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INDEFINABILITY IN THE ARITHMETIC ISOLIC INTEGERS

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This paper is primarily concerned with the theory of the arithmetic isolic integers. The following results are obtained:

(1) No nonfinite member of the arithmetic isolic integers $\Lambda^*(A)$ can be defined in $\Lambda^*(A)$ even by an infinite number of arithmetic formulas (Theorem 4).

(2) The arithmetic isols $\Lambda(A)$ cannot be defined in $\Lambda^*(A)$ even by an infinite number of arithmetic formulas (Theorem 7).

(3) We exhibit some nonstandard models of arithmetic contained within $\Lambda^*(A)$ (Theorem 10).

The first result above follows from recent work of Nerode in the theory of isols, while the second strengthens his results to obtain the desired conclusion.

The ring of arithmetic isolic integers $\Lambda^*(A)$ was introduced by Nerode in [5] where he showed that $\Lambda^*(A)$ is elementarily equivalent to a reduced power Q of the ring of integers and where he adapted the method of Feferman and Vaught [1] to $\Lambda^*(A)$. In [6] Nerode gave a procedure for finding isols and isolic integers which satisfy extensions to isols of recursively enumerable relations. The work which follows here both uses these results and, in some cases, strengthens them. We use the definitions and notation of [4], [5], and [6].

It is possible that a nice structure theorem for $\Lambda^*(A)$ can be proved; Nerode has asked whether $\Lambda^*(A)$ is isomorphic to Q, the reduced power of the integers just mentioned (see remarks following Corollary 5 below). E. Ellentuck has obtained such a result for the ring of Dedekind finite cardinals in the models of a particular extension of Zermelo Fraenkel set theory without the axiom of choice. However even if such a result is obtained for $\Lambda^*(A)$, since $\Lambda(A)$ cannot be defined in $\Lambda^*(A)$ by an infinite number of arithmetic formulas, it will still not be possible to arithmetically define in this way the substructure corresponding to $\Lambda(A)$.

1. We adopt the notation of [4], [5], and [6]. As in [5] and [7], $Q = Z^\omega/D_0$ where Z is the ring of integers and D_0 the filter of cofinite subsets of ω.

THEOREM 1. *Suppose $\{\varphi_i\}_{i<\omega}$ is a collection of arithmetic formulas in the variables $\bar{v} = (v_0, v_1, v_2, \cdots)$ and the collection is finitely satisfiable in Q. Then there exist $A, B \in Q^\omega$ such that*

(i) *for all* i, $Q \models \varphi_i(A)$
(ii) *for all* i, $Q \models \varphi_i(B)$
(iii) *for all* j, *if* $A_j \in E^*$ *then* $A_j = B_j$ *and if* $A_j \notin E^*$ *then* $B_j \notin E^*$ *and* $A_j \neq B_j$.

We give only a sketch of the proof. In [2] it is shown that Q is ω_1-saturated. From this we get $A \in Q^\omega$ satisfying (i). B is constructed from A. If $A_j \in E^*$, $B_j = A_j$. If $A_j \in Q - E^*$, B_j is gotten from A_j by a permutation of the coordinates. The permutation is simply to ensure that $B_j \neq A_j$ and, from the Feferman—Vaught method of [1], the permutation of coordinates does not affect satisfaction in Q.

LEMMA 2. *Suppose* $\varphi(v_1 \cdots, v_n)$ *is an arithmetic formula with* n *free variables,* $A_i \in \Lambda^*(A)$ *for* $1 \leqq i \leqq n$,

$$A_1 \notin E^* , \quad \textit{and} \quad \Lambda^*(A) \models \varphi(A_1, \cdots, A_n) .$$

Then there exist $B_i \in \Lambda^*(A)$ *for* $1 \leqq i \leqq n$, *with*

$$B_1 \neq A_1 \quad \textit{and} \quad \Lambda^*(A) \models \varphi(B_1, \cdots, B_n) .$$

Proof. Assume the conclusion is false. Then we have

$$\Lambda^*(A) \models (Ex_1) \cdots (Ex_n)[\varphi(x_1, \cdots, x_n) \wedge (y_1) \cdots (y_n)$$
$$(\varphi(y_1, \cdots, y_n) \rightarrow y_1 = x_1)] .$$

In [5] Nerode proved that $Q \equiv \Lambda^*(A)$. Hence this statement is also true in Q. For $i = 1$ to n, let $A'_i \in Q$ be the x_i whose existence is thus asserted. It follows from Theorem 1 that $A'_1 \in E^*$. So the arithmetic sentence $(Ex_2) \cdots (Ex_n)\varphi(A'_1, x_2 \cdots, x_n)$ is true in Q and hence in $\Lambda^*(A)$. So by our assumption $A_1 = A'_1 \in E^*$, a contradiction.

