_k(y) - \psi_k(x)]d\mu(x) .$$

If $y > B$ then $I_3 = 0$.

Therefore, we will now consider the case

$$y \leqq B .$$

$$I_3 = e^{cy}\int_{y+\delta}^{\infty} G_N(x, y)\psi_k(y)d\mu(x) - e^{cy}\int_{y+\delta}^{\infty} G_N(x, y)\psi_k(x)d\mu(x)$$

$$|I_3| \leqq M\int_{y+\delta}^{\infty} G_N(x, y)d\mu(x) + m[\sup_{0<y<B} e^{cy}]\int_{y+\delta}^{\infty} G_N(x, y)d\mu(x)$$
$$\leqq [M + m \sup_{0<y\leqq B} e^{cy}]\int_{\delta}^{\infty} G_N(u)d\mu(u) \qquad [1, \text{ p. } 346]$$
$$\to 0 \text{ as } N \to \infty \quad \text{uniformly for all } 0 < y \leqq B .$$

In view of (30) therefore, we have proven that $|I_3| \to 0$ as $N \to \infty$ uniformly for all $y > 0$. Altogether we have proven that

(30a) $$\overline{\lim_{N\to\infty}} |I| \leqq \varepsilon \qquad \delta < y < \infty .$$

When $y \leqq \delta$ we break up the integration in (26) into integrations on $0 < x < y + \delta$, $y + \delta < x < \infty$ and represent the corresponding quantities as J_1 and J_2 respectively.

Clearly $J_2 \to 0$ as $N \to \infty$ uniformly for all $y > 0$. Now,

$$|J_1| \leqq e^{c\delta}\int_0^{y+\delta} G_N(x, y)\,|\psi_k(y) - \psi_k(x)|\,d\mu(x)$$
$$\leqq e^{a_1}\frac{\varepsilon}{B} \sup_{0<\tau<\infty} |\psi_k^{(1)}(\tau)| = \varepsilon' \qquad \text{(say)}$$

(30b) $$\therefore \quad \overline{\lim_{N\to\infty}} |I| \leqq \varepsilon' \qquad \text{for all } 0 < y \leqq \delta .$$

Let $\eta = \text{Max}[\varepsilon, \varepsilon']$. Note that by virtue of the fact that ε is arbitrary η is also arbitrary. Combining (30a) and (30b) we have proven the fact that

$$\overline{\lim_{U\to\infty}} |I| \leqq \eta , \qquad 0 < y < \infty .$$

Since η is arbitrary, our lemma is proven.

We are now ready to prove the main theorem of this paper.

THEOREM 2. *Let* $f \in \mathscr{H}'(I)$ *and let* $F(x)$ *be defined as in* (21). *Then for* $\phi(x) \in D(I)$

$$\langle \mu'()P_N(\Delta_x)F(x), \phi(x)\rangle \rightarrow \langle \mu' f, \phi\rangle \quad as\ N \rightarrow \infty$$

where

$$P_N(\Delta_x) = \prod_{k=1}^{N}\left(1 - \frac{\Delta_x}{a_k^2}\right).$$

Proof. The theorem will be proven by justifying steps in the following manipulations.

$$\begin{aligned}
&(31) \quad \langle \mu'(x)P_N(\Delta_x)F(x), \phi(x)\rangle = \langle \mu'(x)P_N(\Delta_x)\langle \mu'(y)f(y), G(x, y)\rangle, \phi(x)\rangle \\
&(32) \qquad\qquad = \langle \mu'(x)\langle \mu'(y)f(y), P_n(\Delta_x)G(x, y)\rangle, \phi(x)\rangle \\
&(33) \qquad\qquad = \langle \mu'(x)\langle \mu'(y)f(y), G_N(x, y)\rangle, \phi(x)\rangle \\
&(34) \qquad\qquad = \langle \mu'(y)f(y), \langle \mu'(x)G_N(x, y), \phi(x)\rangle\rangle \\
&(35) \qquad\qquad \rightarrow \langle \mu'(y)f(y), \phi(y)\rangle \quad \text{as } N \rightarrow \infty .
\end{aligned}$$

The fact that (31) is true follows at once in view of (21), and that the expressions in (31) are each equal to (32) is an immediate consequence of (21) and (22). That (32) is equal to (33) follows by [1; Lemma 4.1, p. 360].

