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SINGULAR PERTURBATION OF LINEAR PARTIAL DIFFERENTIAL EQUATION WITH CONSTANT COEFFICIENTS

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Let $P_j(z, \varepsilon)$ be a polynomial in z and ε with complex coefficients, where z is in E^m and $\varepsilon > 0$ is a small parameter. Let $L_\varepsilon = \sum_{j=0}^{l} P_{l-j}(\delta_x, \varepsilon)(\delta_t)^j$ be a polynomial in δ_t, δ_x and ε, which is not divisible by the square of a similar nonconstant polynomial. We shall assume that $P_0(z, \varepsilon) = \varepsilon$ and $P_1(z)$ is independent of ε.

In this paper we shall show that under certain conditions the solution $u_\varepsilon(t, x)$ of $L_\varepsilon(u) = f_\varepsilon(t, x)$ converges to the solution $u_0(t, x)$ of $L_0(u) = f_0(t, x)$.

Let $(t, x) = (t, x_1, x_2, \cdots, x_m)$ be a point in $R \times E^m$ where $0 \leqq t \leqq T$, and x in E^m, and E^m denotes an m-dimensional Euclidean space. Let also C_0^∞ be the set of all infinitely times continuously differentiable complex valued functions on E^m with compact support. For any u in C_0^∞, let the norm $\|u\|_p$ be defined for any integer $p < 0$ as follows:

$$(1) \qquad \int_{E^m} \sum_{|\varphi| \leqq p} |\partial_{x_1}^{\varphi_1} \cdots \partial_{x_m}^{\varphi_m} u|^2 \, dx = \|u\|_p^2 \qquad (|\varphi| = \varphi_1 + \cdots + \varphi_m) .$$

It is easy to see that the space C_0^∞ with the norm (1) gives a Hilbert space, which we shall call an H_p-space. We may also notice that $H_p \supset H_q$ and $\|u\|_p \leqq \|u\|_q$ if $p < q$. If for each φ in H_p we denote by $\hat{\varphi}$ the Fourier transform of φ

$$\hat{\varphi}(z) = [1/(2\pi)^{m/2}] \int_{E^m} \exp(-ix.\, z)\varphi(x)dx$$

where,

$$x, z = \sum_1^m x_i z_i$$

then the norm defined in (1) will be equivalent to the norm

$$(2) \qquad \|\varphi\|_p^2 = \int_{E^m} |(1 + |z|^2)^{p/2} \hat{\varphi}(z)|^2 \, dz = \|\hat{\varphi}\|_p^2 .$$

Notice that H_p with respect to the norm defined in (2) is the set of all complex valued measurable functions such that $\|\varphi\|_p < \infty$.

Let D^k be any differential operator with respect to x with constant coefficients of order $k \leqq p$. Then D^k is a bounded linear operator which maps H_p into H_{p-k}.

DEFINITION 1. Let $\varphi(t)$ be a variable element of H_p depending on a real parameter t in a finite interval $J = [0, T]$. We say that $\varphi(t)$ is H_p-continuous in t in J, if the mapping t in $J \longrightarrow \varphi(t)$ in H_p is continuous; That is, $t \longrightarrow t_0$ in the interval J implies $\varphi(t) \longrightarrow \varphi(t_0)$ in H_p. We also maintain that $\varphi(t)$ is H_p-differentiable at $t = t_0$, if there exist a function $g(t)$ in H_p such that

$$(t - t_0)^{-1}[\varphi(t) - \varphi(t_0)] \longrightarrow g(t_0)$$

in H_p as $t \longrightarrow t_0$, then we denote $g(t)$ by $\varphi'(t) = (d/dt)\varphi(t)$.

If D^k is a differential operator in x in E^m with constant coefficients of order k and $\varphi(t)$ is H_p-continuous in t, then $D^k\varphi(t)$ is H_{p-k} continuous, and if $\varphi(t)$ is H_p-differentiable in t then $D^k\varphi(t)$ is H_{p-k} differentiable in t and

$$(d/dt)[D^k\varphi(t)] = D^k[(d/dt)\varphi(t)] \,.$$

Mr. Nagumo in [1] considered a system of linear partial differential equations for an r-vector function with parameter $\varepsilon > 0$,

$$L_\varepsilon(u) = \sum_{j=0}^{l} P_j(\partial_x, \varepsilon)(\partial_t)^j u = f_\varepsilon(t, x) \tag{3}$$

where $P_j(z, \varepsilon)$ are $r \times r$ matrices of polynomials in (z, ε) with constant coefficients and P_l is free from ∂_x such that $\det[P_l(\varepsilon)] \neq 0$ for $\varepsilon = 0$.

Here we are concerned with the case of one equation for one complex valued function $u(t, x)$ containing the parameter $\varepsilon > 0$

$$L_\varepsilon(u) = \sum_{j=0}^{l} P_j(\partial_x, \varepsilon)(\partial_t)^j u = f_\varepsilon(t, x) \tag{4}$$

with the following assumptions:
(0) $L_\varepsilon(\sigma)$ be a polynomial in $\sigma, \partial x$ and ε which is not divisible by the square of a similar nonconstant polynomial for $0 \leqq \varepsilon \leqq \varepsilon_0$ and $f_\varepsilon(t, x)$ is H_p-continuous. $P_j(z, \varepsilon)$ are polynomials in $(z, \varepsilon) = (z_1, \cdots, z_m, \varepsilon)$ with constant coefficients such that $P_l(\varepsilon) \equiv \varepsilon$ and $P_{l-1}(z)$ is independent of ε.