As a consequence of the results in § 3 of Nerode [5] we obtain almost at once the following result. Corresponding to any arithmetic formula φ there is another formula ψ which is a disjunction of conjunctions of equations and their negations, each such equation being of the form $f_{\Lambda^*(A)} = 0$ with f an arithmetic function whose free variables are among those of φ, and such that for

$$X \in \Lambda^*(A)^\omega , \qquad \Lambda^*(A) \models \varphi(X) \leftrightarrow \Lambda^*(A) \models \psi(X) .$$

LEMMA 3. *Suppose* $\{\varphi_i\}_{i<\omega}$ *is a collection of arithmetic formulas with free variables among* $\bar{v} = (v_0, v_1, v_2, \cdots)$, *and suppose any finite subset of this collection is satisfiable in* $\Lambda^*(A)$. *Then the collection is satisfiable in* $\Lambda^*(A)$.

Proof. To each φ_i there corresponds a ψ_i as above. Let us write ψ_i as $\bigvee_{j \in J_i} C_j^i$ where the C_j^i's are the various conjunctions of ψ_i. Since the φ_i's are finitely satisfiable in $\Lambda^*(A)$, so are the ψ_i's. We claim there is a $k \in J_0$ such that the collection $\{\psi_i\}_{i \in \omega - \{0\}} \cup \{C_k^0\}$ is finitely satisfiable in $\Lambda^*(A)$. If not then for each $j \in J_0$ there is a finite $S_j \subseteq \omega - \{0\}$ such that $\{\psi_i\}_{i \in S_j} \cup \{C_j^0\}$ is not satisfiable in $\Lambda^*(A)$. Let $S = \cup_{j \in J_0} S_j$. But $\{\psi_i\}_{i \in S} \cup \{\psi_0\}$*is* satisfiable and hence there is some disjunct C_k^0 of ψ_0 such that $\{\psi_i\}_{i \in S} \cup \{C_k^0\}$ is satisfiable and thus $\{\psi_i\}_{i \in S_k} \cup \{C_k^0\}$ is satisfiable, which is a contradiction. So $\{C_k^0\} \cup \{\psi_i\}_{i \in \omega - \{0\}}$ is finitely satisfiable in $\Lambda^*(A)$. Apply the same procedure to ψ_1 in this last collection, getting $\{C_k^0\} \cup \{C_k^1\} \cup \{\psi_i\}_{i \in \omega - \{0,1\}}$ finitely satisfiable. By induction we obtain in this way a collection $\{C^i\}_{i<\omega}$, with C^i one of the disjuncts of ψ_i, and the collection finitely satisfiable in $\Lambda^*(A)$. Simultaneously satisfying this collection would clearly also simultaneously satisfy $\{\varphi_i\}_{i<\omega}$. So the problem is now reduced to the following. Given a collection $\{f^i_{\Lambda^*(A)} = 0\}_{i<\omega} \cup \{g^i_{\Lambda^*(A)} \neq 0\}_{i<\omega}$, variables among $\bar{v} = (v_0, v_1, v_2, \cdots)$, each f and g an arithmetic function, and the collection finitely satisfiable in $\Lambda^*(A)$: show that the whole collection is satisfiable in $\Lambda^*(A)$.

Let $X_\omega E^*$ be the subset of $E^{*\omega}$ consisting of the finitely nonzero sequences and let $R^i(T^i)$ consist of those $x \in X_\omega E^*$ such that $f^i(x) = 0$ ($g^i(x) = 0$). We shall use Theorem 2.1 of Nerode [4], in the arithmetic case. A first application of this theorem enables us to infer from the finite satisfiability of $\{f^i_{\Lambda^*(A)} = 0\}$ in $\Lambda^*(A)$ that the collection $\{R^i\}_{i<\omega}$ generates a proper filter F^* in the lattice L^* of finitary arithmetic relations (see [6]). So by Theorem 4.7 of [6] in the arithmetic case, there is an $X \in \Lambda^*(A)^\omega$ such that

$$F^* = \{R \in L^* \mid X \in R_{\Lambda^*(A)}\} .$$

With another application of Theorem 2.1 of [4] we get $X \in R^i_{\Lambda^*(A)}$ if and only if $f^i_{\Lambda^*(A)}(X) = 0$. Since for all i, $R^i \in F^*$, X satisfies $\{f^i_{\Lambda^*(A)} = 0\}_{i<\omega}$. Does it also satisfy $\{g^i_{\Lambda^*(A)} \neq 0\}$? Suppose not; then there is some i such that $g^i_{\Lambda^*(A)}(X) = 0$. Then, as above, $X \in T^i_{\Lambda^*(A)}$. Hence $T^i \in F^*$. So there exist $R^{i_0}, \cdots, R^{i_n}$ such that

$$R^{i_0} \cap \cdots \cap R^{i_n} \subseteq T^i .$$

Thus $X_\omega E^* = \overline{R^{i_0}} \cup \cdots \cup \overline{R^{i_n}} \cup T^i$. Yet another application of Theorem 2.1 of [4] shows that the set $\{f^{i_0}_{\Lambda^*(A)} = 0, \cdots, f^{i_n}_{\Lambda^*(A)} = 0, g^i_{\Lambda^*(A)} \neq 0\}$ cannot be satisfied in $\Lambda^*(A)$, which contradicts the hypothesis. Thus X satisfies $\{f^i_{\Lambda^*(A)} = 0\}_{i<\omega} \cup \{g^i_{\Lambda^*(A)} \neq 0\}_{i<\omega}$, as required.

The next theorem is the analogue in $\Lambda^*(A)$ of Theorem 1.