To prove that (33) and (34) are equal we need show the following relation

$$\begin{aligned}
(36) \qquad &\int_A^B \langle \mu'(y)f(y), G_N(x, y)\rangle \phi(x)d\mu(x) \\
&= \left\langle \mu'(y)f(y), \int_A^B G_N(x, y)\phi(x)d\mu(x)\right\rangle .
\end{aligned}$$

As before we assume that the support of $\phi(x)$ is contained in $[A, B]$ where $0 < A < B$.

In view of asymptotic orders of $G_m(x, y)$, $(m = 0, 1, 2, 3, \cdots)$ for fixed x and large y as stated in (15) and using the technique of the Riemann sum, (36) is proved in a way exactly similar to that employed in [4, Th. 2]. Note that (18) is also to be made use of in proving the assertion in (36).

We now proceed to prove that (34) is equal to (35). Our objective will be fulfilled if we can show that

$$(37) \qquad \langle \mu'(x)G_N(x, y), \phi(x)\rangle \rightarrow \phi(y) \text{ in } \mathscr{G}(I) \qquad \text{as } N \rightarrow \infty .$$

Now,

$$\langle \mu'(x) G_N(x, y), \phi(x) \rangle = \int_A^B G_N(x, y)\phi(x) d\mu(x)$$

and

$$\Delta_y^{(k)} \langle \mu'(x) G_N(x, y), \phi(x) \rangle = \int_A^B \{\Delta_y^{(k)} G_N(x, y)\}\phi(x) d\mu(x) \; . \tag{38}$$

By using (15) the differentiation process within the sign of integration in (38) is easily justified. Now,

$$\Delta_x G_N(x, y) = \Delta_y G_N(x, y) = \int_0^\infty \frac{(-t^2) J(xt) J(yt) d\mu(t)}{E_N(t)} \qquad [1;\ \text{p. } 334] . \tag{39}$$

Again

$$\begin{aligned} \int_A^B \{\Delta_x G_N(x, y)\}\phi(x) d\mu(x) &= \int_A^B G_N(x, y)\{\Delta_x \phi(x)\} d\mu(x) \\ &\qquad [\text{integration by parts}] \\ &= \int_0^\infty G_N(x, y)\{\Delta_x \phi(x)\} d\mu(x) \; . \end{aligned} \tag{40}$$

Therefore by repeatedly using (39) and (40) one can readily show that

$$\int_A^B \{\Delta_y^{(k)} G_N(x, y)\}\phi(x) d\mu(x) = \int_0^\infty G_N(x, y)\{\Delta_x^{(k)} \phi(x)\} d\mu(x) \tag{41}$$

$$\begin{aligned} &\therefore \quad e^{cy} \Delta_y^{(k)} [\langle \mu'(x) G_N(x, y), \phi(x) \rangle - \phi(y)] \\ &= e^{cy} \left[\int_0^\infty G_N(x, y)(\{\Delta_x^{(k)} \phi(x)\} - \{\Delta_y^{(k)} \phi(y)\}) d\mu(x) \right. \qquad \text{by (12)}, \\ &= e^{cy} \left[\int_0^\infty G_N(x, y)\{\psi_k(x) - \psi_k(y)\} d\mu(x) \right] \\ &\to 0 \text{ uniformly on } 0 < y < \infty \text{ as } N \to \infty \end{aligned}$$

in view of Lemma 4. Thus the proof of Theorem 2 is complete.

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