System (4) is certainly a special case of system (3). Restricting ourselves to this special case, we will prove a stability theorem somewhat different from that of Mr. Nagumo [1]. Mr. Nagumo proved the convergence of the weak solution to $u_0(t, x)$; where as we shall prove the convergence of the solution $u_\varepsilon(t, x)$ to $u_0(t, x)$.

DEFINITION 2. We say that equation (4) is an H_p-stable equation for $\varepsilon \longrightarrow 0$ in $0 \leqq t \leqq T$ with respect to a particular solution $u_0(t, x)$ of (4) for $\varepsilon = 0$ if and only if $u_\varepsilon(t) \longrightarrow u_0(t)$ in H_p for $0 \leqq t \leqq T$ provided that

$$f_\varepsilon(t, x) \longrightarrow f_0(t, x) \tag{5}$$

in H_p for $0 \leqq t \leqq T$ and $u_\varepsilon(t, x)$ is a solution of the partial differential equation (4) such that

(6)
(i) $\partial_t^{j-1} u_\varepsilon(0) \longrightarrow \partial_t^{j-1} u_0(0)$ in H_p $\quad (j = 1, \cdots, l-1)$.
(ii) There exists a function $F(x)$ in H_p such that $|\partial_t^{l-1} \hat{u}_\varepsilon(0, z)| \leqq |\hat{F}(z)|$ for all small $\varepsilon > 0$.

As in [1] we associate the partial differential equation (4) with the ordinary differential equation

$$\sum_{j=0}^{l} P_j(iz, \varepsilon)(d/dt)^j y = 0 . \tag{7}$$

Let $Y_j(t, z, \varepsilon)$ be the solution of the ordinary differential equation (7) with the initial conditions

$$\partial_t^{k-1} Y_j(0, z, \varepsilon) = \delta_{jk} \qquad (\delta_{jk} \text{ is the Kronecker delta}) . \tag{8}$$

We state here a well known result

THEOREM 1. *Let $P(z)$ be a polynomial in z (z in E^m) with complex coefficients. If $S = (z$ such that $P(z) = 0)$ then S is measurable and has E^m measure zero unless $P(z)$ is identically zero.*

The proof is simple. One approach is to use mathematical induction on m and Fubini's theorem.

COROLLARY. *There exist an $\varepsilon_0 > 0$ such that for each ε in $0 \leqq \varepsilon \leqq \varepsilon_0$ then assumption* (0) *implies that the polynomial equation*

$$\sigma^l + P_{l-1}\sigma^{l-1} + \cdots + P_0(iz, \varepsilon) = 0 \tag{9}$$

has distinct roots except for z belonging to a set of E^m measure zero.

Proof. Notice that the assumption (0) implies that $D(z, \varepsilon)$ the discriminant of equation (9) is not identically zero. $D(z, \varepsilon)$ is a polynomial of $(z_1, \cdots, z_m, \varepsilon)$. Let us write $D(z, \varepsilon)$ as a product of irreducible polynomials in z and ε over the field of complex coefficients. If one or several of the factors do not depend on z explicitly, then they are polynomials in ε; in fact they are linear. All of these have at most finitely many positive zeros, say $\varepsilon_1, \cdots, \varepsilon_n$. Let $\varepsilon_0 = \min(\varepsilon_1, \cdots, \varepsilon_n)$; then for $\varepsilon\ \varepsilon_0$ we can write $D(z, \varepsilon)$ as a product of irreducible polynomials in z and ε none of which vanishes identically. Now by Theorem 1 the zeros of such polynomials for each ε are set of E^m measure zero.

Let $Y_i(t, z, \varepsilon)(i = 1, 2, \cdots, l)$ be the solution of the ordinary differential equation (7) with the initial conditions (8). If $\sigma_1, \cdots, \sigma_l$ are the distinct roots of equation (9) then we can write

$$Y_i(t, z, \varepsilon) = \sum_{j=1}^{l} \alpha_j^i \exp(\sigma_j t) . \tag{10}$$

Here α_j^i are constants to be computed by using the initial conditions (8). Let $V(\sigma_1, \cdots, \sigma_l)$ be the Vandermond determinant of $\sigma_1, \cdots, \sigma_l$, i.e., $V(\sigma_1, \cdots, \sigma_l) = \pi_{q>p}(\sigma_q - \sigma_p)$. Denote by V_i^j the determinant obtained from V by cancelling the i-th column and the j-th row.

THEOREM 2. *$V_i^j = \pi_{qp,q\neq j\neq p}(\sigma_q - \sigma_p)E_{j,i}$ where $E_{j,i}$ is the coefficient of the i-th power of σ_j in the expression*

$$(\sigma_1 - \sigma_j) \cdots (\sigma_{j-1} - \sigma_j)(\sigma_{j+1} - \sigma_j) \cdots (\sigma - \sigma_j)(-1)^j .$$

The proof is simple. Just write $V(\sigma_1, \cdots, \sigma_l)$ in two ways; first as a polynomial in σ_j, and second as a product of linear terms then equate the coefficients of σ_j in the two expressions.

Then initial conditions (8) and further use of Vandermond determinant give the following result

$$Y_i(t, z, \varepsilon) = (-1)^{i-1}\left[\sum_{j=1}^{l} E_{j,(i-1)} \exp(\sigma_j)/A_j\right] \tag{11}$$

where,

$$A_j = (\sigma_l - \sigma_j) \cdots (\sigma_{j-1} - \sigma_j)(\sigma_{j+1} - \sigma_j) \cdots (\sigma_1 - \sigma_j) . \tag{12}$$

Since the preceding result can be computed easily, we shall omit the details.