THEOREM 4. *Suppose* $\{\varphi_i\}_{i<\omega}$ *is a collection of arithmetic formulas in the variables* $\bar{v} = (v_0, v_1, v_2, \cdots)$ *and the collection is finitely satisfiable in* $\Lambda^*(A)$. *Then three exist* $X, Y \in \Lambda^*(A)^\omega$ *such that*

(i) *for all* i, $\Lambda^*(A) \models \varphi_i(X)$

(ii) *for all* i, $\Lambda^*(A) \models \varphi_i(Y)$

(iii) *for all* j, *if* $X_j \in E^*$ *then* $X_j = Y_j$ *and if* $X_j \notin E^*$ *then* $Y_j \notin E^*$ *and* $X_j \neq Y_j$.

Proof. As in Lemma 3 we obtain $X' \in \Lambda^*(A)^\omega$ and a collection $K_v = \{f^i_{\Lambda^*(A)}(v) = 0\}_{i<\omega} \cup \{g^i_{\Lambda^*(A)}(v) \neq 0\}_{i<\omega}$ which X' satisfies and if any X'' satisfies K_v than X'' satisfies (i) above. For the variables $\bar{u} = (u_0, u_1, u_2, \cdots)$, let K_u be the collection gotten from K_v simply by replacing each v_i by u_i. Then $(X', X') \in (\Lambda^*(A)^\omega)^2$ satisfies $K_v \cup K_u$ (where $\varphi(Z, Z')$ for $(Z, Z') \in (\Lambda^*(A)^\omega)^2$ means replace v_i by Z_i and u_i by Z'_i). Using the lattice L^* of finitary arithmetic relations on $(X_\omega E^*)^2$, let R^i_v and R^i_u denote respectively the arithmetic relations corresponding to $f^i(v) = 0$ and $f^i(u) = 0$. Proceeding in the usual way (see [6]) we let F^* be the filter generated by the R^i_v's and R^i_u's and get $(X, Y) \in (\Lambda^*(A)^\omega)^2$. With methods like those of Lemma 3 we show that X satisfies (i) above and Y satisfies (ii). Because for each i, $X'_i = a$ for a in E^* if and only if

$$\{(v, u) \in (X_\omega E^*)^2 \mid v_i = a\} \quad \text{and} \quad \{(v, u) \in (X_\omega E^*)^2 \mid u_i = a\}$$

are in F^*, it follows that if X'_i is in E^* then $X'_i = X_i = Y_i$ and if $X'_i \notin E^*$ then $X_i \notin E^*$ and $Y_i \notin E^*$. Assume i is such that $X'_i \notin E^*$. We claim $X_i \neq Y_i$. Suppose not. Let $T = \{(v, u) \in (X_\omega E^*)^2 \mid v_i = u_i\}$. Then $(X, Y) \in T_{\Lambda^*(A)}$ and hence $T \in F^*$. Proceeding as in Lemma 3 there exist $f^{i_1}, \cdots, f^{i_n}, f^{j_1}, \cdots, f^{j_m}$ such that

$$\{f^{i_1}_{\Lambda^*(A)}(v) = 0, \cdots f^{i_n}_{\Lambda^*(A)}(v) = 0, f^{j_1}_{\Lambda^*(A)}(u) = 0, \cdots, f^{j_m}_{\Lambda^*(A)}(u) = 0, v_i \neq u_i\}$$

cannot be satisfied in $\Lambda^*(A)$. Thus for an appropriate finite set of φ's we have that $\bigwedge_j \varphi_j(v) \wedge \bigwedge_k \varphi_k(u) \wedge (v_i \neq u_i)$ cannot be satisfied in $\Lambda^*(A)$. But (X', X') satisfies $\bigwedge_j \varphi_j(v) \wedge \bigwedge_k \varphi_k(u)$ and $X'_i \notin E^*$. So by Lemma 2 we can find $X'' \in \Lambda^*(A)^\omega$ such that (X'', X'') satisfies $\bigwedge_j \varphi_j(v) \wedge \bigwedge_k \varphi_k(u)$ and $X''_i \neq X'_i$. But then (X', X'') satisfies

$$\bigwedge_j \varphi_j(v) \wedge \bigwedge_k \varphi_k(u) \wedge (v_i \neq u_i) ,$$

a contradiction. Thus $X_i \neq Y_i (T \notin F^*)$, completing the theorem.

COROLLARY 5. *No member of* $\Lambda^*(A) - E^*$ *can be defined in* $\Lambda^*(A)$ *even by an infinite number of arithmetic formulas.*

We conclude this section with another application of the methods used above. Nerode has asked whether $Q \cong \Lambda^*(A)$. (In [5] he showed $Q \equiv \Lambda^*(A)$ and both have power $2^{\aleph_0}$.) The answer is not trivially yes because by the following example, pointed out to the author by M. Morley, the theory of Q (and hence of $\Lambda^*(A)$) is not categorical in any infinite power. Let $p_0, p_1, p_2, \cdots$ be a sequence of distinct positive primes, and $q_0, q_1, q_2, \cdots$ a list of all the other positive primes. Consider $\{p_i \text{ divides } x\}_{i<\omega} \cup \{q_i \text{ does not divide } x\}_{i<\omega}$. This countable collection of arithmetic formulas is clearly finitely satisfiable in Q. Since Q is ω_1-saturated, the collection is simultaneously satisfiable in Q. But there are $2^{\aleph_0}$ such types realized in Q. So by a result of Ehrenfeucht, the theory of Q is not categorical in any infinite power.

