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TRIPLE SERIES EQUATIONS INVOLVING LAGUERRE POLYNOMIALS

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In this paper it is shown that the problem of solving the triple series equations of the first kind

$$(1) \qquad \sum_{n=0}^{\infty} A_n \Gamma(\alpha + 1 + n) L_n(\alpha; x) = 0 , \qquad 0 \leqq x < a ,$$

$$(2) \qquad \sum_{n=0}^{\infty} A_n \Gamma(\alpha + \beta + n) L_n(\alpha; x) = f(x) , \qquad a < x < b,$$

$$(3) \qquad \sum_{n=0}^{\infty} A_n \Gamma(\alpha + 1 + n) L_n(\alpha; x) = 0 , \qquad b < x < \infty ,$$

and the triple series equations of the second kind

$$(4) \qquad \sum_{n=0}^{\infty} B_n \Gamma(\alpha + \beta + n) L_n(\alpha; x) = g(x) , \qquad 0 \leqq x < a ,$$

$$(5) \qquad \sum_{n=0}^{\infty} B_n \Gamma(\alpha + 1 + n) L_n(\alpha; x) = 0 , \qquad a < x < b,$$

$$(6) \qquad \sum_{n=0}^{\infty} B_n \Gamma(\alpha + \beta + n) L_n(\alpha; x) = h(x) , \qquad b < x < \infty ,$$

where $\alpha + \beta > 0$, $0 < \beta < 1$, $L_n(\alpha; x) = L_n^{\alpha}(x)$ is the Laguerre polynomial and $f(x)$, $g(x)$ and $h(x)$ are known functions, can be reduced to that of solving a Fredholm integral equation of the second kind. The analysis is formal and no attempt is made to supply details of rigour.

In a recent paper [1] the present author has solved the dual series equations

$$(7) \qquad \sum_{n=0}^{\infty} C_n \Gamma(\alpha + \beta + n) L_n(\alpha; x) = f(x) , \qquad 0 \leqq x < d ,$$

$$(8) \qquad \sum_{n=0}^{\infty} C_n \Gamma(\alpha + 1 + n) L_n(\alpha; x) = g(x) , \qquad d < x < \infty ,$$

where $\alpha + \beta > 0$, $0 < \beta < 1$. The triple series equations (1) to (6) can be considered to be extensions of the equations (7) and (8). In other papers [2], [3], [5], and in a book by Sneddon [4], the solution of triple integral equations involving Bessel functions and triple series equations involving functions orthogonal on a finite interval have been considered and in every case the solution is expressed in terms of the solution of one or two Fredholm integral equations.

2. In the course of the analysis we shall use the following results.

The orthogonality relation for Laguerre polynomials is

$$(9)\quad \int_0^\infty x^\alpha e^{-x} L_m(\alpha; x) L_n(\alpha; x) dx = \frac{\Gamma(\alpha + 1 + n)}{\Gamma(n + 1)} \delta_{mn}, \qquad \alpha > -1,$$

where δ_{mn} is the Kronecker delta.

Using the results (5) and (20) given in [1] it is easily shown that

$$(10)\qquad S(r, x) = (rx)^\alpha \sum_{n=0}^\infty \frac{\Gamma(n + 1)\Gamma(\alpha + \beta + n)}{\{\Gamma(\alpha + 1 + n)\}^2} L_n(\alpha; x) L_n(\alpha; r)$$

$$(11)\qquad = \frac{1}{\{\Gamma(1 - \beta)\}^2} \int_0^t n(y)(r - y)^{-\beta}(x - y)^{-\beta} dy = \frac{1}{\{\Gamma(1 - \beta)\}^2} S_t(r, x),$$

where $\beta < 1$, $\alpha + \beta > 0$, $n(y) = e^y y^{\alpha+\beta-1}$ and $t = \min(r, x)$.

If $f(x)$ and $f'(x)$ are continuous in $a \leqq x \leqq b$ and if $0 < \sigma < 1$, then the solutions of the Abel integral equations

$$(12)\qquad f(x) = \int_a^x \frac{F(y)}{(x - y)^\sigma} dy,$$

and

$$(13)\qquad f(x) = \int_x^b \frac{F(y)}{(y - x)^\sigma} dy,$$

are given by

$$(14)\qquad F(y) = \frac{\sin(\sigma\pi)}{\pi} \frac{d}{dy} \int_a^y \frac{f(x)}{(y - x)^{1-\sigma}} dx,$$

and

$$(15)\qquad F(y) = -\frac{\sin(\sigma\pi)}{\pi} \frac{d}{dy} \int_y^b \frac{f(x)}{(x - y)^{1-\sigma}} dx,$$

respectively.