THEOREM 3. *If $\sigma_1, \cdots, \sigma_l$ are the roots of equation* (9) *and $Y_1, Y_2, \cdots, Y_l$ are the solutions of the ordinary differential equation* (7) *with the initial conditions* (8). *Then for each $\sigma_j \neq 0 (j = 1, \cdots, l)$ we have*

$$\sigma_j^{l-1} \sum_{i=1}^{l} (Y_i/\sigma_j^{l-1}) = \exp(\sigma_j t) . \tag{13}$$

Proof. The initial conditions (8) shows that the identity (13) is valid for $t = 0$. Furthermore take the 1st, 2nd, $\cdots$, $(l-1)$-th derivatives of both sides of the identity with respect to t and each time apply the initial conditions (8) we get the validity of the identity for $t = 0$. Since the right side of equation (13) is a solution of the ordinary differential equation (7) therefore the identity (13) is valid for all t in $0 \leqq t \leqq T$.

THEOREM 4. *For each fixed z in E^m consider $Y_j(t, z, \varepsilon)$ a function of t and ε only and assume that there exists a number $M_j(z)(j = 1, \cdots, l)$ independent of both t and ε such that*

$$| Y_j(t, z, \varepsilon) | \leqq M_j(z)$$

for $0 < \varepsilon \leqq \varepsilon_0$ and $0 \leqq t \leqq T$. Then the roots of equation (9) *have for each z in E^m a real part bounded from above as $\varepsilon \to 0$.*

Proof. Let z_0 be a fixed point in E^m. Let $\mu = \varepsilon\sigma$ when σ is a root of equation (9) then, equation (9) becomes

$$\mu^l + P_{l-1}(iz_0)\mu^{l-1} + \varepsilon P_2(iz, \varepsilon)\mu^{l-2} + \cdots + \varepsilon^{l-1}P_0(iz, \varepsilon) = 0 . \tag{14}$$

Now assume first that $P_{l-1}(iz_0) \neq 0$ then for $\varepsilon = 0$ equation (14) becomes

$$\mu^l + P_{l-1}(iz_0)\mu^{l-1} = 0 = \mu^{l-1}(\mu + P_{l-1}(iz_0)) . \tag{15}$$

Here we have one simple root $\mu_1 = - P_{l-1}(iz_0)$, if we call this root $\mu_1(0)$ then we can write

$$\mu_1(\varepsilon) = \mu_1(0) + \sum_1^\infty q_{1i}\varepsilon^i, \text{ so } \mu_1(\varepsilon) \to \mu_1(0) \text{ as } \varepsilon \to 0 .$$

Therefore we can write the simple root $\sigma_1(z_0, \varepsilon)$ of equation (14) as follows,

$$\sigma_1 = (1/\varepsilon)\mu_1(\varepsilon) = (-P_{l-1}(iz_0)/\varepsilon) + \sum_{i=1}^\infty q_{li}(z_0)\varepsilon^{i-1} .$$

Hence Real $\sigma_1(z_0, \varepsilon) = -$ Real $P_{l-1}(iz_0)/\varepsilon +$ Real $q_{11}(z_0) + o(\varepsilon)$. Therefore if Real $P_{l-1}(iz_0) \geqq 0$ then obviously Real $\sigma_1(\varepsilon)$ is bounded from above. Now suppose that Real $P_{l-1} < 0$. Then $| \sigma_1(\varepsilon) | \geqq$ Real $(\sigma_1(\varepsilon)) \to \infty$ in turn implies that for $t > 0$ we get

$$\left| \sum_1^l [Y_j(t, z_0, \varepsilon)/\sigma^{l-j}] \right| M_j(z_0)$$

as $\varepsilon \to 0$ for some number $M_j(z_0)$ independent of t and ε. This is so because of the hypothesis of the theorem. Now we use Theorem 3. Then identity (13) shows for small

$$\varepsilon > 0, 2 | P_{l-1} | \geqq | - P_{l-1}(iz_0 + 0(\varepsilon) | \geqq (1/2) | P_{l-1}(iz_0) |$$

since Real $P_{l-1}(iz_0) < 0$ and as $t > 0$,

$$\begin{aligned}&[\exp (t \operatorname{Re} \sigma_1)/\sigma_1^{l-1} |] \geqq [\exp (2_2^t\varepsilon | P_{l-1} |)\varepsilon^{l-1}/(4 | P_{l-1}(iz_0)|)^{l-1}] \\ &\quad \geqq (t/4^l l! \, \varepsilon) \to \infty \text{ as } \varepsilon \to 0\end{aligned}$$

utilizing only the $(l - 1)$ th term of the Taylor series of the exponential function.

But the above result contradicts the boundedness of expression (13) for small $\varepsilon > 0$. Consequently Real $P_{l-1}(iz_0) \geqq 0$ and hence Real $\sigma_1(\varepsilon)$ is bounded from above as $0 < \varepsilon \leqq \varepsilon_0$.