In [7] it is shown that, assuming the continuum hypothesis, $Q \cong \Lambda^*(A)$ if and only if $\Lambda^*(A)$ is saturated. Using the method of Lemma 3 it can be shown that $\Lambda^*(A)$ is saturated if and only if for every collection $\{f^i_{\Lambda^*(A)} = 0\}_{i<\omega} \cup \{g^i_{\Lambda^*(A)} \neq 0\}_{i<\omega}$, where the f's and g's are arithmetic functions having exactly one free variable and possibly constants from $\Lambda^*(A)$, if the collection is finitely satisfiable in $\Lambda^*(A)$ then it is satisfiable in $\Lambda^*(A)$.

2. We shall prove shortly that $\Lambda(A)$ cannot be defined in $\Lambda^*(A)$ by an infinite number of formulas. In particular, one formula will not define $\Lambda(A)$ within $\Lambda^*(A)$. The former result requires a strengthening of the proof of Theorem 3.1 of [6]. But the latter can be proved as a corollary of that theorem.

THEOREM 6. *$\Lambda(A)$ cannot be defined in $\Lambda^*(A)$ by means of one arithmetic formula (and hence by means of a finite number of such formulas).*

Proof. We prove something a little stronger. Suppose $\varphi(x)$ is an arithmetic formula of one free variable, $X \in \Lambda(A) - E$, and $\Lambda^*(A) \models \varphi(X)$. We shall find $Y \in \Lambda^*(A) - (\Lambda(A) \cup - \Lambda(A))$ such that $\Lambda^*(A) \models \varphi(Y)$.

By the remark preceding Lemma 3, we have

$$f^1_{\Lambda^*(A)}(x) = 0 \wedge \cdots \wedge f^n_{\Lambda^*(A)}(x) = 0 \wedge g^1_{\Lambda^*(A)}(x) \neq 0 \wedge \cdots \wedge g^m_{\Lambda^*(A)}(x) \neq 0$$

satisfied by X, and if any $X' \in \Lambda^*(A)$ satisfies this conjunction then $\Lambda^*(A) \models \varphi(X')$. In particular for each j, $1 \leqq j \leqq m$,

$$f^1_{\Lambda^*(A)}(X) = 0 \wedge \cdots \wedge f^n_{\Lambda^*(A)}(X) = 0 \wedge g^j_{\Lambda^*(A)}(X) \neq 0 .$$

Since $X \in \Lambda(A) - E$, we can apply the first part of Theorem 11.1 of [3] to get the existence of an infinite number of members of E which are solutions in E^* to

$$(^*) f^1(x) = 0 \wedge \cdots \wedge f^n(x) = 0 \wedge g^j(x) \neq 0 \,.$$

We can now find a finite set $S, S \subseteqq E^*$, cardinality of $S \geqq 2$, such that for each $j, 1 \leqq j \leqq m$, there is an $a_j \in S$ satisfying (*) in E^*. Of course $S \in L^*$ (see [6], using only one coordinate). Let F^* be the filter of L^* generated by S. By Theorem 4.7 of [6] there is a $Y \in \Lambda^*(A)$ such that $F^* = \{R \in L^* \mid Y \in R_{\Lambda^*(A)}\}$. Since $S \in F^*$, $Y \in S_{\Lambda^*(A)}$. For each i, $1 \leqq i \leqq n$, we also have $S \subseteqq \{x \in E^* \mid f^i(x) = 0\} = T^i$. So $Y \in T^i_{\Lambda^*(A)}$. Again by Theorem 2.1 of [4] we have $f^i_{\Lambda^*(A)}(Y) = 0$, $1 \leqq i \leqq n$. We claim $g^j_{\Lambda^*(A)}(Y) \neq 0$, $1 \leqq j \leqq m$. Say, for some j, $g^j_{\Lambda^*(A)}(Y) = 0$. So for $U^j = \{x \in E^* \mid g^j(x) = 0\}$, again by Theorem 2.1 of [4], $Y \in U^j_{\Lambda^*(A)}$. Hence $U^j \in F^*$ and $U^j \supseteqq S$. Thus

$$E^* \models (x)(x \notin S \vee g^j(x) = 0) \,.$$

Again by Theorem 2.1, $\Lambda^*(A) \models (X)(X \notin S_{\Lambda^*(A)} \vee g^j_{\Lambda^*(A)}(X) = 0)$. But $a_j \in \Lambda^*(A)$, $a_j \in S_{\Lambda^*(A)}$ and $g^j_{\Lambda^*(A)}(a_j) \neq 0$, a contradiction. Thus for all j, $1 \leqq j \leqq m$, $g^j_{\Lambda^*(A)}(Y) \neq 0$. Hence $\Lambda^*(A) \models \varphi(Y)$. We claim $Y \notin E^*$. If $Y = a \in E^*$ then, for $U = \{a\}$, $Y \in U_{\Lambda^*(A)}$ and hence $U \in F^*$ and $U \supseteqq S$ which contradicts S having cardinality $\geqq 2$. So $Y \notin E^*$. But $Y \in S_{\Lambda^*(A)}$. Now by Corollary 5.10 of [4] (in the arithmetic case), $Y \in \Lambda^*(A) - (\Lambda(A) \cup - \Lambda(A))$.

