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EIGENVALUES OF THE ADJACENCY MATRIX OF CUBIC LATTICE GRAPHS

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A cubic lattice graph is defined to be a graph G, whose vertices are the ordered triplets on n symbols, such that two vertices are adjacent if and only if they have two coordinates in common. If $n_2(x)$ denotes the number of vertices y, which are at distance 2 from x and $A(G)$ denotes the adjacency matrix of G, then G has the following properties: (P_1) the number of vertices is n^3. (P_2) G is connected and regular. (P_3) $n_2(x) = 3(n-1)^2$. (P_4) the distinct eigenvalues of $A(G)$ are -3, $n-3$, $2n-3$, $3(n-1)$. It is shown here that if $n > 7$, any graph G (with no loops and multiple edges) having the properties $(P_1)-(P_4)$ must be a cubic lattice graph. An alternative characterization of cubic lattice graphs has been given by the author (J. Comb. Theory, Vol. 3, No. 4, December 1967, 386-401).

We shall consider only finite undirected graphs without loops or multiple edges. A cubic lattice graph with characteristic n is defined to be a graph whose vertices are identified with the n^3 ordered triplets on n symbols, with two vertices adjacent if and only if their corresponding triplets have two coordinates in common. If $d(x, y)$ denotes the distance between two vertices x and y and $\Delta(x, y)$ the number of vertices adjacent to both x and y, then it has been shown by the author [6] that for $n > 7$, the following properties characterize the cubic lattice graph with characteristic n:

(b_1) The number of vertices is n^3.

(b_2) The graph is connected and regular of degree $3(n-1)$.

(b_3) If $d(x, y) = 1$, then $\Delta(x, y) = n - 2$.

(b_4) If $d(x, y) = 2$, then $\Delta(x, y) = 2$.

(b_5) If $d(x, y) = 2$, there exist exactly $n-1$ vertices z, adjacent to x such that $d(y, z) = 3$.

Dowling [4] in a note has shown that the property (b_5) is implied by properties $(b_1) - (b_4)$ for $n > 7$. Hence for $n>7$, $(b_1) - (b_4)$ characterize a cubic lattice graph with characteristic n.

The adjacency matrix $A(G)$ of a graph G is a square (0, 1) matrix whose rows and columns correspond to the vertices of G, and $a_{ij} = 1$ if and only if i and j are adjacent. Let $n_2(x)$ denote the number of vertices y at distance 2 from x.

A cubic lattice graph G with characteristic n has the following properties:

(P_1) The number of vertices is n^3.

(P_2) G is connected and regular.

(P_3) $n_2(x) = 3(n-1)^2$ for all x in G.

(P_4) The distinct eigenvalues of $A(G)$ are $-3, n-3, 2n-3, 3(n-1)$.

(P_1), (P_2), (P_3) are obvious. (P_4) is proved in paragraph 2. We go on to show that (P_1), (P_2), (P_3), (P_4) characterize a cubic lattice graph with characteristic n. Similar characterization for tethrahedral graphs has been given by Bose and Laskar [1].

2. Determination of the eigenvalues of $A(G)$. Given v objects, a relation satisfying the following conditions is said to be an association scheme with m classes:

(a) Any two objects are either 1st, 2nd, $\cdots$, or mth associates, the relation of association being symmetrical.

(b) Each object α has n_i ith associates, the number n_i being independent of α.

(c) If any two objects α and β are ith associates, then the number of objects which are jth associates of α, and kth associates of β, is p^i_{jk} and is independent of the pair of ith associates α and β.

The numbers v, n_i and p^i_{jk}, $i, j, k = 1, 2, \cdots, m$ are the parameters of the association scheme.

The concept of an association scheme was first introduced by Bose and Shimamoto [3].