In order to prove the result for the remaining of the roots of equation (9) we shall give a reasoning which is incidentally does not utilize the condition $P_{l-1}(iz_0) \neq 0$. Equation (15) has a root $\mu(0) = 0$ of multiplicity $(l-1)$. From the Puiseux series expansion we deduce that the $(l-1)$ roots $\mu(\varepsilon)$ of equation (14) other than μ_1 will split into r groups of $m_1, \cdots, m_r, 1 \leqq m_1 \leqq \cdots \leqq m_r$ and $\sum_1^r m_i = l-1$ as follows: each root $\mu_{0j}, \cdots, \mu_{m_{j-1}j}$ can be written as $\mu_{\eta j} = \sum_{j=1}^{\infty} q_{\eta j i} x^i$ where,

$$x = (\varepsilon)^{1/m}j \text{ and } \eta = 0, 1, \cdots, m_{j-1} .$$

Notice that the above series converges for sufficiently small x. We shall here and later understand by $(\varepsilon)^{1/m}j$ the positive m_j-th root of $\varepsilon > 0$. Let $\mu_{\eta j}$ be any one of the $(l-1)$ roots of equation (14) which tend to zero as ε tends to zero, then we can write the corresponding roots σ_η of equation (9) as follows

$$(\mu_\eta/\varepsilon) = \sigma_\eta(x, z_0) = (1/X^m j\left(\sum_{i=1}^{\infty} q_{\eta j i} x^i\right) . \tag{16}$$

Now to simplify notations, let us drop indices j, η of the root $\mu_{\eta j}$ once we are dealing with only one root. Put $q\eta_{ji} = q_i$ and $m_j = m$. Assume that q_k is the first nonzero coefficient in equation (16) if there is any, and q_s is the first nonzero coefficient that has a nonzero real part, if there is any, in the expresion $\mu = \sum q_i x^i$. Evidently, if $s \geqq m$ then, Real $\sigma(x, z_0) = x^{s-m}$ Real $q_s + 0(x)$, and this is bounded from above as $x \to 0$. If there is no s, then $\sigma(x, z_0) \leqq 0$. Let $s < m$, notice that $k \leqq s \leqq m$. Then we can write σ as follows,

$$\sigma(x) = x^{k-m}(q_k + \cdots + q_s x^{s-k} + q_{s+1} x^{s-k+1} + \cdots)$$

and

$$\text{Real}\, \sigma(x) = x^{s-m}(\text{Real}\, q_s + 0(x)) .$$

If Real $q_s < 0$ then Real $\sigma(x) \to -\infty$ as $x \to 0$ implying Real $\sigma \leqq 0$ as $\varepsilon \to 0$. Suppose finaly Real $q_s > 0$, i.e., Real $\sigma(x) \to +\infty$ as $x \to 0$. Then for small $x > 0$, $2\,|\,q_k\,| \geqq |\,q_k + 0(x)\,|$ and Real $q_s + 0(x) \geqq \text{Re}\, q_s/2$, and

$$\begin{aligned}\exp(t \,\text{Re}\, \sigma)/\,|\,\sigma\,|^{l-1} &\geqq \exp(\tfrac{t}{2}\, \text{Re}\, q_s x^{s-m})/2^{l-1}\,|\,q_k\,|^{l-1} x^{(k-m)(l-1)} \\ &> (t^j/j!\, 2^j) x^{j(s-m)}\, (\text{Real}\, q_s)^j/2^{l-1}\,|\,q_k\,|^{l-1} x^{(k-m)(l-1)} .\end{aligned}$$

Here we utilize only the $(j+1)$-th term of the Taylor series of the exponential function where j is the smallest integer greater than zero such that $(m-k)(l-1)+j(s-m)<0$. Therefore

$$\exp(t \operatorname{Real} \sigma)/|\sigma|^{l-1} > t^j(\operatorname{Real} q_s)^j x^{(m-k)(l-1)+j(s-m)}/j!\, 2^{j+l-1}|q_k|^{l-1} \longrightarrow \infty$$

as $x \to 0$ because of the assumption on j.

Now in order to prove that Real σ is bounded from above we use identity (13) once more.

In order to complete the proof of the theorem we should assume that $P_{l-1}(iz_0)=0$. Notice that equation (14) for $P_{l-1}=0$ becomes

$$\mu^l + \varepsilon P_{l-2}(iz_0, \varepsilon) + \cdots + \varepsilon^{l-1} P_0(iz_0, \varepsilon) = 0 .$$

This means that $\mu = 0$ is a root of multiplicity l, not $l-1$, of equation (15). So by Puiseux series expansion we can write the roots

$$\sigma_{1\eta_1} = (1/\varepsilon)\left[\sum_{j=1}^{\infty} \beta_{1j} \exp(2\pi i \eta_1 j/m_1)\varepsilon^{j/m_1}\right] \qquad \eta_1 = 0, \cdots, m_1 - 1$$

$$\vdots$$

$$\sigma r\eta_r = (1/\varepsilon)\left[\sum_{j=1}^{\infty} \beta_{rj} \exp(2\pi i \eta_r j/m_r)\varepsilon^{j/m_r}\right] \qquad \eta_r = 0, \cdots, m_r - 1 .$$

Now we can carry on the same proof as before for any root. The proof of Theorem 4 is completed.

In what follows there will be for each ε certain exceptional sets of z of measure zero for which our conclusion do not apply. In order to be able to draw inferences as $\varepsilon \to 0, \varepsilon > 0$ we wish to be able to disregard these sets. Now let the notion $\varepsilon \to 0$ be henceforth interpreted as meaning "ε tends to zero through an arbitrary sequence of positive numbers." Then all of the corresponding exceptional sets will still be a countable union of sets of measure zero and accordingly has itself measure zero.