The next theorem is the major result of this paper. We use the definitions and notation of [6] but always in the arithmetic rather than the recursive case. Further, for simplicity we take L^* to be the lattice of arithmetic subsets of E^* (since only one variable is needed) and L the lattice of arithmetic subsets of E^2. Consequently we ignore the notion of "support".

THEOREM 7. *The arithmetic isols $\Lambda(A)$ cannot be defined in the arithmetic isolic integers $\Lambda^*(A)$ by an infinite number of arithmetic formulas.*

Proof. We prove a slightly stronger result. Suppose $\{\varphi_i\}_{i<\omega}$ are arithmetic formulas of one free variable, $X \in \Lambda(A) - E$, and for each i, $\Lambda^*(A) \models \varphi_i(X)$. Then we will find $Y \in \Lambda^*(A) - (\Lambda(A) \cup - \Lambda(A))$ such that for each i, $\Lambda^*(A) \models \varphi_i(Y)$.

By the remark preceding Lemma 3 we get a collection

$$\{f^j_{\Lambda^*(A)}(x) = 0\}_{i<\omega} \cup \{g^j_{\Lambda^*(A)}(x) \neq 0\}_{i<\omega} \,,$$

with the f^j's and g^j's arithmetic functions of one variable, such that X satisfies the collection and if any $X' \in \Lambda^*(A)$ satisfies the collection then for each i, $\Lambda^*(A) \models \varphi_i(X')$. Let $T^j \subseteqq E$ be the set of nonnegative integer solutions to $f^j(x) = 0$. Since $f^j_{\Lambda^*(A)}(X) = 0$ for all j, the intersection of any finite number of T^j's is infinite (see [3]). Let

F^* be the filter in L^* consisting of all $R \in L^*$ such that R contains the intersection of some finite number of T^j's, except perhaps for a finite set. If U^j is the set of all integer solutions to $g^j(x) = 0$ then $U^j \notin F^*$ because, by results of [3] again, since X satisfies every $f^j_{A^*(A)}(x) = 0$ and also this particular $g^j_{A^*(A)}(x) \neq 0$, we have that every finite intersection of T^j's contains an infinite subset of E not in U^j. Let $R^0, R^1, R^2, \cdots$ be an enumeration of F^* and $W^0, W^1, W^2, \cdots$ an enumeration of $L^* - F^*$. As in Theorem 4.7 of [6], let $F = \{R^\wedge \mid R \in F^*\}$; F is a realizability filter in L enumerated by $(R^0)^\wedge, (R^1)^\wedge, (R^2)^\wedge, \cdots$. We can now prove a lemma essentially the same as Lemma 3.3 of [6], but using $(R^i)^\wedge$ in place of T^i, $(W^i)^\wedge$ in place of S^i, and $(L^* - F^*)^\wedge$ in place of $L - F$. (Note that $\{(W^i)^\wedge\}_{i<\omega}$ is *not* an enumeration of $L - F$.) Assume this has been done. (The same proof will work.)

We now wish to prove a lemma corresponding to Lemma 3.2 of [6] but with a stronger conclusion. So we shall define inductively (two at a time) $\{x^i\}_{i<\omega}$, $x^i \in E^2$. Let $P_0, P_1, P_2, \cdots$ be an enumeration of the one-one partial arithmetic functions of one variable. We shall also inductively define a set $G \subsetneqq E$ which will contain "integers to be avoided."