If we define

$$B_i = (b^\beta_{\alpha i}) = \begin{pmatrix} b^1_{1i} & b^2_{1i} & \cdots & b^v_{1i} \\ b^1_{2i} & b^2_{2i} & \cdots & b^v_{2i} \\ \cdots & \cdots & \cdots & \cdots \\ b^1_{vi} & b^2_{vi} & \cdots & b^v_{vi} \end{pmatrix},$$

$i = 0, 1, 2, \cdots, m$,
where

$$b^\beta_{\alpha i} = 1, \text{ if the objects } \alpha \text{ and } \beta \text{ are } i\text{th associates}$$
$$= 0, \text{ otherwise},$$

and

$$\mathscr{P}_k = (p^j_{ik}) = \begin{pmatrix} p^0_{0k} & p^1_{0k} & \cdots & p^m_{0k} \\ p^0_{1k} & p^1_{1k} & \cdots & p^m_{1k} \\ \cdots & \cdots & \cdots & \cdots \\ p^0_{mk} & p^1_{mk} & \cdots & p^m_{mk} \end{pmatrix}, \qquad k = 0, 1, \cdots, m,$$

then it has been shown by Bose and Mesner [2], that the matrices $\mathscr{P}_i$, $i = 0, 1, \cdots, m$ are linearly independent and combine in the same way as the B's under addition as well as multiplication. It was further shown that if

$$B = \sum_{i=0}^{m} c_i B_i$$

$$\mathscr{P} = \sum_{i=0}^{m} c_i \mathscr{P}_i ,$$

then B and $\mathscr{P}$ have the same distinct eigenvalues. If in particular we take $c_0 = 0, c_1 = 1, c_2 = c_3 = \cdots = c_m = 0$, it follows that the distinct eigenvalues of B_1 are the same as those of $\mathscr{P}_1$.

Consider a cubic lattice graph G with characteristic n. If a relation of association on the vertices of G is defined, such that two vertices are 1st, 2nd, or 3rd associates if they are at distances 1, 2 or 3 respectively, then it can be easily checked that G yields a three-class association scheme. It may be pointed out that the matrix $A(G)$ is the matrix B_1 and thus the distinct eigenvalues of $A(G)$ are given by those of the matrix

$$\mathscr{P}_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ n_1 & p^1_{11} & p^2_{11} & p^3_{11} \\ 0 & p^1_{12} & p^2_{12} & p^3_{12} \\ 0 & p^1_{13} & p^2_{13} & p^3_{13} \end{pmatrix} .$$

The parameters p^i_{jk} of the association scheme corresponding to G are easily calculated. They are given by

$$\begin{aligned} n_1 = 3(n-1) , \quad & p^1_{11} = n-2 , & p^2_{11} &= 2 , & p^3_{11} &= 0 , \\ & p^1_{12} = 2(n-1) , & p^2_{12} &= 2(n-2) , & p^3_{12} &= 3 , \\ & p^1_{13} = 0 , & p^2_{13} &= n-1 , & p^3_{13} &= 3(n-2) . \end{aligned}$$

Substituting these values in the matrix $\mathscr{P}_1$, the eigenvalues are easily calculated. They are found to be

$$-3, n-3, 2n-3, 3(n-1) .$$

Thus, we have the following lemma:

LEMMA 2.1. *If G is a cubic lattice graph with characteristic n and if $A(G)$ is the adjacency matrix of G, then the distinct eigenvalues of $(A)G$ are*

$$-3, n-3, 2n-3, 3(n-1) . \tag{2.1}$$

3. Some preliminaries on matrices. Before stating the next lemma, we need the concept of the polynomial of a graph introduced by Hoffman [5]. Let J be the matrix all of whose entries are unity. Then for any graph G with adjacency matrix $A = A(G)$, there exists a polynomial $P(x)$ such that $P(A) = J$ if and only if G is regular and connected. The unique polynomial of least degree satisfying this equation is called the polynomial of G, and is calculated as follows: if G has v vertices, it is regular of degree d, and the other distinct eigenvalues of $A(G)$ are $\alpha_1, \alpha_2, \cdots, \alpha_t$, then

$$P(x) = \frac{v \prod_{i=1}^{t} (x - \alpha_i)}{\prod_{i=1}^{t} (d - \alpha_i)} . \tag{3.1}$$

Consider a regular connected graph H (with no loops and multiple edges) on $v = n^3$ vertices such that the adjacency matrix $A = A(H)$ has the distinct eigenvalues $-3, n - 3, 2n - 3, 3(n - 1)$.