THEOREM 5. *Assume in equation* (9) *that $P_{l-1}(iz_0)$ does not vanish identically. Also assume that for $0 \leqq t \leqq T$, $0 < \varepsilon \leqq \varepsilon_0$, there are numbers $M(z)$ and $M_j(z)(j=1, \cdots, l)$ independent of both t and ε such that,*

$$|Y_j(t, z, \varepsilon)| \leqq M_j(z) \text{ for } 0 < \varepsilon \leqq \varepsilon_0 .$$

Then for almost all z in E^m we have

(i) $|Y_l(t, z, \varepsilon)| \to 0$ *as* $\varepsilon \to 0$ *and for* $0 \leqq t \leqq T$.

(ii) $|(1/\varepsilon) Y_l(t, z, \varepsilon)| \leqq M(z)$ *for* $0 \leqq t \leqq T$ *and as* $\varepsilon \to 0$.

Proof. Here we use Theorem 1 in order to be able to assume $P_{l-1}(iz_0) \neq 0$. By the corollary of Theorem 1, equation (9) has for fixed small ε and for almost all z in E^m distinct roots. By letting $\mu = \varepsilon\sigma$ the equation (9) becomes as we have seen before

$$\mu^l + P_{l-1}(iz_0)\mu^{l-1} + \varepsilon P_{l-1}(iz_0, \varepsilon)\mu^{l-2} + \cdots + \varepsilon^{l-1}P_0(iz_0, \varepsilon) = 0 .$$

So for $\varepsilon = 0$ the above equation will be,

$$\mu^{l-1}(\mu + P_{l-1}(iz_0)) = 0 .$$

Therefore by Puiseux series expansion we can write the roots of equation (9) as follows,

$$\sigma_1 = (1/\varepsilon)[-P_{l-1}(iz_0) + 0(\varepsilon)]$$

m_1 roots as follows,

$$\sigma_{1+\eta_1} = (1/\varepsilon)\left[\sum_{j=1}^{\infty} \beta_{1j}\varepsilon^{j/m_1} \exp(2\pi i\eta_1 j/m_1)\right] \qquad n_1 = 1, \cdots, m_1$$

$$\vdots$$

and m_r roots as follows,

$$\sigma_{-m_r+\eta_r} = (1/\varepsilon)\left[\sum_{j=1}^{\infty} \beta_{rj} \exp(2\pi i\eta_r j/m_r)\varepsilon^{j/m_r}\right] \qquad \eta_r = 1, \cdots, m_r .$$

Provided that $l = 1 + m_1 + \cdots + m_r$ where $1 \leqq m_1 \leqq \cdots \leqq m_r$. Let us write $\gamma = [\sigma_1, \cdots, \sigma_l]$. Notice that each σ in γ is a regular function in ε and hence the difference between any two of them is also regular in ε. If $\sigma_1 = (1/\varepsilon)[-P_{l-1}(iz_0) + 0(\varepsilon)]$ then for any j, $1 < j \leqq l$ we have, since $P_{l-1}(iz_0) \neq 0$,

$$|\sigma_j - \sigma_1| = |(1/\varepsilon)[-P_{l-1}(iz_0) + 0(\varepsilon) - 0(\varepsilon^R)]| \longrightarrow \infty \text{ as } \varepsilon \longrightarrow 0 .$$

For $\sigma_j = (1/\varepsilon)[0(\varepsilon^R)]$, R 0 a rational number.

Notice that for each $i = 1, \cdots, l$, $i \neq j$,

$$|\sigma_j - \sigma_i|$$

either; (1) goes to zero as ε tends to zero; (2) tends to some fixed number greater than zero or (3) goes to infinity as ε tends to zero. For any arbitrary σ in γ collect those, and only those, elements σ' of γ such that $|\sigma' - \sigma| \longrightarrow 0$ as $\varepsilon \longrightarrow 0$. Then γ will split into disjoint subsets, namely:

(17) $$\gamma = \gamma_1 U\gamma_2 \cdots U\gamma_r \text{ and } \gamma_j \cap \gamma_k = \varnothing \text{ for } j \neq k ,$$

which incidentally do not necessarily coincide with our previous grouping of the σ's. According to this decomposition of γ and using identity (10) we can write

$$|Y_l(t, z, \varepsilon)| = \left|\sum_{\sigma_j in \gamma_1} [\exp(t\sigma_j)/A_j] + \cdots + \sum_{\sigma_j in \gamma_r} [\exp(t\sigma_j)/A_j]\right| .$$

LEMMA 1. *In the above expression each summation tends to zero in absolute value as ε tends to zero.*

Proof. Denoting by $\sigma_1', \cdots, \sigma_n'$ the elements of γ_1. Then,

$$\sum_{\sigma_j' in \gamma_1} [\exp(\sigma_j' t)/A_j] = \sum_{\delta_j' in \gamma_1} [\exp(\sigma_j' t)/\varphi]$$

$$\varphi = \prod_{\sigma_k in \gamma - \gamma_1} (\sigma_k - \sigma_j') \prod_{\substack{\sigma_k in \gamma_1 \\ k \neq j}} (\sigma_k' - \sigma_j') .$$

Let $F(\sigma_j') = \exp(t\sigma_j') / \prod_{\sigma_k in \gamma - \gamma_1} (\sigma_k - \sigma_j')$ and $F(\sigma_j') = \alpha(\varepsilon)$. Now if γ_1 contains the root $\sigma_1 = (1/\varepsilon)[-P_{l-1}(iz_0) + 0(\varepsilon)]$ then it will contain only σ_1. Since $P_1(iz_0) \neq 0$ it is easily shown that,

$$\exp(t\sigma_1)/A_1 \longrightarrow 0 \text{ as } \varepsilon \longrightarrow 0 .$$