From the definition of F^* it follows that for each $m < \omega$, $(R^0 \cap \cdots \cap R^m) - W^m$ contains an infinite subset of E. We assume that $x^0, x^1, \cdots, x^{2n-1}$ have been defined, that G^{n-1} is the finite part of G defined so far, and that for any index s of the form $(x^0, b^0), (x^1, b^1), \cdots, (x^{2n-1}, b^{2n-1})$ we have $(\alpha_s)_0 \cap G^{n-1} = \varnothing$. $(G^{-1} = \varnothing)$. Choose $z \in (R^0 \cap \cdots \cap R^{2n}) - W^{2n}$ and $z' \in (R^0 \cap \cdots \cap R^{2n+1}) - W^{2n+1}$ such that $z > z'$. These choices are possible because these sets contain an infinite number of positive integers. Define $x^{2n} \in E^2$ to satisfy: (i) $x_0^{2n} - x_1^{2n} = z$, (ii) $\max(x_0^{2n-1}, x_1^{2n-1}) < \min(x_0^{2n}, x_1^{2n})$, and (iii) for any index s with an initial segment of the form $(x^0, b^0), \cdots, (x^{2n}, b^{2n})$, we have $(\alpha_s)_0 \cap G^{n-1} = \varnothing$. Taking into account the induction hypothesis on G^{n-1}, such a x^{2n} exists since there are an infinite number of members of E^2 satisfying (i) and (ii) and since G^{n-1} is finite, almost all of these satisfy (iii). Now let $V = \bigcup_s (\alpha_s)_0$ where the union is taken over all indexes s of the form $(x^0, b^0), \cdots, (x^{2n}, b^{2n})$. Then there exists $x^{2n+1} \in E^2$ such that: (i) $x_0^{2n+1} - x_1^{2n+1} = z'$, (ii) $\max(x_0^{2n}, x_1^{2n}) < \min(x_0^{2n+1}, x_1^{2n+1})$, and (iii) for any index s of the form $(x^0, b^0), \cdots, (x^{2n+1}, b^{2n+1})$ we have $P_n((\beta_s)_1) \cap V = \varnothing$. Again such a x^{2n+1} exists because V is a finite set and of the infinite number of ways to satisfy (i) and (ii), almost all will satisfy (iii). And we have $x^{2n} \in (R^0)^\wedge \cap \cdots \cap (R^{2n})^\wedge - (W^{2n})^\wedge$, $x^{2n+1} \in (R^0)^\wedge \cap \cdots \cap (R^{2n+1})^\wedge - (W^{2n+})^\wedge$. Since $z > z'$, we have $x_0^{2n} - x_1^{2n} > x_0^{2n+1} - x_1^{2n+1}$ and thus

$$x_1^{2n+1} - x_1^{2n} > x_0^{2n+1} - x_0^{2n} .$$

Let s be an index of the form $(x^0, b^0), \cdots, (x^{2n+1}, b^{2n+1})$. Then $(\beta_s)_1$

has cardinality $x_1^{2n+1} - x_1^{2n}$ and $(\beta_s)_0$ has the smaller cardinality $x_0^{2n+1} - x_0^{2n}$. For this s three cases may then occur.

Case I. P_n is not defined on all of $(\beta_s)_1$.

Case II. P_n is defined on all of $(\beta_s)_1$ and $P_n((\beta_s)_1) \subseteqq \bigcup_r (\beta_r)_0$ where the union is taken over all indexes r of the form $(x^0, b^0), \cdots, (x^{2n+1}, b^{2n+1})$. In this case, since $P_n((\beta_s)_1)$ has larger cardinality than $(\beta_s)_0$, there is an $r \neq s$ and an integer u such that $u \in P_n((\beta_s)_1) \cap (\beta_r)_0$.

Case III. Cases I and II do not hold. In this case, again since $P_n((\beta_s)_1)$ has larger cardinality than $(\beta_s)_0$, there is a

$$u \in P_n((\beta_s)_1 - \bigcup_r (\beta_r)_0 .$$

Define G^n to consist of G^{n-1} together with the u's which come from indexes s satisfying Case III. Then for any index s of the form $(x^0, b^0), \cdots, (x^{2n+1}, b^{2n+1})$ we have $(\alpha_s)_0 \cap G^n = \varnothing$ (using part (iii) of the definition of x^{2n+1}). This completes the inductive definitions of $\{x_i\}_{i<\omega}$ and G. We have at least the conclusions of Lemma 3.2 of [6] with $(R_i)^\wedge$ replacing T^i, $(W^i)^\wedge$ replacing S^i, and $(L^* - F^*)^\wedge$ replacing $L - F$.

Proceeding now as in [6] we obtain the infinite sequence (x^0, y^0), $(x^1, y^1), \cdots$ whose initial segments are the indexes

$$t_d = (x^0, y^0), \cdots, (x^d, y^d) ,$$

and such that $\alpha = \alpha_{t_0} \vee \alpha_{t_1} \vee \cdots$. Let $Y = \langle \alpha_0 \rangle - \langle \alpha_1 \rangle$. We can now prove (as in [6]) lemmas corresponding to Lemmas 3.4 and 3.6 of [6]. Assume this has been done. We claim α_0 and α_1 are arithmetically isolated sets. If one of them is not, then proceeding as in Lemma 3.5 of [6] we could show that α_0 and α_1 are both arithmetically enumerable. Let $c_0, c_1, c_2, \cdots$ and $e_0, e_1, e_2, \cdots$ be arithmetic enumerations of α_0 and α_1 respectively. Let H consist of all pairs of sets of the form

$$(\{c_0, \cdots, c_{2n+2}\}, \{e_0, \cdots, e_{2n}\}), \qquad n = 0, 1, 2, \cdots .$$

Then α is attainable from the arithmetic $\{2\}$-frame H. But $\{2\} \notin F^*$. So $\{2\} = W^i$ for some i and by our version of Lemma 3.4 of [6], α could not be attainable from H, a contradiction. So

$$Y = \langle \alpha_0 \rangle - \langle \alpha_1 \rangle \in \Lambda^*(A) .$$

We wish to show $Y \notin \Lambda(A) \cup -\Lambda(A)$. This is equivalent to showing that $\langle \alpha_0 \rangle \nleqq \langle \alpha_1 \rangle$ and $\langle \alpha_1 \rangle \nleqq \langle \alpha_0 \rangle$. Suppose $\langle \alpha_0 \rangle \leqq \langle \alpha_1 \rangle$. If