3. Equations of the first kind. In order to solve the triple series equations of the first kind we set

$$(16)\qquad \sum_{n=0}^\infty A_n \Gamma(\alpha + 1 + n) L_n(\alpha; x) = \phi(x), \qquad a < x < b,$$

and using the orthogonality relation (9) we find from equations (1), (16) and (3)

$$A_n = \frac{\Gamma(n+1)}{\{\Gamma(\alpha+1+n)\}^2}\int_a^b r^\alpha e^{-r}\phi(r)L_n(\alpha; r)dr\,, \qquad \alpha > -1\,. \tag{17}$$

Substituting for A_n in equation (2) and interchanging the order of summation and integration we have

$$\int_a^b e^{-r}\phi(r)S(r, x)dr = x^\alpha f(x)\,, \qquad a < x < b\,, \tag{18}$$

where $S(r, x)$ is defined by equation (10).

Using the notation of equation (11) this can be written as

$$\int_a^x e^{-r}\phi(r)S_r(r, x)dr + \int_x^b e^{-r}\phi(r)S_x(r, x)dr = \{\Gamma(1-\beta)\}^2 x^\alpha f(x)\,, \tag{19}$$

where $a < x < b$, $\alpha + \beta > 0$, $\beta < 1$.

Inverting the order of integration in equation (19) we find that it becomes

$$\begin{aligned}\int_a^x \frac{n(y)}{(x-y)^\beta}\Phi(y)dy &= \{\Gamma(1-\beta)\}^2 x^\alpha f(x)\\ &\quad - \int_0^a \frac{n(y)}{(x-y)^\beta}dy\int_a^b \frac{e^{-r}\phi(r)}{(r-y)^\beta}dr\,,\end{aligned} \tag{20}$$

where

$$\Phi(y) = \int_y^b \frac{e^{-r}\phi(r)}{(r-y)^\beta}dr\,. \tag{21}$$

When $0 < \beta < 1$, equation (21) is an Abel integral equation and using the results (13) and (15), we find

$$e^{-r}\phi(r) = -\frac{\sin(\beta\pi)}{\pi}\frac{d}{dr}\int_r^b \frac{\Phi(y)}{(y-r)^{1-\beta}}dy\,. \tag{22}$$

Also, from the results (12) and (14), we see that equation (20) can be written as

$$n(y)\Phi(y) = F(y) - \frac{\sin(\beta\pi)}{\pi}\int_0^a n(\xi)l(y, \xi)d\xi\int_a^b \frac{e^{-r}\phi(r)}{(r-\xi)^\beta}dr\,, \tag{23}$$

for $a + \beta > 0$, $0 < \beta < 1$, $a < y < b$, where

$$F(y) = \frac{\Gamma(1-\beta)}{\Gamma(\beta)}\frac{d}{dy}\int_a^y \frac{x^\alpha f(x)}{(y-x)^{1-\beta}}dx\,, \tag{24}$$

is a known function, and

$$l(y, \xi) = \frac{d}{dy}\int_a^y \frac{dx}{(y-x)^{1-\beta}(x-\xi)^\beta}\,. \tag{25}$$

It is shown in [2] that

$$l(y, \xi) = \frac{(a-\xi)^{1-\beta}}{(y-\xi)(y-a)^{1-\beta}}, \qquad 0 < \beta < 1, \tag{26}$$

and hence equation (23) becomes

$$n(y)\Phi(y) = F(y) - \frac{\sin(\beta\pi)}{\pi(y-a)^{1-\beta}}\int_0^a \frac{(a-\xi)^{1-\beta}}{y-\xi} n(\xi)d\xi \int_a^b \frac{e^{-r}\phi(r)}{(r-\xi)^\beta} dr. \tag{27}$$

Using equation (22) we can write

$$\begin{aligned} \int_a^b \frac{e^{-r}\phi(r)}{(r-\xi)^\beta}dr &= -\frac{\sin(\beta\pi)}{\pi}\int_a^b \frac{dr}{(r-\xi)^\beta}\frac{d}{dr}\int_r^b \frac{\Phi(y)}{(y-r)^{1-\beta}}dy \\ &= \frac{\sin(\beta\pi)}{\pi}\left\{\frac{1}{(a-\xi)^\beta}\int_a^b \frac{\Phi(y)}{(y-a)^{1-\beta}}dy \right. \\ &\quad \left. - \beta\int_a^b \frac{dr}{(r-\xi)^{1+\beta}}\int_r^b \frac{\Phi(y)}{(y-r)^{1-\beta}}dy\right\}, \end{aligned} \tag{28}$$

after an integration by parts.