Now suppose that γ_1 does not contain σ_1. Then $\prod_{\sigma_k in \gamma - \gamma_1} (\sigma_k - \sigma_j')$ will have a factor $(\sigma_1 - \sigma_j') = 0(\varepsilon)$ and hence tends to infinity as ε tends to zero while no factor of $\prod_{\sigma_k in \gamma - \gamma_1} (\sigma_k - \sigma_j')$ tends to zero as ε tends to zero. Therefore,

$$\prod_{\sigma_k in \gamma - \gamma_1} (\sigma_k - \sigma_j') \longrightarrow \infty \text{ as } \varepsilon \longrightarrow 0 .$$

This in turn implies that $\alpha(\varepsilon)$ tends to zero as ε tends to zero. Let $\alpha = \min(\text{limit}_{\varepsilon \to 0} |\sigma_k - \sigma_j'|)$. The minimum is taken over all σ_k in $\gamma - \gamma_1$. Notice that by the definition of $\gamma_1, \delta > 0$. Now chose a circle C of radius $\delta/2$ about one of the points $\sigma_1', \cdots, \sigma_n'$ of γ_1. Then for sufficiently small $\varepsilon > 0, C$ will contain those, and only those, σ_j' which belong to γ_1. We may likewise assume that for any point w on the circumference of C, that $|w - \sigma_j'| > \delta/4, (j = 1, \cdots, n)$. Let

$$I = (1/2\pi i) \int_C F(\eta) d\eta / \prod_{\delta_k' in \gamma_1} (\sigma_k' - \eta)$$

then,

$$I < 4^n \alpha(\varepsilon)/\delta^n \longrightarrow 0 \text{ as } \varepsilon \longrightarrow 0 . \tag{18}$$

On the other hand, I equal the absolute value of the sum of the residues of the integrand at $\sigma_1', \cdots, \sigma_n'$. Notice that for each σ_j' the residue of I at σ_j' is $F(\sigma_j') / \prod_{\sigma_k \neq \sigma_j'} (\sigma_k - \sigma_j')$ and hence the sum of the residues of I at the $\sigma_1', \cdots \sigma_n'$ is equal to

$$\sum_{\sigma_j' in \gamma_1} [\exp(t\sigma_j')/A_j] .$$

Hence by (18) the above expression tends to zero as ε tends to zero and this ends the proof of the lemma. Now in order to finish the proof of the theorem we just write,

$$| Y_l(t, z\varepsilon) | \leqq \left| \sum_{\sigma_j in \gamma_1} [\exp(t\sigma_j)/A_j] + \cdots + \sum_{\sigma_j in \gamma_r} \exp(t\sigma_j)/A_j \right| .$$

Using Lemma 1 one sees for all positive $t \leqq T$ that

$$| Y_l(t, z, \varepsilon) | \to 0 \text{ as } \varepsilon \to 0 .$$

Now for proving the second part of the theorem, one notices from the preceding discussion that $Y_l(t, z, \varepsilon) = 0(\varepsilon)$, i.e., there exist a number $M(z)$ such that $| Y_l(t, z, \varepsilon)/\varepsilon | \leqq M(z)$.

LEMMA A. *Limit*$_{\varepsilon \to 0} | E_{1,(i-1)}/A_1 | = 0.$

Proof. $|E_{1,(i-1)}/A_1| = |\sigma_2 \cdots \sigma_{l-i+1} + \cdots + \sigma_{i+1} \cdots \sigma_{l-1}/j = 2(\sigma_j - \sigma_1)|$ $(j = 1, \cdots, l)$. Notice that as ε tends to zero σ_1 tends to infinity while $\sigma_j(\varepsilon)$ tends to $\sigma_j(0)$ and hence this proves the lemma.

LEMMA B. *Limit*$_{\varepsilon \to 0} (E_{j(i-1)}/\sigma_1) = E'_{j,(i-1)}$ *where* $E'_{j,(i-1)}$ *is the coefficient of the* $(i-1)$*th power of* $\sigma_j(0)$ *in the expression,*

$$\prod_{\substack{k \neq 2 \\ k \neq j}}^{l} [\sigma_k(0) - \sigma_j(0)](-1)^j .$$

Proof. Notice that $E_{j,(i-1)}$ is the sum of the product of σ taken $l - (i + 1)$ at a time, i.e.,

$$E_{j,(i-1)} = [\sigma_1 \cdots \sigma_{l-i+1} + \cdots + \sigma_{i+1} \cdots \sigma_{l-1}](-)^{j-1}(-1)^{i-1} .$$

Therefore,

$$E_{j,(i-1)}/\sigma_1 = \sigma_2 \cdots \sigma_{l-i+1} + \cdots + (\sigma_{i+1} \cdots \sigma_{l-1}/\sigma_1)(-1)^{i+j2} .$$

Now it is easy to see that $E_{j,(i-1)}$ tends to $E'_{j,(i-1)}$ as ε tends to zero.

LEMMA C. *Limit*$_{\varepsilon \to 0} [E_{j,(i-1)} \exp(t\sigma_j)/A_j] = E'_{j,(i-1)} \exp(\sigma(0)t)/A'_j$ $(j = 2, \cdots)$ *and* $A'_j = k = 2[\sigma_k(0) - \sigma_j(0)]$.