W is the set of nonpositive integers then $W \notin F^*$ and so $W = W^i$ for some i. But $W^\wedge = \{(x_0, x_1) \in E^2 \mid x_0 \leqq x_1\}$, and since $\langle\alpha_0\rangle \leqq \langle\alpha_1\rangle$ we must have $\langle\alpha_0\rangle - \langle\alpha_1\rangle \in W_{\Lambda^*(A)}$, a contradiction. Now assume $\langle\alpha_0\rangle \geqq \langle\alpha_1\rangle$. So there is a one-one partial arithmetic function of one variable, call it P_m, such that P_m is defined on α_1 and $P_m(\alpha_1) \subseteqq \alpha_0$. Recall the definitions of x^{2m} and x^{2m+1}. For the index t_{2m+1} Case I could not have held since P_m is defined on all of α_1. If Case II held then we have the u as described there. But by property (3.10) of [6] it follows that although $u \in P_m(\alpha_1)$, u could not be in α_0. So Case II could not have held and Case III must have held. So there is a $u \in P_m(\alpha_1) \cap G$. But by the construction of G, any $u \in G$ could not be in any $(\alpha_{t_d})_0$ and thus $u \notin \alpha_0$, a contradiction. So

$$\langle\alpha_0\rangle \ngeqq \langle\alpha_1\rangle, \text{ and } Y \in \Lambda^*(A) - (\Lambda(A) \cup - \Lambda(A)) .$$

Now by Theorem 2.1 of [4] (arithmetic case),

$$Y \in T^j_{\Lambda^*(A)} \rightarrow f^j_{\Lambda^*(A)}(Y) = 0 .$$

But $T^j \in F^*$, and thus $Y \in T^j_{\Lambda^*(A)}$. So for all $j < \omega, f^j_{\Lambda^*(A)}(Y) = 0$. Suppose $g^j_{\Lambda^*(A)}(Y) = 0$. By the same theorem, $Y \in U^j_{\Lambda^*(A)} \leftrightarrow g^j_{\Lambda^*(A)}(Y) = 0$. So $Y \in U^j_{\Lambda^*(A)}$. But we showed $U^j \notin F^*$; thus U^j is some W^i and so $Y \notin U^j_{\Lambda^*(A)}$. Hence $g^j_{\Lambda^*(A)}(Y) \neq 0$ and the proof is complete.

COROLLARY 8. *If F^*, a filter in L^*, cannot be generated by a singleton set, then there is a $Y \in \Lambda^*(A) - (\Lambda(A) \cup - \Lambda(A))$ which realizes F^*.*

Proof. First of all, if F^* is generated by some singleton set $\{a\}$ for $a \in E^*$, then by Theorem 4.7 of [6], a is the one and only member of $\Lambda^*(A)$ which realizes F^*. Now assume F^* cannot be generated by a singleton set. Again by Theorem 4.7 of [6], there is an $X \in \Lambda^*(A)$ which realizes F^*. If $X \in E^*$ it would follow that F^* is generated by $\{X\}$. So $X \in \Lambda^*(A) - E^*$. If

$$X \in \Lambda^*(A) - (\Lambda(A) \cup - \Lambda(A)) ,$$

we would be done. If not, then using this X we can proceed as in the theorem to get $Y \in \Lambda^*(A) - (\Lambda(A) \cup - \Lambda(A))$ realizing F^*.

We remark that the corollary holds also in the recursive case.

3. In this section we prove the analogue for $\Lambda^*(A)$ of one of the major results of [7].

DEFINITION 9. A subset S of $\Lambda^*(A)$ is said to be indecomposable if for every arithmetic relation $R(x_1, \cdots, x_n)$ and every $X_1, \cdots, X_n \in S$, either $(X_1, \cdots, X_n) \in R_{\Lambda^*(A)}$ or $(X_1, \cdots, X_n) \in \bar{R}_{\Lambda^*(A)}$ ($\bar{R}$ is the complement of R).

If we apply Theorem 4.7 of [6], using just one variable and taking an ultrafilter F^*, then the $X \in \Lambda^*(A)$ so realized is indecomposable (that is, $\{X\}$ is indecomposable). So indecomposable subsets of $\Lambda^*(A)$ exist, and maximal indecomposable subsets properly contain E^*.

THEOREM 10. *Maximal indecomposable subsets of $\Lambda^*(A)$ are proper elementary extensions of E^*.*

Proof. Let S be a maximal indecomposable subset. By the remarks above we need only show that S is an elementary extension of E^*.