LEMMA 3.1. *The matrix A satisfies the equation*

$$\begin{aligned} A^3 - A^2(3n - 9) + A(2n^2 - 18n + 27) \\ +(6n^2 - 27n + 27)I = 6J , \end{aligned} \tag{3.2}$$

where J is a $v \times v$ matrix all of whose entries are 1, *and I is the $v \times v$ identity matrix.*

Proof. It follows immediately by calculating the polynomial of the graph as given in (3.1).

LEMMA 3.2. *For any two vertices x, y in H, $d(x, y) \leq 3$.*

Proof. If in (3.2) we set $A_{ij} = 0$, $A^2_{ij} = 0$, then $A^3_{ij} = 6$, but this implies that $d(i, j) \leq 3$ for all vertices i, j in H.

LEMMA 3.3. *Consider the matrix*

$$B = \tfrac{1}{2}\{A^2 - (n - 2)A - 3(n - 1)I\} .$$

Let $n_2(i)$ denote the number of vertices j, such that $d(i, j) = 2$, and $n_3(i)$ denote the number of vertices k, such that $d(i, k) = 3$. If $n_2(i) = 3(n - 1)^2$ for all vertices i in H, then

(i) *B is a* (0, 1) *matrix,*
(ii) *$\Delta(x, y) = n - 2$, for all vertices x, y in H, such that $d(x, y) = 1$,*
(iii) *$\Delta(x, y) = 2$, for all vertices x, y in H, such that $d(x, y) = 2$.*

Proof. Since H is regular and $3(n-1)$ is the dominant eigenvalue, it follows H is regular of degree $n_1 = 3(n - 1)$.

Divide the set of vertices of H, with respect to a particular vertex i into four subsets S_0, S_1, S_2, S_3 as follows:

$S_0 : i$

$S_1 : j_1, j_2, \cdots, j_t \cdots, j_{n_1}$, such that $d(i, j_t) = 1, t = 1, 2, \cdots, n_1$

$S_2 : k_1, k_2, \cdots, k_s, \cdots, k_{n_2(i)}$, such that $d(i, k_s) = 2, s = 1, 2, \cdots, n_2(i)$

$S_3 : l_1, l_2, \cdots, l_r, \cdots, l_{n_3(i)}$, such that $d(i, l_r) = 3, r = 1, 2, \cdots, n_3(i)$.

Thus the vertices in S_t are tth associates of the vertex i. The following relations can be deduced easily from (3.2) by noting that $AJ = JA$.

$$A^3_{ii} = \sum_{t=1}^{n_1} A^2_{ij_t} = 3(n-1)(n-2) . \tag{3.3}$$

$$A^4_{ii} = \sum_{t=1}^{n_1} A^3_{ij_t} = 3(n-1)(n^2+3n-3) . \tag{3.4}$$

Also, since $A^tJ = \{3(n-1)\}^tJ$, we get

$$\sum_{j=1}^{v} A^2_{ij} = (A^2J)_{ii} = 9(n-1)^2 , \tag{3.5}$$

$$A^2_{ii} = \sum_{t=1}^{n_1} A_{ij_t} = 3(n-1) . \tag{3.6}$$

Also

$$\sum_{r=1}^{n_3(i)} A^2_{il_r} = 0 . \tag{3.7}$$

Hence it follows from (3.3), (3.5), (3.6), (3.7) that

$$\sum_{s=1}^{n_2(i)} A^2_{ik_s} = \sum_{j=1}^{v} A^2_{ij} - \sum_{t=1}^{n_1} A^2_{ij_t} - \sum_{r=1}^{n_3(i)} A^2_{il_r} - A^2_{ii} = 6(n-1)^2 . \tag{3.8}$$

Consider

$$X_i = b^2_{ii} + \sum_{t=1}^{n_1(i)} b^2_{ij_t} + \sum_{s=1}^{n_2(i)} (b_{ik_s} - 1)^2 + \sum_{r=1}^{n_3(i)} b^2_{il_r} = \sum_{j=1}^{v} b^2_{ij} - 2\sum_{s=1}^{n_2(i)} b_{ik_s} + n_2(i) . \tag{3.9}$$