Proof. Let us write

$$A_j = (\sigma_1 - \sigma_j) \prod_{\substack{k=2 \\ k \neq j}}^{l} (\sigma_k - \sigma_j) = \sigma_1 \prod_{\substack{k=2 \\ k \neq j}}^{l} (\sigma_k - \sigma_j) - \sigma_j \prod_{\substack{k=2 \\ k \neq j}}^{l} (\sigma_k - \sigma_j) .$$

Therefore A_j/σ_1 tends to A'_j as ε tends to zero. Now we use Lemma B and the proof of Lemma C will be completed.

Notice that $Y_i(t, z, 0) = \sum_{j=2}^{l} E'_{j,(i-1)} \exp(\sigma_j(0)t)/A'_j$

is the solution of the ordinary differential equation $L_0(D) Y = 0$ with the initial conditions $\sigma_t^{k-1} Y_i(0, z, 0) = \delta_{jk}\ 1 \leqq j, k \leqq l - 1$. Now we may sum the results of Lemmas A, B, and C in the following theorem.

THEOREM 6. *Let* $Y_1(t, z, \varepsilon), \cdots, Y_l(t, z, \varepsilon)$ *be the solutions of the ordinary differential equation* (7) *with the initial conditions* (8). *Assume that;* (1) $P_{l-1}(iz)$ *is not identically zero.* (2) *assumption* (0) *on page* (3). *And finally* (3) *There exist numbers* $M_i(z)$ *and* ε_0 *such that for* $0 < \varepsilon \leqq \varepsilon_0$ *and for almost all* z *in* E^m,

$$| Y_i(t, z, \varepsilon) | \leqq M_i(z) \qquad 1 \leqq i \leqq l, l > 2 .$$

Let ε *tend to zero on a sequence, then* $Y_i(t, z, \varepsilon)$ *tends to* $Y_i(0, z, 0)$ *for* $0 \leqq t \leqq T, 1 \leqq i \leqq l-1$ *and for almost all* z *in* E^m. *Where* $Y_i(t, z, 0)$ *are the solution of the ordinary differential equation* $L_0(D)Y = 0$ *with the initial conditions* $\partial_t^{k-1} Y_i(0, z, 0) = \delta_{ik}, 1 \leqq i, k \leqq l-1$.

THEOREM 7. *Assuming all the hypothesis of Theorem* (5) *then for each* z *in* E^m *we have,*

$$| (1/\varepsilon) Y_l(t, z, \varepsilon) - (1/P_{l-1}(iz)) Y_{l-1}(t, z, \varepsilon) | \longrightarrow 0$$
$$\text{as } \varepsilon \longrightarrow 0 \text{ and for all } 0 \leqq t \leqq T .$$

Proof.

$$\begin{aligned} &| \varepsilon^{-1} Y_l(t, z, \varepsilon) - P_{l-1}(iz)^{-1} Y_{l-1}(t, z, \varepsilon) | \\ &\leqq \left| (\varepsilon A_1)^{-1} \exp(\sigma_1 t) + (A_1 P_{l-1})^{-1} \exp(\sigma_1 t) \sum_2^l \sigma_i \right| \\ &+ \left| \sum_{j=2}^l [(\varepsilon A_j)^{-1} \exp(\sigma_j t) + (P_{l-1} A_j)^{-1} \exp(\sigma_j t) \sum_{\substack{i=1 \\ i \neq j}}^l \sigma_i \right| . \end{aligned}$$

It is clear that the first term of the above sum tends to zero as ε tends to zero. Before dealing with the second term we shall reduce it into a simpler form. Notice that $\sum_{i=1, i \neq j} \sigma_i = \sum_{i=1} \sigma_i - \sigma_j = -\varepsilon^{-1} P_{l-1}(iz) - \sigma_j$.

Then it is easy to see that

$$\begin{aligned} &\sum_{j=2}^l \left[(\varepsilon A_j)^{-1} \exp(\sigma_j t) + (A_j P_{l-1})^{-1} \exp(\sigma_j t) \sum_{\substack{i=1 \\ i \neq j}} \sigma_i \right] \\ &= - \sum_{j=2}^l (A_j P_{l-1})^{-1} \sigma_j \exp(\sigma_j t) . \end{aligned}$$

Now if, $| \sigma_i - \sigma_j | \geqq \delta > 0 \ (i, j = 2, \cdots, l) \ i \neq j$ then it is clear that

$$\left| \sum_{j=2}^l (A_j P_{l-1})^{-1} \sigma_j \exp(\sigma_j t) \right| \longrightarrow 0 \text{ as } \varepsilon \longrightarrow 0 .$$

On the other hand, if for some i and $i \neq j$ we have $| \sigma_i - \sigma_j |$ tends to zero as ε tends to zero then we use the residue theorem to prove that

$$\sum_{j=2} (A_j P_{l-1})^{-1} \sigma_j \exp(\tau_j t) \longrightarrow 0 \text{ as } \varepsilon \longrightarrow 0$$

in the same way used before. The proof of Theorem 7 is ended.

Now we arrive at the main theorem of this paper.