We shall first show that S is closed under arithmetic functions extended to $\Lambda^*(A)$. Suppose $X_1, \cdots, X_n \in S$ and $f(x_1, \cdots, x_n)$ is an arithmetic function. Let $f_{\Lambda^*(A)}(X_1, \cdots, X_n) = X \in \Lambda^*(A)$. Consider $S' = S \cup \{X\}$. If it is not indecomposable, there are $Y_1, \cdots, Y_m \in S$ and an arithmetic relation $R(x_1, \cdots, x_{m+1})$ such that

$$(X, Y_1, \cdots, Y_m) \notin R_{\Lambda^*(A)} \cup \bar{R}_{\Lambda^*(A)} .$$

The following statement is true in E^*: $(x_1) \cdots (x_n)(x)(y_1) \cdots (y_m)$

$$[f(x_1, \cdots, x_n) = x \wedge R(f(x_1, \cdots, x_n), y_1, \cdots, y_m) \rightarrow R(x, y_1, \cdots, y_m)] .$$

And, of course, the same statement is true with R replaced by $\bar{R}$. Since f is an arithmetic function and R and $\bar{R}$ are arithmetic relations we can apply Theorem 2.1 (ii) of [4] in the arithmetic case. This gives:

$$\Lambda^*(A) \models \sim R_{\Lambda^*(A)}(X, Y_1, \cdots, Y_m) \rightarrow \sim R_{\Lambda^*(A)}(f_{\Lambda^*(A)}(X_1, \cdots, X_n), Y_1, \cdots, Y_m)$$

and the same statement with R replaced by $\bar{R}$. Define an arithmetic relation $R'(x_1, \cdots, x_n, y_1, \cdots, y_m) \leftrightarrow R(f(x_1, \cdots, x_n), y_1, \cdots, y_m)$. (Note that $\bar{R}'(x_1, \cdots, x_n, y_1, \cdots, y_m) \leftrightarrow \bar{R}(f(x_1, \cdots, x_n), y_1, \cdots, y_m)$.) Then $R'_{\Lambda^*(A)}(X_1, \cdots, X_n, Y_1, \cdots, Y_m) \leftrightarrow R_{\Lambda^*(A)}(f_{\Lambda^*(A)}(X_1, \cdots, X_n), Y_1, \cdots, Y_m)$. Combining now we get $(X_1, \cdots, X_n, Y_1, \cdots, Y_m) \notin R'_{\Lambda^*(A)} \cup \bar{R}'_{\Lambda^*(A)}$. Since $X_1, \cdots, X_n, Y_1, \cdots, Y_m \in S$, this contradicts the indecomposability of S. Hence S' is indecomposable and since S is maximal, $S' = S$ and $X \in S$. So S is closed under extended arithmetic functions.

The theory of arithmetic has definable Skolem functions. So every statement of arithmetic is equivalent to a universal statement in which the matrix is in disjunctive normal form, $\vee \wedge P$, where P is of the form $u + v = w$, $u \cdot v = w$, or $u \neq v$ and u, v, w can be variables, integers, or arithmetic Skolem functions of variables.

Suppose φ is a true statement of arithmetic. We claim φ is true in S, with Skolem functions which are the extensions to $\Lambda^*(A)$ of the arithmetic Skolem functions for φ. φ has the form $(x_1) \cdots (x_n)[\bigvee_{i=1}^{m} R_i]$ where each R_i is a conjunction of P's as described above. So assume $X_1, \cdots, X_n \in S$. Suppose $(X_1, \cdots, X_n) \notin (R_i)_{\Lambda^*(A)}$ for $i = 1$ to $m-1$. Then by the indecomposability of S, $(X_1, \cdots, X_n) \in (\bar{R}_i)_{\Lambda^*(A)}$ for $i = 1$ to $m-1$ and thus $(X_1, \cdots, X_n) \in (\bigcap_{i=1}^{m-1} \bar{R}_i)_{\Lambda^*(A)}$. But because φ is true in E^*, we have $\bigcap_{i=1}^{m-1} \bar{R}_i \subseteqq R_m$. Thus $(X_1, \cdots, X_n) \in (R_m)_{\Lambda^*(A)}$. Hence there is some R_i, call it R, such that $(X_1, \cdots, X_n) \in R_{\Lambda^*(A)}$. Let P be one of the conjuncts in R. Then $(X_1, \cdots, X_n) \in P_{\Lambda^*(A)}$. But a triple from $\Lambda^*(A)$ which is in the extension of the addition relation in E^* satisfies the addition relation in $\Lambda^*(A)$; similarly for multiplication and inequality. And since extension commutes with composition, each P in R is satisfied in $\Lambda^*(A)$ when X_i is substituted for the variable x_i. But by the earlier part of this proof, $f_{\Lambda^*(A)}(X_1, \cdots, X_n) \in S$ for the Skolem function $f(x_1, \cdots, x_n)$. Thus φ is true in S. Since φ was any statement true in arithmetic we have the converse and hence S is an elementary extension of E^*, as required.

Suppose S is a maximal indecomposable subset of $\Lambda^*(A)$. Let T be a subset of S, $T \supseteqq E^*$, and let $\bar{T}$ be the closure of T in S under arithmetic functions extended to $\Lambda^*(A)$ (S being closed under such functions). From the proof of the theorem, these functions were exactly the Skolem functions for S. Hence $\bar{T}$ is an elementary extension of E^* and $\bar{T}$ and S are elementarily equivalent. (Clearly $\bar{T}$ is contained in every maximal indecomposable S which contains T.) In particular, if $X \in \Lambda^*(A)$ is indecomposable, then

$$\{f_{\Lambda^*(A)}(X) \mid f \text{ is a one-place arithmetic function}\}$$

is an elementary extension of E^*, which *is* E^* if X is finite and which properly extends E^* if $X \in \Lambda^*(A) - E^*$.

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