Inverting the order of integration in the last term of equation (28) and using the result given in [2]

$$\beta\int_a^y \frac{dr}{(r-\xi)^{1+\beta}(y-r)^{1-\beta}} = \frac{(y-a)^\beta}{(y-\xi)(a-\xi)^\beta}, \qquad 0 < \beta < 1, \tag{29}$$

we get

$$\int_a^b \frac{e^{-r}\phi(r)}{(r-\xi)^\beta}dr = \frac{\sin(\beta\pi)}{\pi(a-\xi)^{\beta-1}}\int_a^b \frac{\Phi(y)}{(y-\xi)(y-a)^{1-\beta}}dy. \tag{30}$$

Substituting for this expression in equation (27) we find that $\Phi(y)$ is given by the equation

$$n(y)\Phi(y) = F(y) - \int_a^b \Phi(x)M(x, y)dx, \qquad 0 < \beta < 1, \quad a < y < b, \tag{31}$$

where $M(x, y)$ is the symmetric kernel

$$M(x, y) = \frac{\sin^2(\beta\pi)}{\pi^2[(x-a)(y-a)]^{1-\beta}}\int_0^a \frac{n(\xi)(a-\xi)^{2(1-\beta)}}{(x-\xi)(y-\xi)}d\xi. \tag{32}$$

Equation (31) is a Fredholm integral equation which determines $\Phi(y)$, $\phi(r)$ is then obtained from equation (22) and the coefficients A_n which satisfy the equations (1), (2) and (3), when $\alpha + \beta > 0$, $0 < \beta < 1$, can be found from equation (17).

If we put $a = 0$ the triple series equations (1), (2) and (3) reduce to the dual series equations considered in [1] and it is easily shown that the above solution reduces to the solution obtained in that paper.

4. Equations of the second kind. To solve the triple series equations (4), (5) and (6) we put

$$\begin{aligned}(33)\qquad \sum_{n=0}^{\infty} B_n\Gamma(\alpha+1+n)L_n(\alpha; x) &= \psi_1(x)\ , \qquad 0 \leqq x < a\ ,\\ &= \psi_2(x)\ , \qquad b < x < \infty\ ,\end{aligned}$$

and use the orthogonality relation (9) to find that equations (5) and (33) give

$$(34)\quad B_n = \frac{\Gamma(n+1)}{\{\Gamma(\alpha+1+n)\}^2}\left\{\int_0^a \psi_1(r) + \int_b^\infty \psi_2(r)\right\} r^\alpha e^{-r} L_n(\alpha; r)dr\ , \qquad \alpha > -1\ .$$

Substituting for B_n in equations (4) and (6) and interchanging the order of summation and integration we find

$$\begin{aligned}(35)\qquad \left\{\int_0^a \psi_1(r) + \int_b^\infty \psi_2(r)\right\} e^{-r}S(r, x)dr &= x^\alpha g(x)\ , \qquad 0 \leqq x < a\ ,\\ &= x^\alpha h(x)\ , \qquad b < x < \infty\ ,\end{aligned}$$

where $S(r, x)$ is given by equation (10).

In the notation of equation (11) these equations may be written as

$$\begin{aligned}(36)\qquad \int_0^x e^{-r}\psi_1(r)S_r(r, x)dr &+ \int_x^a e^{-r}\psi_1(r)S_x(r, x)dr\\ &+ \int_b^\infty e^{-r}\psi_2(r)S_x(r, x)dr = \{\Gamma(1-\beta)\}^2 x^\alpha g(x)\ ,\end{aligned}$$

where $0 \leqq x < a$,

$$\begin{aligned}(37)\qquad \int_0^a e^{-r}\psi_1(r)S_r(r, x)dr &+ \int_b^x e^{-r}\psi_2(r)S_r(r, x)dr\\ &+ \int_x^\infty e^{-r}\psi_2(r)S_x(r, x)dr = \{\Gamma(1-\beta)\}^2 x^\alpha h(x)\ ,\end{aligned}$$

where $b < x < \infty$, $\alpha + \beta > 0$, $\beta < 1$.

Inverting the order of integration in the above equations we have

$$(38)\quad \int_0^x \frac{n(y)}{(x-y)^\beta}\left\{\Psi_1(y) + \int_b^\infty \frac{e^{-r}\psi_2(r)}{(r-y)^\beta}dr\right\}dy = \{\Gamma(1-\beta)\}^2 x^\alpha g(x)\ ,$$

for $0 \leqq x < a$, and

$$\begin{aligned}(39)\quad \int_b^x \frac{n(y)}{(x-y)^\beta}\Psi_2(y)dy &= \{\Gamma(1-\beta)\}^2 x^\alpha h(x) - \int_0^a \frac{n(y)}{(x-y)^\beta}\Psi_1(y)dy\\ &\quad - \int_0^b \frac{n(y)}{(x-y)^\beta}dy\int_b^\infty \frac{e^{-r}}{(r-y)^\beta}\psi_2(r)dr\ ,\end{aligned}$$

where $b < x < \infty$,

$$(40)\qquad \text{(i)}\quad \Psi_1(y) = \int_y^a \frac{e^{-r}}{(r-y)^\beta}\psi_1(r)dr\ ,$$

$$\text{(ii)}\quad \Psi_2(y) = \int_y^\infty \frac{e^{-r}}{(r-y)^\beta}\psi_2(r)dr\ .$$