THEOREM 8. *Let the degree of $P_j(iz)$ be at most k in z $(j = 1, \cdots l)$ and assume that $P_{l-1}(iz)$ not identically zero. Denote by $u_0(t)$ the $l-1$ times H_{p+k} continuously differentiable solution of the partial differential equation* (4) *for $\varepsilon = 0$ in $0 \leqq t \leqq T$. If there exist two constants $\varepsilon_0 > 0$ and C such that*

$$(19, \text{i}) \quad \operatorname*{Sup}_{z \text{ in } E^m} | Y_j(t, z, \varepsilon) | \leqq C \text{ for } 0 \leqq t \leqq T \text{ and } 0 < \varepsilon \leqq 0 \quad (j = 1, \cdots l)$$

$$(19, \text{ii}) \quad \operatorname*{Sup}_{z \text{ in } E^m} \int_0^T | \varepsilon^{-1} Y_l(t, z, \varepsilon) | \, dt \leqq C \text{ for } 0 < \varepsilon \leqq \varepsilon_0$$

where $y = Y_j$ the solutions of equation (7) *with the initial conditions* (8). *Then equation* (4) *is an H_p-stable equation with respect to $u_0\,(t)$.*

Proof. Let $u_\varepsilon(t, x)$ be l-times H_{p+k} continuously differentiable solution of the partial differential equation (4) with the initial conditions (6). Then from Theorem 2 in [1] we may write

$$u_\varepsilon(t, x) = \sum_{j=1}^{l} (2\pi)^{-m/2} \int_{E^m} \exp(ix, z) Y_j(t, z, \varepsilon) \partial_t{}^{j-1} \hat{u}_\varepsilon(0, z) dz$$
$$+ 2\pi^{-m/2} \int_{E^m} \exp(ix, z) \int_0^t \varepsilon^{-1} Y(t - \tau, z\varepsilon) \hat{f}_\varepsilon(\tau, z) d \, dz$$

and

$$u_0(t, x) = \sum_{j=1}^{l-1} (2\pi)^{-m/2} \int_{E^m} \exp(ix, z) Y_j(t, z, 0) \partial_t^{j-1} \hat{u}_0(0, z) dz$$
$$+ (2\pi)^{-m/2} \int_{E^m} \exp(ix, z) \int_0^t P_{l-1}{}^{-1} Y_l(t - \tau, z) \hat{f}_0(\tau, z) d \, dz \, .$$

We have to prove that

$$\| u_\varepsilon(t, x) - u_0(t, x) \|_p \longrightarrow 0 \text{ as } \varepsilon \longrightarrow 0 \, .$$

Let us write

$$\| u_\varepsilon(t, x) - u_0(t, x) \|_p = M(x) + N(x) + Q(x)$$

$$M(x) = \| (2\pi)^{-m/2} \sum_{j=1}^{l} \int_{E^m} \exp(ix, z) Y_j(t, z, \varepsilon) \partial_t^{j-1} [\hat{u}_\varepsilon(0, z) - \hat{u}_0(0, z)]$$
$$+ (2\pi)^{-m/2} \int_{E^m} \exp(ix, z) Y_{l \partial t}^{l-1} \hat{u}_\varepsilon(0, z) \|_p$$

$$N(x) = \int_{E^m} |(1 + |z|^2)^{m/2} \int_0^t \varepsilon^{-1} Y_l(t - \tau, z, \varepsilon)\hat{g}(\tau, z)d\tau|^2 dz$$
$$\times \hat{g}(\tau, z) = \hat{f}_\varepsilon(\tau, z) - \hat{f}_0(\tau, z)$$

$$Q(x) = \int_{E^m} (1 + |z|^2)^{m/2} \int_0^t (\varepsilon^{-1} Y_l(t - \tau, z, \varepsilon) - P_{l-1}^{-1} Y_{l-1}(t - \tau, z, 0))$$
$$\times \hat{f}_0(\tau, z)d\tau|^2 dz .$$

To prove the convergence of $M(x)$ we proceed as follows,

$$M(x) \leqq ||(2\pi)^{-m/2} \sum_{j=1}^{l-1} \int_{E^m} \exp(ix, z) Y_j(t, z, \varepsilon)\partial_t^{j-1}[\hat{u}_\varepsilon(0, z) - \hat{u}_0(0, z)]||_p$$
$$+ ||(2\pi)^{-m/2} \int_{E^m} \exp(ix, z) Y_l(t, z, \varepsilon)\partial_t^{l-1}\hat{u}_\varepsilon(0, z)||_p .$$

Using the condition (19, i) and Ascoli's theorem we conclude

$$M(x) \longrightarrow 0 \text{ as } \varepsilon \longrightarrow 0 .$$

Condition (19, ii) and Ascoli's theorem imply that

$$N(x) \longrightarrow 0 \text{ as } \varepsilon \longrightarrow 0 .$$

For $Q(x)$ we proceed as follows. Notice that Theorem 6 shows that

$$(1/P_{l-1}(iz)) Y_{l-1}(t, z, \varepsilon) \longrightarrow (1/P_{l-1}(iz)) Y_{l-1}(t, z, 0)$$

and Theorem 7 shows that

$$|\varepsilon^{-1} Y_l(t, z, \varepsilon) - P_{l-1}^{-1} Y_{-1}(t, z, \varepsilon)| \longrightarrow 0 \text{ as } \varepsilon \longrightarrow 0 .$$

Therefore

$$|\varepsilon^{-1} Y_l(t, z, \varepsilon) - P_{l-1}^{-1}(iz) Y_{l-1}(t, z, 0)| \longrightarrow 0 \text{ as } \varepsilon \longrightarrow 0 .$$

Consequently, using Ascoli's theorem once more we get

$$Q(x) \longrightarrow 0 \text{ as } \varepsilon \longrightarrow 0 .$$

This ends the proof of Theorem 8.

References

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