We first show that

$$X_i = n_2(i) - 3(n-1)^2 .$$

Since

$$B = \tfrac{1}{2}[A^2 - (n-2)A - 3(n-1)I], \text{ we get} \tag{3.10}$$

(3.11) $$B^2_{ii} = \tfrac{1}{4}[A^4_{ii} - 2(n-2)A^3_{ii} + (n^2 - 10n + 10)A^2_{ii} + 6(n^2 - 3n + 2)A_{ii} + 9(n-1)^2 I_{ii}] \,.$$

Substituting values from (3.3), (3.4), (3.6) in (3.11) we get

$$B^2_{ii} = 3(n-1)^2 \,.$$

But

$$\sum_{j=1}^{v} b^2_{ij} = B^2_{ii} \,.$$

Hence

(3.12) $$\sum_{j=1}^{v} b^2_{ij} = 3(n-1)^2 \,.$$

Also from (3.10)

$$\sum_{s=1}^{n_2(i)} b_{ik_s} = \tfrac{1}{2} \sum_{s=1}^{n_2(i)} A^2_{ik_s} \,.$$

It follows from (3.8) that

(3.13) $$\sum_{s=1}^{n_2(i)} b_{ik_s} = 3(n-1)^2 \,.$$

Substituting values from (3.12), (3.13) in (3.9) we get

$$X_i = n_2(i) - 3(n-1)^2 \,.$$

Now if $n_2(i) = 3(n-1)^2$ for all i in H, then $X_i = 0$ for all i in H. Then it follows from (3.9) that B is a (0, 1) matrix which proves (i).

To prove (ii), we note that if $A_{ij_t} = 1$, then from (3.10), (3.3) and (3.6) it follows

$$\sum_{t=1}^{n_1} b_{ij_t} = 0 \,.$$

But since $b_{ij} = 0$ or 1, this implies $b_{ij_t} = 0$, and hence from (3.10) it follows that $A^2_{ij} = n - 2$.

To prove (iii) we note that if $A_{ij} = 0$, $A^2_{ij} \neq 0$, then $b_{ij} \neq 0$ and hence $A^2_{ij} = 2$.

4. THEOREM. *If H is a graph satisfying the following properties:*

(P_1) *The number of vertices is n^3.*

(P_2) *H is connected and regular.*

(P_3) *$n_2(x) = 3(n-1)^2$ for all x in H.*

(P_4) *The distinct eigenvalues of $A(H)$ are $-3, n-3, 2n-3, 3(n-1)$. Then, for $n > 7$, H is cubic lattice.*

Proof. From Lemmas (3.1) – (3.3) and the hypothesis H clearly satisfies the following conditions:

(b_1) The number of vertices is n^3.

(b_2) H is connected and regular of degree $3(n-1)$.

(b_3) $\Delta(x, y) = n - 2$ for $d(x, y) = 1$.

(b_4) $\Delta(x, y) = 2$, for $d(x, y) = 2$.

Hence if $n > 7, H$ is cubic lattice [6], [4].

Note. It is conjectured that the property (P_3) of the theorem is implied by other properties (P_1), (P_2), (P_4).

It may be pointed out that the main purpose of assuming (P_3) is to prove that B is a $(0, 1)$ matrix. If we replace (P_3) by (P_3') and (P_3'') as follows:

(P_3'). H is edge-regular, i.e., $\Delta(x, y) = \Delta$ for all x, y, such that $\Delta(x, y) = 1$,

(P_3'') $\Delta(x, y) =$ even, for all x, y, such that $d(x, y) = 2$,

then it can be shown that B is a $(0, 1)$ matrix. The proof goes like this: From (P_3') and (3.3) it follows that $\Delta = n-2$. Substituting value for Δ in (3.10) and noting (P_3'') we get $b_{ij} = 0$ if $A_{ij} = 1$, and $b_{ij} =$ an integer if $A_{ij} = 0$. Again from (3.10) and (3.12) it follows that

$$\sum_{j=1}^{v} b_{ij} = \sum_{j=1}^{v} b_{ij}^2 .$$

Thus B is a matrix whose entries are either 0 or integer such that for any row, sum of the elements is equal to the sum of the squares of the elements, but this implies that B is a $(0, 1)$ matrix.

Hence we can also state that for $n > 7$, (P_1), (P_2), (P_3'), (P_3''), (P_4) characterize a cubic lattice graph with characteristic n.

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