When $0 < \beta < 1$ we may use the results (12) to (15) and deal with equations (38), (39) and (40) in a similar manner to that used to obtain equations (22) and (27) and find that

$$(41)\qquad n(y)\Psi_1(y) = G(y) - n(y)\int_b^\infty \frac{e^{-r}\psi_2(r)}{(r-y)^\beta}dr\ ,$$

$$(42)\qquad n(y)\Psi_2(y) = H(y) - \frac{\sin(\beta\pi)}{\pi(y-b)^{1-\beta}}\int_0^a \frac{(b-\xi)^{1-\beta}}{y-\xi}n(\xi)\Psi_1(\xi)d\xi - \frac{\sin(\beta\pi)}{\pi(y-b)^{1-\beta}}\int_0^b \frac{(b-\xi)^{1-\beta}}{y-\xi}n(\xi)d\xi\int_b^\infty \frac{e^{-r}\psi_2(r)}{(r-\xi)^\beta}dr\ ,$$

$$(43)\qquad e^{-r}\psi_1(r) = -\frac{\sin(\beta\pi)}{\pi}\frac{d}{dr}\int_r^a \frac{\Psi_1(y)}{(y-r)^{1-\beta}}dy\ ,\qquad 0 < r < a\ ,$$

and

$$(44)\qquad e^{-r}\psi_2(r) = -\frac{\sin(\beta\pi)}{\pi}\frac{d}{dr}\int_r^\infty \frac{\Psi_2(y)}{(y-r)^{1-\beta}}dy\ ,\qquad b < r < \infty\ ,$$

where $G(y)$ and $H(y)$ are the known functions

$$(45)\qquad G(y) = \frac{\Gamma(1-\beta)}{\Gamma(\beta)}\frac{d}{dy}\int_0^y \frac{x^\alpha g(x)}{(y-x)^{1-\beta}}dx\ ,\qquad 0 < y < a\ ,$$

and

$$(46)\qquad H(y) = \frac{\Gamma(1-\beta)}{\Gamma(\beta)}\frac{d}{dy}\int_b^y \frac{x^\alpha h(x)}{(y-x)^{1-\beta}}dx\ ,\qquad b < y < \infty\ .$$

From equation (41), using the result (43), it can be shown that $\psi_1(r)$ and $\psi_2(r)$ are related by the equation

$$(47)\qquad e^{-r}\psi_1(r) = -\frac{\sin(\beta\pi)}{\pi}\frac{d}{dr}\int_r^a \frac{G(y)}{n(y)(y-r)^{1-\beta}}\mathrm{d}y + \frac{\sin(\beta\pi)}{\pi(a-r)^{1-\beta}}\int_b^\infty \frac{e^{-\xi}(\xi-a)^{1-\beta}}{r-\xi}\psi_2(\xi)d\xi\ ,$$

where $0 < r < a$.

By a similar method to that used to obtain equation (30) we can show that

$$(48)\qquad \int_b^\infty \frac{e^{-r}\psi_2(r)}{(r-y)^\beta}dr = \frac{\sin(\beta\pi)}{\pi(b-y)^{\beta-1}}\int_b^\infty \frac{(\xi-b)^{\beta-1}}{\xi-y}\Psi_2(\xi)d\xi\ .$$

Using this result and equation (41) it can be shown, after some manipulation, that equation (42) can be written in the form

$$(49) \qquad n(y)\Psi_2(y) + \int_b^\infty \Psi_2(x)N(x, y)dx = H(y) - \frac{\sin(\beta\pi)}{\pi(y-b)^{1-\beta}} \int_0^a \frac{(b-\xi)^{1-\beta}}{y-\xi} G(\xi)d\xi ,$$

where $N(x, y)$ is the symmetric kernel

$$(50) \quad N(x, y) = \frac{\sin^2(\beta\pi)}{\pi^2[(x-b)(y-b)]^{1-\beta}} \int_a^b \frac{n(\xi)(b-\xi)^{2(1-\beta)}}{(x-\xi)(y-\xi)} d\xi , \quad b < y < \infty .$$

Equation (49) is a Fredholm integral equation of the second kind which determines $\Psi_2(y)$, $\psi_2(r)$ can then be found from equation (44) and $\psi_1(r)$ from equation (47). Finally, the coefficients B_n which satisfy the triple series equations of the second kind when $\alpha + \beta > 0$, $0 < \beta < 1$, are given by equation (34).

If we let $b \to \infty$ in equations (4), (5) and (6) they reduce to the dual series equations considered in [1] and the above solution can be shown to agree with the solution obtained in that paper.

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